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Probabilistic Degenerate Poly r-Stirling Numbers of the Second Kind and r-Bell Polynomials

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Abstract. We introduce degenerate poly r-Stirling numbers of the second kind and poly r-Bell polynomials by using degenerate polyexponential function and investigate some properties of these number and polynomials.

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Key Words and Phrases: Stirling numbers of the second kind, Bell polynomials

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1. Introduction

Recently, degenerate Sitrling numbers of the second kind and degenerate Bell polynomials have been studied by many researchers (see [1],[2],[3],[4],[5],[6],[7],[8]). These numbers and polynomials were explored from a view of probabilistic perspective (see [9],[10],[11],[12],[13],[14],[15],[16],[17],[18],[19]). The outline of the paper is as follows. In section 1, we recall some definitions. In section 2, we consider a probabilistic polyexponential function using a degenerate polyexponential function, and then define the probabilistic degenerate poly r-Stirling numbers of the second kind and probabilistic degenerate poly r-Bell polynomials. In Theorem 2.1, we derive an expression for $S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)$. In Theorem 2.2, we get an expression for $S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)$ as sum of the products. In Theorem 2.3, we get expression for $S_{2,\lambda}^{(r,k,Y)}(n+r,l+2r)$. In Theorem 2.4, we find some relation for $S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)$. In Theorem 2.5 we get an expression for $Bel_{n,\lambda}^{(r,k,Y)}(x)$. In Theorem 2.6 we derive an expression for $Bel_{n,\lambda}^{(r,k,Y)}(x)$.

In ([20],[4],[13],[21],[14],[22]) and ([15],[23],[24],[25],[19]) researchers studied degenerate exponential function.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials $e_\lambda^x(t)$, which are defined by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}. \tag{1}$$

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where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.

The degenerate Stirling numbers of the second kind are defined

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_\lambda \frac{t^n}{n!}, \quad \text{(see[26], [27], [5], [6], [7], [28]).} \tag{2}$$

The degenerate r-Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k e_\lambda^r(t) = \sum_{n=k}^{\infty} S_{2,\lambda}^{(r)}(n+r, k+r)\frac{t^n}{n!}, \quad \text{(see[20], [29]).} \tag{3}$$

The degenerate r-Bell polynomials are defined by

$$e^{x(e_\lambda(t)-1)} e_\lambda^r(t) = \sum_{n=0}^{\infty} Bel_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see[3], [30], [29]).} \tag{4}$$

The degenerate polyexponential function is defined by

$$Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n, (k \in \mathbb{Z}, |x| < 1), \quad \text{(see[[31], [4], [5], [10], [16], [17], [18], [32], [33]).} \tag{5}$$

In this paper, let Y be a random variable such that the moment generating function of Y and satisfy

$$E[e^{Yt}] = \sum_{n=0}^{\infty} E[Y^n]\frac{t^n}{n!}, (|t| < r), \quad \text{(see[1], [9], [2], [34], [35], [8], [36]).} \tag{6}$$

for some $r > 0$. Let $(Y_k)_{k\geq 1}$ be a sequence of mutually copies of random variable Y and let $S_k = Y_1 + Y_2 + \cdots + Y_k$, $(k \geq 1)$ with $S_0 = 0$.

The probabilistic degenerate Stirling numbers of the second kind associated with Y are defined by

$$\frac{1}{k}\left(E[e_\lambda^Y(t)] - 1\right)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} \frac{t^n}{n!}, \quad \text{(see[1], [11], [15], [37], [8], [38], [19]).} \tag{7}$$

2. Probabilistic degenerate poly r-Stirling numbers of the second kind and r-Bell polynomials

In this section, we consider degenerate probabilistic polyexponential function which is given by

$$Ei_{k,\lambda}^Y(t) = \sum_{n=1}^{\infty} \frac{E[(Y)_{n,\lambda}]}{(n-1)!n^k} t^n. \tag{8}$$

