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Characterization of Multiplicative Mixed Jordan-Type Derivations on Ring with Involution

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Abstract. Let $\mathcal{B}$ be a unital $\varnothing$-ring with a 2-torsion free that contains non-trivial symmetric idempotent. For any $B_1, B_2, B_3, \ldots, B_n \in \mathcal{B}$, a product $B_1 \circ B_2 = B_1B_2 + B_2B_1$ is called Jordan product and $B_1 \bullet B_2 = B_1B_2 + B_2B_1^{\varnothing}$ is recognized as a skew Jordan product. Characterize mixed Jordan triple product as $Q_3(B_1, B_2, B_3) = B_1 \circ B_2 \bullet B_3$ and mixed Jordan n-product as $Q_n(B_1, B_2, \ldots, B_n) = B_1 \circ B_2 \circ \cdots \bullet B_n$ for all integer $n \geq 3$. The present paper deals that a mapping which is called multiplicative mixed Jordan n-derivation, $\Psi\colon \mathcal{B} \to \mathcal{B}$ satisfies $\Psi(Q_n(B_1, B_2, \ldots, B_n)) = \sum_{i=1}^{n} Q_n(B_1, \ldots, B_{i-1}, \Psi(B_i), B_{i+1}, \ldots, B_n)$ for all $B_1, B_2, \ldots, B_n \in \mathcal{B}$ if and only if Ψ is an additive $\varnothing$-derivation. Finally, primary outcome is applicable in various specific categories of unital $\varnothing$-rings and $\varnothing$-algebras including prime $\varnothing$-rings, prime $\varnothing$-algebras and factor von Neumann algebras.

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Key Words and Phrases: Additive $\varnothing$-derivation; Mixed Jordan-type derivation; $\varnothing$-rings; factor von Neumann algebra.

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1. Introduction

Throughout this work, $\mathcal{B}$ is taken to be an associative ring having $Z(\mathcal{B})$ as its center. An involution '$\varnothing$' is described as an anti-automorphism of order 1 or 2 and $\mathcal{B}$ along with an involution '$\varnothing$' is known as a $\varnothing$-ring. If $\alpha^2 = \alpha = \alpha^{\varnothing}$, then $\alpha \in \mathcal{B}$ is termed a symmetric idempotent and it is considered non-trivial if α is neither 0 nor I. A mapping $\Psi\colon \mathcal{B} \to \mathcal{B}$ is called a *derivation* if it satisfies $\Psi(B_1+B_2) = \Psi(B_1)+\Psi(B_2)$ and $\Psi(B_1B_2) = \Psi(B_1)B_2 + B_1\Psi(B_2)$ for all $B_1, B_2 \in \mathcal{B}$. For a ring $\mathcal{B}$ with an involution '$\varnothing$' a mapping $\Psi : \mathcal{B} \to \mathcal{B}$ is called an *additive $\varnothing$-derivation*, if it is a derivation and $\Psi(B_1^{\varnothing}) = \Psi(B_1)^{\varnothing}$ holds for all $B_1, B_2 \in \mathcal{B}$. The expressions $B_1 \circ B_2 = B_1B_2 + B_2B_1$ and $B_1 \bullet B_2 = B_1B_2 + B_2B_1^{\varnothing}$,

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define the Jordan product and skew Jordan product of B_1 and B_2 respectively. An additive mapping $\Psi : \mathcal{B} \to \mathcal{B}$ is known as a *Jordan derivation* (respectively *Jordan triple derivation*) if

$$\begin{aligned} \Psi(B_1 \circ B_2) &= \Psi(B_1) \circ B_2 + B_1 \circ \Psi(B_2) \\ \text{(resp. } \Psi((B_1 \circ B_2) \circ B_3) &= (\Psi(B_1) \circ B_2) \circ B_3 + (B_1 \circ \Psi(B_2)) \circ B_3 + (B_1 \circ B_2) \circ \Psi(B_3)) \end{aligned}$$

holds for all $B_1, B_2, B_3 \in \mathcal{B}$. Analogously, an additive mapping $\Psi : \mathcal{B} \to \mathcal{B}$ is said to be a *skew-Jordan derivation* (respectively *skew-Jordan triple derivation*) if

$$\begin{aligned} \Psi(B_1 \bullet B_2) &= \Psi(B_1) \bullet B_2 + B_1 \bullet \Psi(B_2) \\ \text{(resp. } \Psi((B_1 \bullet B_2) \bullet B_3) &= (\Psi(B_1) \bullet B_2) \bullet B_3 + (B_1 \bullet \Psi(B_2)) \bullet B_3 + (B_1 \bullet B) \bullet \Psi(B_3)) \end{aligned}$$

holds for all $B_1, B_2, B_3 \in \mathcal{B}$. In case the mappings Ψ is not necessarily additive in the above definitions, then Ψ is called a *multiplicative Jordan derivation* (respectively *multiplicative Jordan triple derivation*) and *multiplicative skew Jordan derivation* (respectively *multiplicative skew Jordan triple derivation.* In recent years, many mathematicians have been interested in mappings with different kinds of algebras and rings. These findings to be becoming more and more important in a variety of research areas, and many authors have taken an interest in studying them. (see [1–6]).

Define a sequence of polynomials as follows: $Q_n(B_1, B_2, \ldots, B_{n-1}, B_n) = Q_{n-1}(B_1, B_2, \ldots, B_{n-1}) \bullet B_n$ $(n \geq 3)$ where $Q_n(B_1, B_2, \ldots, B_{n-1}, B_n) = Q_{n-1}(B_1, B_2, \ldots B_{n-1}) \bullet B_n$ is recognized as mixed-Jordan n-product. We have $Q_3(B_1, B_2, B_3) = B_1 \circ B_2 \bullet B_3$ is referred to as the mixed-Jordan triple product (see [7]), for $n = 3$. We define a mapping $\Psi : \mathcal{B} \to \mathcal{B}$ (not necessarily additive) is said to be *multiplicative mixed-Jordan n-derivation* if

$$\begin{aligned} \Psi(Q_n(B_1, B_2, \ldots, B_n)) &= Q_n(\Psi(B_1), B_2, \ldots, B_n) + Q_n(B_1, \Psi(B_2), \ldots, B_n) \\ &\quad + \cdots + Q_n(B_1, B_2, \ldots, \Psi(B_n)) \end{aligned}$$

for all $B_1, B_2, \ldots, B_n \in \mathcal{B}$.
Every multiplicative mixed-Jordan 3-derivations on $\varnothing$-rings is a multiplicative mixed-Jordan n-derivation, in general the converse need not be true. Multiplicative mixed-Jordan $(3, 4, n)$-derivation are en masse recognized as multiplicative mixed-Jordan type derivations. Recently, many researchers paid more attention to the study of Lie (Jordan) mappings involving two different kinds of products at the same time in the nonlinear settings in [8–10]. Zhou in [11] studied that: Every nonlinear mixed Lie-triple derivation on unital prime $\varnothing$-algebra is an additive $\varnothing$-derivation. Rehman in [7], proved that: On $\varnothing$-algebras, all nonlinear mixed Jordan-triple derivations qualify as additive $\varnothing$-derivation. The work of Ferreira and Wei in [12] focused on characterizing nonlinear mixed $\varnothing$-Jordan-type derivations on $\varnothing$-algebras. They established that any nonlinear mixed $\varnothing$-Jordan n-derivation is an additive $\varnothing$-derivation. In their study presented in [13], Yu et al. demonstrated that all skew Lie derivations on factor von Neumann algebras are equivalent to additive $\varnothing$-derivations. In a recent work, Kong and Zhang [14] generalized this result to

prime $\varnothing$-rings and proved that every skew Lie derivation on 2-torsion prime $\varnothing$-ring is an additive $\varnothing$-derivation. The assumption $\frac{1}{2} \in \mathcal{B}$ holds throughout the discussion.

Inspired by the preceding studies, this paper examines the relationship between multiplicative mixed-Jordan n-derivations and additive $\varnothing$-derivation in arbitrary $\varnothing$-rings. Our results demonstrate that, under specific conditions, every multiplicative mixed-Jordan n-derivation defined is an additive $\varnothing$-derivation, on $\varnothing$-ring with unity.

2. The principal conclusion

The key result of this work can be summarized as follows:

Theorem 1. *Suppose that $\mathcal{B}$ is a 2-torsion free $\varnothing$-ring having unity u containing a non-trivial symmetric idempotent α_1. Write $\alpha_2 = u - \alpha_1$ and assume that $\mathcal{B}$ fullfils*

$$Y\mathcal{B}\alpha_k = 0 \Longrightarrow Y = 0 \quad for\ k = 1, 2 \tag{$\spadesuit$}$$

where $Y \in \mathcal{B}$. Then a map $\Psi : \mathcal{B} \to \mathcal{B}$ (need not necessarily additive) satisfies

$$\Psi(Q_n(B_1, B_2, \ldots, B_n)) = \sum_{i=1}^{n} Q_n(B_1, \ldots, B_{i-1}, \Psi(B_i), B_{i+1}, \ldots, B_n) \tag{2.1}$$

for all $B_1, B_2, \ldots, B_n \in \mathcal{B}$ if and only if Ψ is an additive $\varnothing$-derivation.

Assume $\mathcal{B}_{ij} = P_i\mathcal{B}P_j$ with $i, j = 1, 2$, then by the Peirce decomposition, we have $\mathcal{B} = \mathcal{B}_{11} \oplus \mathcal{B}_{12} \oplus \mathcal{B}_{21} \oplus \mathcal{B}_{22}$. Clearly any $R \in \mathcal{B}$ can be written as $R = B_{11} + B_{12} + B_{21} + B_{22}$, where $B_{ij} \in \mathcal{B}_{ij}$ for $i, j = 1, 2$. To conclude the proof of the stated theorem, a number of following lemmas are necessary:

Lemma 1. $\Psi(0) = 0$.

Proof. We have

$$\begin{aligned}
\Psi(0) &= \Psi(Q_n(0, 0, \ldots, 0)) \\
&= Q_n(\Psi(0), 0, \ldots, 0) + Q_n(0, \Psi(0), \ldots, 0) + \cdots + Q_n(0, 0, \ldots, \Psi(0)) = 0.
\end{aligned}$$

Lemma 2. *For any $B_{12} \in \mathcal{B}_{12}$ and $B_{21} \in \mathcal{B}_{21}$, we have*

$$\Psi(B_{12} + B_{21}) = \Psi(B_{12}) + \Psi(B_{21}).$$

Proof. Let $\mathcal{H} = \Psi(B_{12} + B_{21}) - \Psi(B_{12}) - \Psi(B_{21})$. We need to prove that $\mathcal{H} = 0$. Applying $Q_n(u, u, \ldots, B_{12}, \alpha_2, \alpha_1) = 0$ and Lemma 1 to get

$$\begin{aligned}
&\Psi(Q_n(u, u, \ldots, B_{12} + B_{21}, \alpha_2, \alpha_1)) \\
&= \Psi(Q_n(u, u, \ldots, B_{12}, \alpha_2, \alpha_1)) + \Psi(Q_n(u, u, \ldots, B_{21}, \alpha_2, \alpha_1))
\end{aligned}$$

$$
\begin{aligned}
&= Q_n(\Psi(u), u, \ldots, B_{12}, \alpha_2, \alpha_1) + Q_n(u, \Psi(u), \ldots, B_{12}, \alpha_2, \alpha_1) + \cdots + \\
&\quad Q_n(u, u, \ldots, \Psi(B_{12}), \alpha_2, \alpha_1) + Q_n(u, u, \ldots, B_{12}, \Psi(\alpha_2), \alpha_1) + Q_n(u, u, \ldots, B_{12}, \alpha_2, \\
&\quad \Psi(\alpha_1)) + Q_n(\Psi(u), u, \ldots, B_{21}, \alpha_2, \alpha_1) + Q_n(u, \Psi(u), \ldots, B_{21}, \alpha_2, \alpha_1) + \cdots + \\
&\quad Q_n(u, u, \ldots, \Psi(B_{21}), \alpha_2, \alpha_1) + Q_n(u, u, \ldots, B_{21}, \Psi(\alpha_2), \alpha_1) + Q_n(u, u, \ldots, B_{21}, \alpha_2, \\
&\quad \Psi(\alpha_1)) \\
&= Q_n(\Psi(u), u, \ldots, B_{12} + B_{21}, \alpha_2, \alpha_1) + Q_n(u, \Psi(u), \ldots, B_{12} + B_{21}, \alpha_2, \alpha_1) + \cdots + \\
&\quad Q_n(u, u, \ldots, \Psi(B_{12}) + \Psi(B_{21}), \alpha_2, \alpha_1) + Q_n(u, u, \ldots, B_{12} + B_{21}, \Psi(\alpha_2), \alpha_1) + \\
&\quad Q_n(u, u, \ldots, B_{12} + B_{21}, \alpha_2, \Psi(\alpha_1)).
\end{aligned}
$$

Alternatively, we obtain

$$
\begin{aligned}
&\Psi(Q_n(u, u, \ldots, B_{12} + B_{21}, \alpha_2, \alpha_1)) \\
&\quad = Q_n(\Psi(u), u, \ldots, B_{12} + B_{21}, \alpha_2, \alpha_1) + Q_n(u, \Psi(u), \ldots, B_{12} + B_{21}, \alpha_2, \alpha_1) + \cdots + \\
&\qquad Q_n(u, u, \ldots, \Psi(B_{12} + B_{21}), \alpha_2, \alpha_1) + Q_n(u, u, \ldots, B_{12} + B_{21}, \Psi(\alpha_2), \alpha_1) \\
&\qquad + Q_n(u, u, \ldots, B_{12} + B_{21}, \alpha_2, \Psi(\alpha_1)).
\end{aligned}
$$

A comparison of the aforementioned expressions reveals that $Q_n(u, u, \ldots, \mathcal{H}, \alpha_2, \alpha_1) = 0$. This futher implies that $\mathcal{H}_{21} = 0$. Similarly, we can show that $\mathcal{H}_{12} = 0$. Based on the fact $Q_n(u, u, \ldots, u, (\alpha_1 - \alpha_2), B_{21}) = 0$ and use Lemma 1 to have

$$
\begin{aligned}
&\Psi(Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, B_{12} + B_{21})) \\
&\quad = \Psi(Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, B_{12})) + \Psi(Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, B_{21})) \\
&\quad = Q_n(\Psi(u), u, \ldots, u, \alpha_1 - \alpha_2, B_{12}) + Q_n(u, \Psi(u), \ldots, u, \alpha_1 - \alpha_2, B_{12}) + \cdots + \\
&\qquad Q_n(u, u, \ldots, \Psi(u), \alpha_1 - \alpha_2, B_{12}) + Q_n(u, u, \ldots, u, \Psi(\alpha_1 - \alpha_2), B_{12}) + Q_n(u, u, \\
&\qquad \ldots, u, \alpha_1 - \alpha_2, \Psi(B_{12})) + Q_n(\Psi(u), u, \ldots, u, \alpha_1 - \alpha_2, B_{21}) + \\
&\qquad Q_n(u, \Psi(u), \ldots, u, \alpha_1 - \alpha_2, B_{21}) + \cdots + Q_n(u, u, \ldots, \Psi(u), \alpha_1 - \alpha_2, B_{21}) \\
&\qquad + Q_n(u, u, \ldots, u, \Psi(\alpha_1 - \alpha_1), B_{21}) + Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, \Psi(B_{21})) \\
&\quad = Q_n(\Psi(u), u, \ldots, u, \alpha_1 - \alpha_2, B_{12} + B_{21}) + Q_n(u, \Psi(u), \ldots, u, \alpha_1 - \alpha_2, B_{12} + B_{21}) \\
&\qquad + \cdots + Q_n(u, u, \ldots, \Psi(u), \alpha_1 - \alpha_2, B_{12} + B_{21}) + Q_n(u, u, \ldots, u, \Psi(\alpha_1 - \alpha_2), B_{12} \\
&\qquad + B_{21}) + Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, \Psi(B_{12}) + \Psi(B_{21})).
\end{aligned}
$$

Alternatively, we have

$$
\begin{aligned}
&\Psi(Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, B_{12} + B_{21})) \\
&\quad = Q_n(\Psi(u), u, \ldots, u, \alpha_1 - \alpha_2, B_{12} + B_{21}) + Q_n(u, \Psi(u), \ldots, u, \alpha_1 - \alpha_2, B_{12} + B_{21}) + \\
&\qquad \cdots + Q_n(u, u, \ldots, \Psi(u), \alpha_1 - \alpha_2, B_{12} + B_{21}) + Q_n(u, u, \ldots, u, \Psi(\alpha_1 - \alpha_2), B_{12} + B_{21}) \\
&\qquad + Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, \Psi(B_{12} + B_{21}).
\end{aligned}
$$