From (8), we note that

$$Ei_{1,\lambda}^{Y}(t)=\sum_{n=1}^{\infty}\frac{E[(Y)_{n,\lambda}]}{n!}t^{n}=E[e_{\lambda}^{Y}(t)]-1 \tag{9}$$

and

$$Ei_{1,\lambda}^{1}(t)=\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{n!}t^{n}=Ei_{1,\lambda}(t). \tag{10}$$

Now, we consider probabilistic degenerate poly r-Stirling numbers of the second kind which are given by

$$\frac{1}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^{l}\left(E[e_{\lambda}^{Y}(t)]\right)^{r}=\sum_{n=l}^{\infty}S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^{n}}{n!}. \tag{11}$$

When $k=1,Y=1$, $S_{2,\lambda}^{(r,1,1)}(n+r,l+r)=S_{2,\lambda}^{(r)}(n+r,l+r)$.

From (11), we have

$$\begin{aligned}\sum_{n=l}^{\infty}S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^{n}}{n!}&=\frac{1}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^{l}\left(E[e_{\lambda}^{Y}](t)\right)^{r} \\ &=\frac{1}{l!}\left(\sum_{i_1=1}^{\infty}\frac{E[(Y)_{i_1,\lambda}]}{(i_1-1)!i_1^{k}}t^{i_1}\right)\cdots\left(\sum_{i_l=1}^{\infty}\frac{E[(Y)_{i_l,\lambda}]}{(i_l-1)!i_l^{k}}t^{i_l}\right)\left(E[e_{\lambda}^{Y}](t)\right)^{r} \\ &=\frac{1}{l!}\sum_{m=l}^{\infty}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\cdots,i_l}\frac{E[(Y)_{i_1,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{i_1^{k-1}i_2^{k-1}\cdots i_l^{k-1}}\frac{t^{m}}{m!}\sum_{j=0}^{\infty}E[(S_r)_{j,\lambda}]\frac{t^{j}}{j!} \\ &=\frac{1}{l!}\sum_{n=l}^{\infty}\sum_{m=l}^{n}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\cdots,i_l}\binom{n}{m}\frac{E[(Y)_{i_1,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{i_1^{k-1}i_2^{k-1}\cdots i_l^{k-1}}E[(S_r)_{j,\lambda}]\frac{t^{n}}{n!}.\end{aligned} \tag{12}$$

Thus, we have the following theorem.

Theorem 1. *For $n\geq l$, we have*

$$S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)=\sum_{n=l}^{\infty}\sum_{m=l}^{n}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\cdots,i_l}\binom{n}{m}\frac{E[(Y)_{i_1,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{i_1^{k-1}i_2^{k-1}\cdots i_l^{k-1}}E[(S_r)_{j,\lambda}].$$

From (11), we get

$$\begin{aligned}\sum_{n=l}^{\infty}S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^{n}}{n!}&=\frac{1}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^{l}\left(E[e_{\lambda}^{Y}(t)]-1+1\right)^{r} \\ &=\frac{1}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^{l}\sum_{i=0}^{\infty}\binom{r}{i}\left(E[e_{\lambda}^{Y}(t)]-1\right)^{i}\end{aligned} \tag{13}$$

$$
\begin{aligned}
&= \frac{1}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^{l}\sum_{i=0}^{\infty}(r)_i\sum_{m=i}^{\infty}\left\{\begin{matrix} m \\ i \end{matrix}\right\}_{\lambda,Y}\frac{t^m}{m!} \\
&= \sum_{j=l}^{\infty}S_{2,\lambda}^{(k,Y)}(j,l)\frac{t^j}{j!}\sum_{m=i}^{\infty}\sum_{i=0}^{m}(r)_i\left\{\begin{matrix} m \\ i \end{matrix}\right\}_{\lambda,Y}\frac{t^m}{m!} \\
&= \sum_{n=l+i}^{\infty}\sum_{m=i}^{n}\sum_{i=0}^{m}\binom{n}{m}(r)_i\left\{\begin{matrix} m \\ i \end{matrix}\right\}_{\lambda,Y}S_{2,\lambda}^{(k,Y)}(n-m,l)\frac{t^n}{n!}.
\end{aligned}
$$

Thus, we get the following theorem.