From the above, $Q_n(u, u, \ldots, u, \alpha_1 - \alpha_2, \mathcal{H}) = 0$ that implies $\mathcal{H}_{11} = 0$ and $\mathcal{H}_{22} = 0$. Therefore, $\mathcal{H} = 0$, i.e.,

$$
\Psi(B_{12} + B_{21}) = \Psi(B_{12}) + \Psi(B_{21}).
$$

Lemma 3. *For any* $B_{11} \in \mathcal{B}_{11}, B_{12} \in \mathcal{B}_{12}, B_{21} \in \mathcal{B}_{21},$ *and* $B_{22} \in \mathcal{B}_{22}$*, we have*

$$\Psi(B_{11} + B_{12} + B_{21}) = \Psi(B_{11}) + \Psi(B_{12}) + \Psi(B_{21})$$

and

$$\Psi(B_{12} + B_{21} + B_{22}) = \Psi(B_{12}) + \Psi(B_{21}) + \Psi(B_{22}).$$

Proof. Let $\mathcal{H} = \Psi(B_{11}+B_{12}+B_{21})-\Psi(B_{11})-\Psi(B_{12})-\Psi(B_{21})$. The aim to prove that $\mathcal{H} = 0$. Applying $Q_n(u, u, \ldots, B_{11}, \alpha_1, \alpha_2) = Q_n(u, u, \ldots, B_{21}, \alpha_1, \alpha_2) = 0$ and Lemma 1, we find

$$\begin{aligned}
&\Psi(Q_n(u, u, \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \alpha_2)) \\
&\quad= \Psi(Q_n(u, u, \ldots, B_{11}, \alpha_1, \alpha_2)) + \Psi(Q_n(u, u, \ldots, B_{12}, \alpha_1, \alpha_2)) + \Psi(Q_n(u, u, \ldots, B_{21}, \alpha_1, \\
&\qquad \alpha_2)) \\
&\quad= Q_n(\Psi(u), u, \ldots, B_{11}, \alpha_1, \alpha_2) + Q_n(u, \Psi(u), \ldots, B_{11}, \alpha_1, \alpha_2) + \cdots + Q_n(u, u, \ldots, \\
&\qquad \Psi(B_{11}), \alpha_1, \alpha_2) + Q_n(u, u, \ldots, B_{11}, \Psi(\alpha_1), \alpha_2) + Q_n(u, u, \ldots, B_{11}, \alpha_1, \Psi(\alpha_2)) + \\
&\qquad Q_n(\Psi(u), u, \ldots, B_{12}, \alpha_1, \alpha_2) + Q_n(u, \Psi(u), \ldots, B_{12}, \alpha_1, \alpha_2) + \cdots + Q_n(u, u, \ldots, \\
&\qquad \Psi(B_{12}), \alpha_1, \alpha_2) + Q_n(u, u, \ldots, B_{12}, \Psi(\alpha_1), \alpha_2) + Q_n(u, u, \ldots, B_{12}, \alpha_1, \Psi(\alpha_2)) + \\
&\qquad Q_n(\Psi(u), u, \ldots, B_{21}, \alpha_1, \alpha_2) + Q_n(u, \Psi(u), \ldots, B_{21}, \alpha_1, \alpha_2) + \cdots + Q_n(u, u, \ldots, \\
&\qquad \Psi(B_{21}), \alpha_1, \alpha_2) + Q_n(u, u, \ldots, B_{21}, \Psi(\alpha_1), \alpha_2) + Q_n(u, u, \ldots, B_{21}, \alpha_1, \Psi(\alpha_2)) \\
&\quad= Q_n((\Psi(u), u, \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \alpha_2) + Q_n((u, \Psi(u), \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \alpha_2) \\
&\qquad + \cdots + Q_n((u, u, \ldots, \Psi(B_{11}) + \Psi(B_{12}) + \Psi(B_{21}), \alpha_1, \alpha_2) + Q_n((u, u, \ldots, B_{11} + \\
&\qquad B_{12} + B_{21}, \Psi(\alpha_1), \alpha_2) + Q_n(u, u, \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \Psi(\alpha_2)).
\end{aligned}$$

Alternatively, we obtain

$$\begin{aligned}
&\Psi(Q_n(u, u, \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \alpha_2)) \\
&\quad= Q_n(\Psi(u), u, \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \alpha_2) + Q_n(u, \Psi(u), \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \alpha_2) \\
&\qquad + \cdots + +Q_n(u, u, \ldots, \Psi(B_{11} + B_{12} + B_{21}), \alpha_1, \alpha_2) + Q_n(u, u, \ldots, B_{11} + B_{12} + B_{21}, \\
&\qquad \Psi(\alpha_1), \alpha_2) + Q_n(u, u, \ldots, B_{11} + B_{12} + B_{21}, \alpha_1, \Psi(\alpha_2)).
\end{aligned}$$

Comparing the last expressions for $\Psi(Q_n(u, u, \ldots, B_{11}+B_{12}+B_{21}, \alpha_1, \alpha_2))$ to get $Q_n(u, u, \ldots, \mathcal{H}, \alpha_1, \alpha_2) = 0$. That implies $\mathcal{H}_{12} = 0$, and $\mathcal{H}_{21} = 0$. Based on the facts $Q_n(u, u \ldots, (\alpha_1 - \alpha_2), B_{12}) = Q_n(u, u, \ldots (\alpha_1 - \alpha_2), B_{21}) = 0$ and use Lemma 1 to get that

$$\begin{aligned}
&\Psi(Q_n(u, u, \ldots, (\alpha_1 - \alpha_2), (B_{11} + B_{12} + B_{21}))) \\
&\quad= \Psi(Q_n(u, u, \ldots, (\alpha_1 - \alpha_2), B_{11})) + \Psi(Q_n(u, u, \ldots, (\alpha_1 - \alpha_2), B_{12}) \\
&\qquad +\Psi(Q_n(u, u, \ldots, (\alpha_1 - \alpha_2), B_{21})) \\
&\quad= Q_n(\Psi(u), u, \ldots, (\alpha_1 - \alpha_2), B_{11}) + Q_n(u, \Psi(u), \ldots, (\alpha_1 - \alpha_2), B_{11}) \\
&\qquad + \cdots + Q_n(u, u, \ldots, \Psi(\alpha_1 - \alpha_2), B_{11}) + Q_n(u, u, \ldots, (\alpha_1 - \alpha_2), \Psi(B_{11})) \\
&\qquad +Q_n(\Psi(u), u, \ldots, (\alpha_1 - \alpha_2), B_{12}) + Q_n(u, \Psi(u), \ldots, (\alpha_1 - \alpha_2), B_{12})
\end{aligned}$$

$$
\begin{aligned}
&+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1-\alpha_2),B_{12})+Q_n(u,u,\ldots,(\alpha_1-\alpha_2),\Psi(B_{12}))\\
&+Q_n(\Psi(u),u\ldots,(\alpha_1-\alpha_2),B_{21})+Q_n(u,\Psi(u),\ldots,(\alpha_1-\alpha_2),B_{21})\\
&+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1-\alpha_2),B_{21})+Q_n(u,u,\ldots,(\alpha_1-\alpha_2),\Psi(B_{21}))\\
=\;& Q_n(\Psi(u),u,\ldots,(\alpha_1-\alpha_2),(B_{11}+B_{12}+B_{21}))+Q_n(u,\Psi(u),\ldots,(\alpha_1-\alpha_2),(B_{11}\\
&+B_{12}+B_{21}))+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1-\alpha_2),(B_{11}+B_{12}\\
&+B_{21}))+Q_n(u,u,\ldots,(\alpha_1-\alpha_2),\Psi(B_{11})+\Psi(B_{12})+\Psi(B_{21})).
\end{aligned}
$$

However, we also have

$$
\begin{aligned}
&\Psi(Q_n(u,u,\ldots,(\alpha_1-\alpha_2),B_{11}+B_{12}+B_{21}))\\
&\quad= Q_n(\Psi(u),u,\ldots,(\alpha_1-\alpha_2),(B_{11}+B_{12}+B_{21}))+Q_n(u,\Psi(u),\ldots,(\alpha_1-\alpha_2),(B_{11}+\\
&\qquad B_{12}+B_{21}))+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1-\alpha_2),(B_{11}+B_{12}+B_{21}))\\
&\qquad+Q_n(u,u,\ldots,(\alpha_1-\alpha_2),\Psi(B_{11}+B_{12}+B_{21})).
\end{aligned}
$$

Examine the two relations for $\Psi(Q_n(u,u,\ldots,(\alpha_1-\alpha_2),B_{11}+B_{12}+B_{21}))$ to have $Q_n(u,u,\ldots,(\alpha_1-\alpha_2),\mathcal{H})=0$ that gives $\mathcal{H}_{11}=0$ and $\mathcal{H}_{22}=0$. Hence $\mathcal{H}=0$, that is,

$$\Psi(B_{11}+B_{12}+B_{21})=\Psi(B_{11})+\Psi(B_{12})+\Psi(B_{21}).$$

Similarly, we can show that $\Psi(B_{12}+B_{21}+B_{22})=\Psi(B_{12})+\Psi(B_{21})+\Psi(B_{22})$.

Lemma 4. *For any $B_{11}\in\mathcal{B}_{11}, B_{12}\in\mathcal{B}_{12}, B_{21}\in\mathcal{B}_{21}$, and $B_{22}\in\mathcal{B}_{22}$, we have*

$$\Psi(B_{11}+B_{12}+B_{21}+B_{22})=\Psi(B_{11})+\Psi(B_{12})+\Psi(B_{21})+\Psi(B_{22}).$$

Proof. Let $\mathcal{H}=\Psi(B_{11}+B_{12}+B_{21}+B_{22})-\Psi(B_{11})-\Psi(B_{12})-\Psi(B_{21})-\Psi(B_{22})$. Applying $Q_n(u,u,\ldots,\alpha_1,B_{22})=0$ with the above lemmas, we get the following

$$
\begin{aligned}
&\Psi(Q_n(u,u,\ldots,\alpha_1,(B_{11}+B_{12}+B_{21}+B_{22})))\\
&\quad= \Psi(Q_n(u,u,\ldots,\alpha_1,(B_{11}+B_{12}+B_{21})))+\Psi(Q_n(u,u,\ldots,\alpha_1,B_{22}))\\
&\quad= Q_n(\Psi(u),u,\ldots,\alpha_1,(B_{11}+B_{12}+B_{21}))+Q_n(u,\Psi(u),\ldots,\alpha_1,(B_{11}+B_{12}+B_{21}))\\
&\qquad+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1),(B_{11}+B_{12}+B_{21}))+Q_n(u,u,\ldots,\alpha_1,(\Psi(B_{11})+\\
&\qquad\Psi(B_{12})+\Psi(B_{21})))+Q_n(\Psi(u),u,\ldots,\alpha_1,B_{22})+Q_n(u,\Psi(u),\ldots,\alpha_1,B_{22})\\
&\qquad+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1),B_{22})+Q_n(u,u,\ldots,\alpha_1,\Psi(B_{22}))\\
&\quad= Q_n(\Psi(u),u,\ldots,\alpha_1,(B_{11}+B_{12}+B_{21}+B_{22}))+Q_n(u,\Psi(u),\ldots,\alpha_1,(B_{11}+B_{12}\\
&\qquad+B_{21}+B_{22}))+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1),(B_{11}+B_{12}+B_{21}+B_{22}))\\
&\qquad+Q_n(u,u,\ldots,\alpha_1,(\Psi(B_{11})+\Psi(B_{12})+\Psi(B_{21})+\Psi(B_{22}))).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\Psi(Q_n(u,u,\ldots,\alpha_1,(B_{11}+B_{12}+B_{21}+B_{22})))\\
&\quad= Q_n(\Psi(u),u,\ldots,\alpha_1,(B_{11}+B_{12}+B_{21}+B_{22}))+Q_n(u,\Psi(u),\ldots,\alpha_1,(B_{11}+B_{12}
\end{aligned}
$$

$$+B_{21}+B_{22}))+\cdots+Q_n(u,u,\ldots,\Psi(\alpha_1),(B_{11}+B_{12}+B_{21}+B_{22}))$$
$$+Q_n(u,u,\ldots,\alpha_1,(\Psi(B_{11}+B_{12}+B_{21}+B_{22}))).$$

Similarly, for $\Psi(Q_n(u,u,\ldots,\alpha_1,(B_{11}+B_{12}+B_{21}+B_{22})))$, we get $Q_n(u,u,\ldots,\alpha_1,\mathcal{H})=0$ This leads to the conclusion that $\mathcal{H}_{11}=\mathcal{H}_{12}=\mathcal{H}_{21}=0$. Similarly, we can show that $\mathcal{H}_{22}=0$. Thus $\mathcal{H}=0$, that is,

$$\Psi(B_{11}+B_{12}+B_{21}+B_{22})=\Psi(B_{11})+\Psi(B_{12})+\Psi(B_{21})+\Psi(B_{22}).$$

Lemma 5. *For any $B_{12},B_{12}^{'}\in\mathcal{B}_{12}$ and $B_{21},B_{21}^{'}\in\mathcal{B}_{21}$ we have*

$$\Psi(B_{12}+B_{12}^{'})=\Psi(B_{12})+\Psi(B_{12}^{\prime})\quad\textit{and}\quad\Psi(B_{21}+B_{21}^{'})=\Psi(B_{21})+\Psi(B_{21}^{\prime}).$$

Proof. Using the fact that $Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},(\alpha_1+B_{12}),(\alpha_2+B_{12}^{'}))=B_{12}+B_{12}^{'}+B_{12}^{\varnothing}+B_{12}^{'}B_{12}^{\varnothing}$ and Lemma 4, we have

$$\begin{aligned}
&\Psi(B_{12}+B_{12}^{'})+\Psi(B_{12}^{\varnothing})+\Psi(B_{12}^{'}B_{12}^{\varnothing})\\
&= \Psi\big(B_{12}+B_{12}^{'}+B_{12}^{\varnothing}+B_{12}^{'}B_{12}^{\varnothing})\\
&= \Psi(Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},(\alpha_1+B_{12}),(\alpha_2+B_{12}{'})))\\
&= Q_n(\Psi(\frac{u}{2}),\frac{u}{2},\ldots,\frac{u}{2},(\alpha_1+B_{12}),(\alpha_2+B_{12}^{'}))+Q_n(\frac{u}{2},\Psi(\frac{u}{2}),\ldots,\frac{u}{2},(\alpha_1+B_{12}),\\
&\quad (\alpha_2+B_{12}^{'}))+\cdots+Q_n(\frac{u}{2},\frac{u}{2},\ldots,\Psi(\frac{u}{2}),(\alpha_1+B_{12}),(\alpha_2+B_{12}^{'}))+Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},\\
&\quad \Psi(\alpha_1+B_{12}),(\alpha_2+B_{12}^{'}))+Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},(\alpha_1+B_{12}),\Psi(\alpha_2+B_{12}^{'}))\\
&= \Psi(Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},\alpha_1,\alpha_2))+\Psi(Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},\alpha_1,B_{12}^{'}))+\Psi(Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},\\
&\quad B_{12},\alpha_2))+\Psi(Q_n(\frac{u}{2},\frac{u}{2},\ldots,\frac{u}{2},B_{12},B_{12}^{'}))\\
&= \Psi(B_{12})+\Psi(B_{12}^{'})+\Psi(B_{12}^{\varnothing})+\Psi(B_{12}^{'}B_{12}^{\varnothing}).
\end{aligned}$$

Hence $\Psi(B_{12}+B_{12}^{'})=\Psi(B_{12})+\Psi(B_{12}^{'})$ for any $B_{12},B_{12}^{'}\in\mathcal{B}_{12}$. The proof for the other part follows similarly.

Lemma 6. *For any $B_{ii},B_{ii}^{'}\in\mathcal{B}_{ii}$ for $(i=1,2)$, we have*

$$\Psi(B_{11}+B_{11}^{'})=\Psi(B_{11})+\Psi(B_{11}^{'})\quad\textit{and}\quad\Psi(B_{22}+B_{22}^{'})=\Psi(B_{22})+\Psi(B_{22}^{'}).$$

Proof. Let $\mathcal{H}=\Psi(B_{11}+B_{11}^{'})-\Psi(B_{11})-\Psi(B_{11}^{'})$. Using the fact that $Q_n(u,u,\ldots,u,\alpha_2,B_{11})=Q_n(u,u,\ldots,u,\alpha_2,B_{11}^{'})=0$. Lemma 1 gives

$$\begin{aligned}
&\Psi(Q_n(u,u,\ldots,u,\alpha_2,(B_{11}+B_{11}^{'})))\\
&= \Psi(Q_n(u,u,\ldots,u,\alpha_2,B_{11}))+\Psi(Q_n(u,u,\ldots,u,\alpha_2,B_{11}^{'}))
\end{aligned}$$

$$\begin{aligned}
&= Q_n(\Psi(u), u, \ldots, u, \alpha_2, B_{11}) + Q_n(u, \Psi(u), \ldots, u, \alpha_2, B_{11}) + \cdots + \\
&\quad Q_n(u, u, \ldots, \Psi(u), \alpha_2, B_{11}) + Q_n(u, u, \ldots, u, \Psi(\alpha_2), B_{11}) + Q_n(u, u, \ldots, u, \alpha_2, \\
&\quad \Psi(B_{11})) + Q_n(\Psi(u), u, \ldots, u, \alpha_2, B_{11}^{'}) + Q_n(u, \Psi(u), \ldots, u, \alpha_2, B_{11}^{'}) + \cdots + \\
&\quad Q_n(u, u, \ldots, \Psi(u), \alpha_2, B_{11}^{'}) + Q_n(u, u, \ldots, u, \Psi(\alpha_2), B_{11}^{'}) + Q_n(u, u, \ldots, u, \alpha_2, \\
&\quad \Psi(B_{11}^{'})) \\
&= Q_n(\Psi(u), u, \ldots, u, \alpha_2, (B_{11} + B_{11}^{'})) + Q_n(u, \Psi(u), \ldots, u, \alpha_2, (B_{11} + B_{11}^{'})) + \cdots + \\
&\quad Q_n(u, u, \ldots, \Psi(u), \alpha_2, (B_{11} + B_{11}^{'})) + Q_n(u, u, \ldots, u, \Psi(\alpha_2), (B_{11} + B_{11}^{'})) \\
&\quad + Q_n(u, u, \ldots, u, \alpha_2, (\Psi(B_{11}) + \Psi(B_{11}^{'}))).
\end{aligned}$$