Theorem 2. *For $l \geq i$, we have*

$$
S_{2,\lambda}^{(r,k,Y)}(n+r,k+r) = \sum_{m=i}^{n}\sum_{i=0}^{m}\binom{n}{m}(r)_i\left\{\begin{matrix} m \\ i \end{matrix}\right\}_{\lambda,Y}S_{2,\lambda}^{(k,Y)}(n-m,l).
$$

From (11), we have

$$
\begin{aligned}
\sum_{n=k}^{\infty}S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^n}{n!}\left(Ei_{k,\lambda}^{Y}(t)\right) &= \frac{1}{l!}\left(Ei_{l,\lambda}^{Y}(t)\right)^{l+1}\left(E[e_\lambda^Y(t)]\right)^{r} \qquad (14) \\
&= \frac{(l+1)!}{l!}\sum_{n=k}^{\infty}S_{2,\lambda}^{(r,k,Y)}(n+r,l+r+1)\frac{t^n}{n!} \\
&= (l+1)\sum_{n=k}^{\infty}S_{2,\lambda}^{(r,k,Y)}(n+r,l+r+1)\frac{t^n}{n!}.
\end{aligned}
$$

The left hand side of (14), we have

$$
\begin{aligned}
&\sum_{n=k}^{\infty}S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^n}{n!}\left(Ei_{k,\lambda}^{Y}(t)\right) = \sum_{m=k}^{\infty}S_{2,\lambda}^{(r,k,Y)}(m+r,l+r)\frac{t^m}{m!}\sum_{i=1}^{\infty}\frac{E[(Y)_{i,\lambda}]}{(i-1)!i^k}t^i \qquad (15) \\
&= \sum_{n=k+1}^{\infty}\sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}^{(r,k,Y)}(m+r,l+r)\frac{E[(Y)_{n-m,\lambda}]}{(n-m-1)!(n-m)^k}\frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients of (14) and (15), we get the following theorem.

Theorem 3. *For $n \geq k+1$, we have*

$$
S_{2,\lambda}^{(r,k,Y)}(n+r,l+r+1) = \frac{1}{l+1}\sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}^{(r,k,Y)}(m+r,l+r)\frac{E[(Y)_{n-m,\lambda}]}{(n-m-1)!(n-m)^k}.
$$

From (11), we have

$$\begin{aligned}\sum_{n=l}^{\infty} S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^n}{n!} &= \frac{1}{l!}\left(Ei_{l,\lambda}^{Y}(t)\right)^l \left(E[e_\lambda^Y(t)]\right)^r \qquad (16)\\ &= \frac{1}{r!}\left(Ei_{k,\lambda}^{Y}(t)\right)^r \left(E[e_\lambda^Y(t)]\right)^r \left(Ei_{k,\lambda}^{Y}(t)\right)^{l-r}\\ &= \sum_{m=r}^{\infty} S_{2,\lambda}^{(r,k,Y)}(m+r,2r)\frac{t^m}{m!}(l-r)!\sum_{i=l-r}^{\infty} S_{2,\lambda}^{(k,Y)}(i,l-r)\frac{t^i}{i!}\\ &= \sum_{n=l}^{\infty}\sum_{m=r}^{n}(l-r)!\binom{n}{m} S_{2,\lambda}^{(r,k,Y)}(m+r,2r) S_{2,\lambda}^{(k,Y)}(n-m,l-r)\frac{t^n}{n!}.\end{aligned}$$

Thus, we get the following theorem.