Alternatively, we have

$$\begin{aligned}
&\Psi(Q_n(u, u, \ldots, u, \alpha_2, (B_{11} + B_{11}^{'}))) \\
&= Q_n(\Psi(u), u, \ldots, u, \alpha_2, (B_{11} + B_{11}^{'})) + Q_n(u, \Psi(u), \ldots, u, \alpha_2, (B_{11} + B_{11}^{'})) + \cdots + \\
&\quad Q_n(u, u, \ldots, \Psi(u), \alpha_2, (B_{11} + B_{11}^{'})) + Q_n(u, u, \ldots, u, \Psi(\alpha_2), (B_{11} + B_{11}^{'})) \\
&\quad + Q_n(u, u, \ldots, u, \alpha_2, (\Psi(B_{11} + B_{11}^{'}))).
\end{aligned}$$

Compare the above for $\Psi(Q_n(u, u, \ldots, u, \alpha_2, (B_{11} + B_{11}^{'})))$, we find that $Q_n(u, u, \ldots, u, \alpha_2, \mathcal{H}) = 0$, This suggests that $\mathcal{H}_{12} = \mathcal{H}_{21} = \mathcal{H}_{22} = 0$. Next, we show that $\mathcal{H}_{11} = 0$. Let $X_{12} \in \mathcal{B}_{12}$ and it is straightforward to observe that $Q_n(u, u, \ldots, \alpha_1, B_{11}, X_{12})$, $Q_n(u, u, \ldots, \alpha_1, B_{11}^{'}, X_{12}) \in \mathcal{B}_{12}$. Hence, apply Lemma 5 to get

$$\begin{aligned}
&\Psi(Q_n(u, u, \ldots, \alpha_1, (B_{11} + B_{11}^{'}), X_{12})) \\
&= \Psi(Q_n(u, u, \ldots, \alpha_1, B_{11}, X_{12})) + \Psi(Q_n(u, u, \ldots, \alpha_1, B_{11}^{'}, X_{12})) \\
&= Q_n(\Psi(u), u, \ldots, \alpha_1, B_{11}, X_{12}) + Q_n(u, \Psi(u), \ldots, \alpha_1, B_{11}, X_{12}) \\
&\quad + \cdots + Q_n(u, u, \ldots, \Psi(\alpha_1), B_{11}, X_{12}) + Q_n(u, u, \ldots, \alpha_1, \Psi(B_{11}), X_{12}) \\
&\quad + Q_n(u, u, \ldots, \alpha_1, B_{11}, \Psi(X_{12})) + Q_n(\Psi(u), u, \ldots, \alpha_1, B_{11}^{'}, X_{12}) \\
&\quad + Q_n(u, \Psi(u), \ldots, \alpha_1, B_{11}^{'}, X_{12}) + \cdots + Q_n(u, u, \ldots, \Psi(\alpha_1), B_{11}^{'}, X_{12}) \\
&\quad + Q_n(u, u, \ldots, \alpha_1, \Psi(B_{11}^{'}), X_{12}) + Q_n(u, u, \ldots, \alpha_1, B_{11}^{'}, \Psi(X_{12})) \\
&= Q_n(\Psi(u), u, \ldots, \alpha_1, (B_{11} + B_{11}^{'}), X_{12}) + Q_n(u, \Psi(u), \ldots, \alpha_1, (B_{11} + B_{11}^{'}), X_{12}) \\
&\quad + \cdots + Q_n(u, u, \ldots, \Psi(\alpha_1), (B_{11} + B_{11}^{'}), X_{12}) \\
&\quad + Q_n(u, u, \ldots, \alpha_1, (\Psi(B_{11}) + \Psi(B_{11}^{'})), X_{12}) + Q_n(u, u, \ldots, \alpha_1, (B_{11} + B_{11}^{'}), \Psi(X_{12})).
\end{aligned}$$

However, in contrast, we get

$$\begin{aligned}
&\Psi(Q_n(u, u, \ldots, \alpha_1, (B_{11} + B_{11}^{'}), X_{12})) \\
&= Q_n(\Psi(u), u, \ldots, \alpha_1, (B_{11} + B_{11}^{'}), X_{12}) + Q_n(u, \Psi(u), \ldots, \alpha_1, (B_{11} + B_{11}^{'}), X_{12}) \\
&\quad + \cdots + Q_n(u, u, \ldots, \Psi(\alpha_1), (B_{11} + B_{11}^{'}), X_{12}) \\
&\quad + Q_n(u, u, \ldots, \alpha_1, (\Psi(B_{11} + B_{11}^{'})), X_{12}) + Q_n(u, u, \ldots, \alpha_1, (B_{11} + B_{11}^{'}), \Psi(X_{12})).
\end{aligned}$$

Examine just the last two statements for $\Psi(Q_n(u, u, \ldots, \alpha_1, (B_{11} + B_{11}^{'}), X_{12}))$ to get $Q_n(u, u, \ldots, \alpha_1, \mathcal{H}, X_{12}) = 0$, implies that $\alpha_1 T X_{12} + \mathcal{H}\alpha_1 X_{12} + X_{12}\mathcal{H}^{\varnothing}\alpha_1 = 0$. Multiplying it both sides by α_1 and α_2 from left and right respectively, we obtain $\alpha_1 T\alpha_1 X\alpha_2 = 0$ for all $Y \in \mathcal{B}$. Utilizing condition (♠) yields $\mathcal{H}_{11} = 0$. Hence $\mathcal{H} = 0$, that is, $\Psi(B_{11} + B_{11}^{'}) = \Psi(B_{11}) + \Psi(B_{11}^{'})$. Symmetrically, one can prove that $\Psi(B_{22} + B_{22}^{'}) = \Psi(B_{22}) + \Psi(B_{22}^{'})$.

Lemma 7. *The mapping Ψ is additive on $\mathcal{B}$.*

Proof. For any element $L, R \in \mathcal{B}$, we get $L = L_{11} + L_{12} + L_{21} + L_{22}$ and $R = B_{11} + B_{12} + B_{21} + B_{22}$. Applying the Lemmas, we get

$$\begin{aligned}
\Psi(L+R) &= \Psi(L_{11} + L_{12} + L_{21} + L_{22} + B_{11} + B_{12} + B_{21} + B_{22}) \\
&= \Psi(L_{11} + B_{11}) + \Psi(L_{12} + B_{12}) + \Psi(L_{21} + B_{21}) + \Psi(L_{22} + B_{22}) \\
&= \Psi(L_{11}) + \Psi(B_{11}) + \Psi(L_{12}) + \Psi(B_{12}) + \Psi(L_{21}) + \Psi(B_{21}) + \Psi(L_{22}) + \\
&\quad \Psi(B_{22}) \\
&= \Psi(L_{11} + L_{12} + L_{21} + L_{22}) + \Psi(B_{11} + B_{12} + B_{21} + B_{22}) \\
&= \Psi(L) + \Psi(R).
\end{aligned}$$

Lemma 8. (1) $\alpha_1\Psi(\alpha_1)\alpha_2 = -\alpha_1\Psi(\alpha_2)\alpha_2$.
(2) $\alpha_2\Psi(\alpha_1)\alpha_1 = -\alpha_2\Psi(\alpha_2)\alpha_1$.
(3) $\alpha_1\Psi(\alpha_2)\alpha_1 = \alpha_2\Psi(\alpha_1)\alpha_2 = 0$.

Proof. Using the fact that $Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_2) = 0$ and Lemma 1 , we obtain

$$\begin{aligned}
0 &= \Psi(Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_2)) \\
&= Q_n(\Psi(\alpha_1), \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_2) + Q_n(\alpha_1, \Psi(\alpha_1), \alpha_1, \ldots, \alpha_1, \alpha_2) + Q_n(\alpha_1, \alpha_1, \Psi(\alpha_1), \\
&\quad \ldots, \alpha_1, \alpha_2) + \cdots + Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \Psi(\alpha_1), \alpha_2) + Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_1, \Psi(\alpha_2)) \\
&= \alpha_1\Psi(\alpha_1)\alpha_2 + \alpha_2\Psi(\alpha_1)^{\varnothing}\alpha_1 + \alpha_1\Psi(\alpha_1)\alpha_2 + \alpha_2\Psi(\alpha_1)^{\varnothing}\alpha_1 + 2\alpha_1\Psi(\alpha_1)\alpha_2 + 2\alpha_2\Psi(\alpha_1)^{\varnothing} \\
&\quad \alpha_1 + \cdots + 2^{n-3}\alpha_1\Psi(\alpha_1)\alpha_2 + 2^{n-3}\alpha_2\Psi(\alpha_1)^{\varnothing}\alpha_1 + 2^{n-2}\alpha_1\Psi(\alpha_2) + 2^{n-2}\Psi(\alpha_2)\alpha_1.
\end{aligned}$$

By multiplying the above relation by α_1 on the left and α_2 on the right, we obtain

$$\alpha_1\Psi(\alpha_1)\alpha_2 = -\alpha_1\Psi(\alpha_2)\alpha_2.$$

(2) Since $Q_n(\alpha_2, \alpha_2, \alpha_2, \ldots, \alpha_2, \alpha_1) = 0$ and Lemma 1 , we obtain

$$\begin{aligned}
0 &= \Psi(Q_n(\alpha_2, \alpha_2, \alpha_2, \ldots, \alpha_2, \alpha_1)) \\
&= Q_n(\Psi(\alpha_2), \alpha_2, \alpha_2, \ldots, \alpha_2, \alpha_1) + Q_n(\alpha_2, \Psi(\alpha_2), \alpha_2, \ldots, \alpha_2, \alpha_1) + Q_n(\alpha_2, \alpha_2, \Psi(\alpha_2), \\
&\quad \ldots, \alpha_2, \alpha_1) + \cdots + Q_n(\alpha_2, \alpha_2, \alpha_2, \ldots, \Psi(\alpha_2), \alpha_1) + Q_n(\alpha_2, \alpha_2, \alpha_2, \ldots, \alpha_2, \Psi(\alpha_1)) \\
&= \alpha_2\Psi(\alpha_2)\alpha_1 + \alpha_1\Psi(\alpha_2)^{\varnothing}\alpha_2 + \alpha_2\Psi(\alpha_2)\alpha_1 + \alpha_1\Psi(\alpha_2)^{\varnothing}\alpha_2 + 2\alpha_2\Psi(\alpha_2)\alpha_1 + 2\alpha_1\Psi(\alpha_2)^{\varnothing} \\
&\quad \alpha_2 + \cdots + 2^{n-3}\alpha_2\Psi(\alpha_2)\alpha_1 + 2^{n-3}\alpha_1\Psi(\alpha_2)^{\varnothing}\alpha_2 + 2^{n-2}\alpha_2\Psi(\alpha_1) + 2^{n-2}\Psi(\alpha_1)\alpha_2.
\end{aligned}$$

After multiplying the final relation by α_2 from the left and by α_1 from the right, we get

$$\alpha_2\Psi(\alpha_1)\alpha_1 = -\alpha_2\Psi(\alpha_2)\alpha_1.$$

(3) un (1), we have

$$\begin{aligned} 0 &= \alpha_1\Psi(\alpha_1)\alpha_2 + \alpha_2\Psi(\alpha_1)^{\varnothing}\alpha_1 + \alpha_1\Psi(\alpha_1)\alpha_2 + \alpha_2\Psi(\alpha_1)^{\varnothing}\alpha_1 + 2\alpha_1\Psi(\alpha_1)\alpha_2 + 2\alpha_2\Psi(\alpha_1)^{\varnothing} \\ &\quad \alpha_1 + \cdots + 2^{n-3}\alpha_1\Psi(\alpha_1)\alpha_2 + 2^{n-3}\alpha_2\Psi(\alpha_1)^{\varnothing}\alpha_1 + 2^{n-2}\alpha_1\Psi(\alpha_2) + 2^{n-2}\Psi(\alpha_2)\alpha_1. \end{aligned}$$

Taking the left and right sides and multiplying them by α_1, respectively, yields

$$\alpha_1\Psi(\alpha_2)\alpha_1 = 0.$$

Similarly, in (2), we have

$$\begin{aligned} 0 &= \alpha_2\Psi(\alpha_2)\alpha_1 + \alpha_1\Psi(\alpha_2)^{\varnothing}\alpha_2 + \alpha_2\Psi(\alpha_2)\alpha_1 + \alpha_1\Psi(\alpha_2)^{\varnothing}\alpha_2 + 2\alpha_2\Psi(\alpha_2)\alpha_1 + 2\alpha_1\Psi(\alpha_2)^{\varnothing} \\ &\quad \alpha_2 + \cdots + 2^{n-3}\alpha_2\Psi(\alpha_2)\alpha_1 + 2^{n-3}\alpha_1\Psi(\alpha_2)^{\varnothing}\alpha_2 + 2^{n-2}\alpha_2\Psi(\alpha_1) + 2^{n-2}\Psi(\alpha_1)\alpha_2. \end{aligned}$$

When we take the left and right sides and multiply them by α_2, respectively, we get

$$\alpha_2\Psi(\alpha_1)\alpha_2 = 0.$$

Lemma 9. $\alpha_1\Psi(\alpha_1)\alpha_1 = \alpha_2\Psi(\alpha_2)\alpha_2 = 0$.

Proof. For $B_{12} \in \mathcal{B}_{12}$, we have $2^{n-2}B_{12} = Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_1, B_{12})$ and Using Lemma 7, we obtain

$$\begin{aligned} 2^{n-2}\Psi(B_{12}) &= \Psi(Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, B_{12})) \\ &= Q_n(\Psi(\alpha_1), \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, B_{12}) + Q_n(\alpha_1, \Psi(\alpha_1), \alpha_1, \ldots, \alpha_1, \alpha_1, B_{12}) \\ &\quad + Q_n(\alpha_1, \alpha_1, \Psi(\alpha_1), \ldots, \alpha_1, \alpha_1, B_{12}) + \cdots + Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \Psi(\alpha_1), \alpha_1, \\ &\quad B_{12}) + Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_1, \Psi(\alpha_1), B_{12}) + Q_n(\alpha_1, \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, \\ &\quad \Psi(B_{12})) \\ &= \Psi(\alpha_1)\alpha_1 B_{12} + (2^{n-2} - 2)\{\alpha_1\Psi(\alpha_1)\alpha_1 B_{12}\} + \alpha_1\Psi(\alpha_1)B_{12} + B_{12}\Psi(\alpha_1)^{\varnothing} \\ &\quad \alpha_1 + \Psi(\alpha_1)\alpha_1 B_{12} + (2^{n-2} - 2)\{\alpha_1\Psi(\alpha_1)\alpha_1 B_{12}\} + \alpha_1\Psi(\alpha_1)B_{12} + \\ &\quad B_{12}\Psi(\alpha_1)^{\varnothing}\alpha_1 + 2\Psi(\alpha_1)\alpha_1 B_{12} + (2^{n-2} - 4)\{\alpha_1\Psi(\alpha_1)\alpha_1 B_{12}\} + 2\alpha_1\Psi(\alpha_1) \\ &\quad B_{12} + 2B_{12}\Psi(\alpha_1)^{\varnothing}\alpha_1 + \cdots + 2^{n-4}\Psi(\alpha_1)\alpha_1 B_{12} + (2^{n-2} - 2^{n-3}) \\ &\quad \{\alpha_1\Psi(\alpha_1)\alpha_1 B_{12}\} + 2^{n-4}\alpha_1\Psi(\alpha_1)B_{12} + 2^{n-4}B_{12}\Psi(\alpha_1)^{\varnothing}\alpha_1 + \\ &\quad 2^{n-3}\alpha_1\Psi(\alpha_1)B_{12} + 2^{n-3}\Psi(\alpha_1)\alpha_1 B_{12} + 2^{n-3}B_{12}\Psi(\alpha_1)^{\varnothing}\alpha_1 + \\ &\quad 2^{n-2}\alpha_1\Psi(B_{12}) + 2^{n-2}\Psi(B_{12})\alpha_1. \end{aligned}$$

Multiplying both sides by α_1 and α_2 from left and right, respectively and using 2-torsion freeness of $\mathcal{B}$. We obtain $\alpha_1\Psi(\alpha_1)\alpha_1 B_{12} = 0$, implies $\alpha_1\Psi(\alpha_1)\alpha_1 S\alpha_2 = 0$ for all $S \in \mathcal{B}$. It follows from ($\spadesuit$) that $\alpha_1\Psi(\alpha_1)\alpha_1 = 0$. Similarly, we can prove that $\alpha_2\Psi(\alpha_2)\alpha_2 = 0$.

Lemma 10. $\Psi(u) = 0$.

Proof. Applying the last three lemmas to get the desired outcome.

Lemma 11. *For every* $S \in \mathcal{B}$, $\Psi(S^{\varnothing}) = \Psi(S)^{\varnothing}$.

Proof. Notice that $Q_n(u, u, \ldots, S, u) = 2^{n-2}(S + S^{\varnothing})$, for all $S \in \mathcal{B}$. Apply Lemmas 7, 10 and using the condition that rings $\mathcal{B}$ is 2-torsion free to get

$$\begin{aligned} 2^{n-2}(\Psi(S) + \Psi(S^{\varnothing})) &= \Psi(Q_n(u, u, \ldots, S, u, u)) \\ &= Q_n(u, u, \ldots, \Psi(S), u, u) \\ &= 2^{n-2}(\Psi(S) + \Psi(S)^{\varnothing}) \end{aligned}$$

which implies

$$\Psi(S^{\varnothing}) = \Psi(S)^{\varnothing}.$$

Now, let $M = \alpha_1\Psi(\alpha_1)\alpha_2 - \alpha_2\Psi(\alpha_1)\alpha_1$, then $M^{\varnothing} = -M$. Define a mapping $\zeta : \mathcal{B} \to \mathcal{B}$ by $\zeta(L) = \Psi(L) - (LM - ML)$ for every $L \in \mathcal{B}$. It is simple to confirm that $\zeta(Q_n(L_1, L_2, \ldots, L_n)) = \sum_{i=1}^{n} Q_n(L_1, \ldots, L_{i-1}, \zeta(L_i), L_{i+1}, \ldots, L_n)$ for all $L_1, L_2, \ldots, L_n \in \mathcal{B}$,

Remark 2.1. ζ *possesses the following behaviors*
(*a*) $\zeta(L^{\varnothing})$ *and* $\zeta(L)^{\varnothing}$ *identical.*
(*b*) ζ *is additive.*
(*c*) $\zeta(\alpha_1)$ *and* $\zeta(\alpha_2)$ *are vanish.*
(*d*) $\zeta(u)$ *is zero.*
(*e*) ζ *is a* $\varnothing$*-derivation iff* Ψ *is a* $\varnothing$*-derivation.*

Lemma 12. $\zeta(A_{ij}) \subseteq A_{ij}$ *i,j=1,2.*

Proof. From $Q_n(u, u, \ldots, u, \alpha_1, A_{12}) = 2^{n-2}A_{12}$ and the above remark, we get

$$\begin{aligned} 2^{n-2}\zeta(A_{12}) &= \zeta(Q_n(u, u, \ldots, u, \alpha_1, A_{12})) \\ &= Q_n(u, u, \ldots, u, \alpha_1, \zeta(A_{12}) \\ &= 2^{n-2}\{\alpha_1\zeta(A_{12}) + \zeta(A_{12})\alpha_1\}. \end{aligned}$$

This implies that $\alpha_1\zeta(A_{12})\alpha_1 = 0$ and $\alpha_2\zeta(A_{12})\alpha_2 = 0$, applying $Q_n(u, u, \ldots, u, A_{12}, \alpha_1) = 0$ and the last remark 2.1, we find

$$\begin{aligned} 0 &= \zeta(Q_n(u, u, \ldots, u, A_{12}, \alpha_1)) \\ &= Q_n(u, u, \ldots, u, \zeta(A_{12}), \alpha_1) \\ &= 2^{n-2}\{\zeta(A_{12})\alpha_1 + \alpha_1\zeta(A_{12})^{\varnothing}\}. \end{aligned}$$

This implies that $\alpha_2\zeta(A_{12})\alpha_1 = 0$, thus $\zeta(A_{12}) \subseteq A_{12}$. Similarly, we can show that $\zeta(A_{21}) \subseteq A_{21}$. Similar as last two steps,

$$0 = \zeta(Q_n(u, u, \ldots, u, \alpha_2, A_{11}))$$

$$\begin{aligned} &= Q_n(u,u,\ldots,u,\alpha_2,\zeta(A_{11})) \\ &= 2^{n-2}\{\alpha_2\zeta(A_{11})+\zeta(A_{11})\alpha_2\}. \end{aligned}$$

This implies that $\alpha_2\zeta(A_{11})\alpha_2=\alpha_1\zeta(A_{11})\alpha_2=\alpha_2\zeta(A_{11})\alpha_1=0$, thus $\zeta(A_{11})\subseteq A_{11}$. Similarly, we can show that $\zeta(A_{22})\subseteq A_{22}$.

Lemma 13. *For any* $A_{ij},B_{ij}\in\mathcal{B}_{ij}, 1\leq i,j\leq 2$, *we get*
(1) $\zeta(A_{11}B_{12})=\zeta(A_{11})B_{12}+A_{11}\zeta(B_{12})$ *and* $\zeta(A_{22}B_{21})=\zeta(A_{22})B_{21}+A_{22}\zeta(B_{21})$.
(2) $\zeta(A_{12}B_{21})=\zeta(A_{12})B_{21}+A_{12}\zeta(B_{21})$ *and* $\zeta(A_{21}B_{12})=\zeta(A_{21})B_{12}+A_{21}\zeta(B_{12})$.
(3) $\zeta(A_{11}B_{11})=\zeta(A_{11})B_{11}+A_{11}\zeta(B_{11})$ *and* $\zeta(A_{22}B_{22})=\zeta(A_{22})B_{22}+A_{22}\zeta(B_{22})$.
(4) $\zeta(A_{12}B_{22})=\zeta(A_{12})B_{22}+A_{12}\zeta(B_{22})$ *and* $\zeta(A_{21}B_{11})=\zeta(A_{21})B_{11}+A_{21}\zeta(B_{11})$.

Proof. (1) From $Q_n(u,u,\ldots,A_{11},B_{12})=2^{n-2}(A_{11}B_{12})$ and Remark 2.1, we find

$$\begin{aligned} 2^{n-2}\zeta(A_{11}B_{12}) &= \zeta(Q_n(u,u,\ldots,A_{11},B_{12})) \\ &= Q_n(u,u,\ldots,\zeta(A_{11}),B_{12})+Q_n(u,u,\ldots,A_{11},\zeta(B_{12})). \end{aligned}$$

Using the last lemma, we find

$$\zeta(A_{11}B_{12})=\zeta(A_{11})B_{12}+A_{11}\zeta(B_{12}).$$

Similarly, we can prove that $\zeta(A_{22}B_{21})=\zeta(A_{22})B_{21}+A_{22}\zeta(B_{21})$.

(2) Again, $Q_n(u,u,\ldots,A_{12},B_{21})=2^{n-2}A_{12}B_{21}$ and last remark, we get

$$\begin{aligned} 2^{n-2}\zeta(A_{12}B_{21}) &= \zeta(Q_n(u,u,\ldots,A_{12},B_{21})) \\ &= Q_n(u,u,\ldots,\zeta(A_{12}),B_{21})+Q_n(u,u,\ldots,A_{12},\zeta(B_{21})) \end{aligned}$$

Applying the last Lemma, we have

$$\zeta(A_{12}B_{21})=\zeta(A_{12})B_{21}+A_{12}\zeta(B_{21}).$$

Similarly, we can prove that $\zeta(A_{21}B_{12})=\zeta(A_{21})B_{12}+A_{21}\zeta(B_{12})$.

(3) For any $X_{12}\in\mathcal{B}_{12}$ and $Q_n(u,u,\ldots,A_{11}B_{11},X_{12})=2^{n-2}(A_{11}B_{11}X_{12})$ and making use of the last remark, we get

$$\begin{aligned} 2^{n-2}\zeta(A_{11}B_{11}X_{12}) &= \zeta(Q_n(u,u,\ldots,A_{11}B_{11},X_{12}) \\ &= Q_n(u,u,\ldots,\zeta(A_{11}B_{11}),X_{12})+Q_n(u,u,\ldots,A_{11}B_{11},\zeta(X_{12})) \\ &= 2^{n-2}\{\zeta(A_{11}B_{11})X_{12}+X_{12}\zeta(A_{11}B_{11})+A_{11}B_{11}\zeta(X_{12})+\zeta(X_{12}) \\ &\quad A_{11}B_{11}\}. \end{aligned}$$

Lemma 12 yields that

$$\zeta(A_{11}B_{11}X_{12}) \;=\; \zeta(A_{11}B_{11})X_{12} + A_{11}B_{11}\zeta(X_{12}).$$

Further, $Q_n(u,u,\ldots,A_{11},B_{11}X_{12}) = 2^{n-2}(A_{11}B_{11}X_{12})$ and making use of Remark 2.1 implies the following

$$\begin{aligned}2^{n-2}\zeta(A_{11}B_{11}X_{12}) &= \zeta(Q_n(u,u,\ldots,A_{11},B_{11}X_{12})\\ &= Q_n(u,u,\ldots,\zeta(A_{11}),B_{11}X_{12}) + Q_n(u,u,\ldots,A_{11},\zeta(B_{11}X_{12}))\\ &= 2^{n-2}\{\zeta(A_{11})B_{11}X_{12} + A_{11}\zeta(B_{11}X_{12})\}.\end{aligned}$$

Using Lemma 13(1), we have

$$\zeta(A_{11}B_{11}X_{12}) \;=\; \zeta(A_{11})B_{11}X_{12} + A_{11}\zeta(B_{11})X_{12} + A_{11}B_{11}\zeta(X_{12}).$$

Comparing the above two expressions for $\zeta(A_{11}B_{11}X_{12})$, we get $(\zeta(A_{11}B_{11}) - \zeta(A_{11})B_{11} - A_{11}\zeta(B_{11}))X_{12} = 0$, implies $(\zeta(A_{11}B_{11}) - \zeta(A_{11})B_{11} - A_{11}\zeta(B_{11}))Y\alpha_2 = 0$ for all $Y \in \mathcal{B}$. it follows from (♠) that $\zeta(A_{11}B_{11}) = \zeta(A_{11})B_{11} + A_{11}\zeta(B_{11})$. Similarly, we can prove that

$$\zeta(A_{22}B_{22}) = \zeta(A_{22})B_{22} + A_{22}\zeta(B_{22}).$$

(4) Apply $Q_n(u,u,\ldots,u,\alpha_1,A_{12},B_{22}) = 2^{n-3}(A_{12}B_{22} + B_{22}A_{12}^{\varnothing})$ and making use of Remark 2.1 to get

$$\begin{aligned}&2^{n-3}\{\zeta(A_{12}B_{22}) + \zeta(B_{22}A_{12}^{\varnothing})\}\\ &= \zeta(Q_n(u,u,\ldots,u,\alpha_1,A_{12},B_{22}))\\ &= Q_n(u,u,\ldots,u,\alpha_1,\zeta(A_{12}),B_{22}) + Q_n(u,u,\ldots,u,\alpha_1,A_{12},\zeta(B_{22})).\end{aligned}$$

Lemma 12 yields

$$\zeta(A_{12}B_{22}) + \zeta(B_{22}A_{12}^{\varnothing}) \;=\; \zeta(A_{12})B_{22} + B_{22}\zeta(A_{12})^{\varnothing} + A_{12}\zeta(B_{22}) + \zeta(B_{22})A_{12}^{\varnothing}.$$

From Lemmas 11 and 13(1), we obtain

$$\begin{aligned}\zeta(A_{12}B_{22}) + \zeta(B_{22})A_{12}^{\varnothing} + B_{22}\zeta(A_{12}^{\varnothing}) &= \zeta(A_{12})B_{22} + B_{22}\zeta(A_{12})^{\varnothing} + A_{12}\zeta(B_{22})\\ &\quad + \zeta(B_{22})A_{12}^{\varnothing}.\end{aligned}$$

Hence

$$\zeta(A_{12}B_{22}) = \zeta(A_{12})B_{22} + A_{12}\zeta(B_{22}).$$

Similarly, we can prove that $\zeta(A_{21}B_{11}) = \zeta(A_{21})B_{11} + A_{21}\zeta(B_{11})$.

Lemma 14. *$\zeta(LR) = \zeta(L)R + L\zeta(R)$, for all $L, R \in \mathcal{B}$.*

Proof. For any $L, R \in \mathcal{B}$, write $L = L_{11}+L_{12}+L_{21}+L_{22}$ and $R = B_{11}+B_{12}+B_{21}+B_{22}$. Use the additivity of ζ Lemma 13 to get

$$\begin{aligned}
\zeta(LR) &= \zeta(L_{11}B_{11} + L_{11}B_{12} + L_{12}B_{21} + L_{12}B_{22} \\
&\quad +L_{21}B_{11} + L_{21}B_{12} + L_{22}B_{21} + L_{22}B_{22}) \\
&= \zeta(L_{11}B_{11}) + \zeta(L_{11}B_{12}) + \zeta(L_{12}B_{21})\zeta(L_{12}B_{22}) \\
&\quad +\zeta(L_{21}B_{11}) + \zeta(L_{21}B_{12}) + \zeta(L_{22}B_{21}) + \zeta(L_{22}B_{22}) \\
&= \zeta(L_{11} + L_{12} + L_{21} + L_{22})(B_{11} + B_{12} + B_{21} + B_{22}) \\
&\quad +(L_{11} + L_{12} + L_{21} + L_{22})\zeta(B_{11} + B_{12} + B_{21} + B_{22}). \\
&= \zeta(L)R + L\zeta(R).
\end{aligned}$$

Lemma 14 and Remark 2.1 show that ζ is an additive $\varnothing$-derivation. As a result, it deduces that Ψ will be an additive $\varnothing$- derivation. Which completes the proof of the Theorem 1.

3. Corollaries

Remember that a ring $\mathcal{B}$ is said to be prime if, $B_1, B_2 \in \mathcal{B}$, $B_1\mathcal{B}B_2 = \{0\}$ implies that either $B_1 = 0$ or $B_2 = 0$. it's easy to observed that every prime $\varnothing$-rings satisfies property (♠). Therefore, Theorem 1 directly leads to the following conclusion:

Corollary 3.1. *Let $\mathcal{B}$ be a 2-torsion free unital prime $\varnothing$-rings having a non-trivial symmetric idempotent. Then a map $\Psi : \mathcal{B} \to \mathcal{B}$ (not necessarily additive) satisfies*

$$\Psi(Q_n(B_1, B_2, \ldots, B_n)) = \sum_{i=1}^{n} Q_n(B_1, \ldots, B_{i-1}, \Psi(B_i), B_{i+1}, \ldots, B_n) \tag{3.1}$$

for all $B_1, B_2, \ldots, B_n \in \mathcal{B}$ iff Ψ is an additive $\varnothing$-derivation.

Remember that a algebra $\mathcal{B}$ is said to be prime if, $B_1, B_2 \in \mathcal{B}$, $B_1\mathcal{B}B_2 = \{0\}$ implies that either $B_1 = 0$ or $B_2 = 0$. it's easy to observed that every prime $\varnothing$-algebra satisfies property (♠). Thus, we can infer the following outcome as ac immediate implication of Theorem 1:

Corollary 3.2. *If $\mathcal{B}$ is a unital prime $\varnothing$-algebra containing a non-trivial projection α_1 and $\alpha_2 = u - \alpha_1$, then a mapping $\Psi : \mathcal{B} \to \mathcal{B}$ (not necessarily additive) satisfies*

$$\Psi(Q_n(B_1, B_2, \ldots, B_n)) = \sum_{i=1}^{n} Q_n(B_1, \ldots, B_{i-1}, \Psi(B_i), B_{i+1}, \ldots, B_n) \tag{3.2}$$

for all $B_1, B_2, \ldots, B_n \in \mathcal{B}$ if and only if Ψ is an additive $\varnothing$-derivation.

Since a factor von Neumann algebra is also prime, it always satisfies (♠). The following result follows as a consequence this corollary 3.2:

Corollary 3.3. *If $\mathcal{B}$ is a factor von Neumann algebra having a dimension greater than or equal to 2, then a mapping $\Psi : \mathcal{B} \to \mathcal{B}$ (not necessarily additive) fulfills*

$$\Psi(Q_n(B_1, B_2, \ldots, B_n)) = \sum_{i=1}^{n} Q_n(B_1, \ldots, B_{i-1}, \Psi(B_i), B_{i+1}, \ldots, B_n) \tag{3.3}$$

for all $B_1, B_2, \ldots, B_n \in \mathcal{B}$ iff Ψ is an additive $\varnothing$-derivation.

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