Theorem 4. *For $n \geq l$, we have*

$$S_{2,\lambda}^{(r,k,Y)}(n+r,l+r) = \sum_{m=r}^{n}\binom{n}{m}(l-r)! S_{2,\lambda}^{(r,k,Y)}(m+r,2r) S_{2,\lambda}^{(k,Y)}(n-m,l-r).$$

Now, we consider probabilistic degenerate poly r-Bell polynomials associated with Y which are given by

$$e^{x(Ei_{k,\lambda}^{Y}(t))}\left(E[e_\lambda^Y(t)]\right)^r = \sum_{n=0}^{\infty} Bel_{n,\lambda}^{(r,k,Y)}(x)\frac{t^n}{n!}. \qquad (17)$$

From (17), we have

$$\begin{aligned}\sum_{n=0}^{\infty} Bel_{n,\lambda}^{(r,k,Y)}(x)\frac{t^n}{n!} &= e^{x(Ei_{k,\lambda}^{Y}(t))}\left(E[e_\lambda^Y(t)]\right)^r \qquad (18)\\ &= \sum_{l=0}^{\infty} x^l \frac{\left(Ei_{k,\lambda}^{Y}(t)\right)^l}{l!}\left(E[e_\lambda^Y(t)]\right)^r\\ &= \sum_{l=0}^{\infty} x^l \sum_{n=l}^{\infty} S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^n}{n!}\\ &= \sum_{n=l}^{\infty}\sum_{l=0}^{n} x^l S_{2,\lambda}^{(r,k,Y)}(n+r,l+r)\frac{t^n}{n!}.\end{aligned}$$

Thus, we have the following theorem.

Theorem 5. *For $n \geq l$, we have*

$$Bel_{n,\lambda}^{(r,k,Y)}(x) = \sum_{l=0}^{n} x^l S_{2,\lambda}^{(r,k,Y)}(n+r, l+r).$$

From (17), we ovserve that

$$\sum_{n=0}^{\infty} Bel_{n,\lambda}^{(r,k,Y)}(x)\frac{t^n}{n!} = e^{\frac{x}{2}\left(Ei_{k,\lambda}^{Y}(t)\right)} \sum_{k=0}^{\infty} Bel_{k,\lambda}^{(r,k,Y)}\left(\frac{x}{2}\right)\frac{t^n}{n!} \tag{19}$$

$$= \sum_{l=0}^{\infty}\left(\frac{x}{2}\right)^l \frac{1}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^l \sum_{m=0}^{\infty} Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{2}\right)\frac{t^n}{n!}$$

$$= \sum_{l=0}^{\infty}\frac{x^l}{2^l l!}\left(\sum_{i_1=1}^{\infty}\frac{E[E(Y)_{i_1,\lambda}]}{(i_1-1)!i_1^k}t^{i_1}\right)\left(\sum_{i_2=1}^{\infty}\frac{E[E(Y)_{i_2,\lambda}]}{(i_2-1)!i_2^k}t^{i_2}\right)\cdots\left(\sum_{i_l=1}^{\infty}\frac{E[E(Y)_{i_l,\lambda}]}{(i_1-1)!i_1^k}t^{i_l}\right)$$

$$\times \sum_{m=0}^{\infty} Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{2}\right)\frac{t^m}{m!}$$

$$= \sum_{l=0}^{\infty}\frac{x^l}{2^l l!}\sum_{j=l}^{\infty}\sum_{i_1+i_2+\cdots+i_l=j}\binom{j}{i_1 i_2\cdots i_l}\frac{E[(Y)_{i_1,\lambda}]E[(Y)_{i_2,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{i_1^{k-1}i_2^{k-2}\cdots i_l^{k-1}}\frac{t^j}{j!}\sum_{m=0}^{\infty}$$

$$\times Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{2}\right)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j}\sum_{i_1+i_2+\cdots+i_l=j}\binom{n}{j}\binom{j}{i_1 i_2\cdots i_l}\frac{x^l}{2^l l!}\frac{E[(Y)_{i_1,\lambda}]E[(Y)_{i_2,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{(i_1 i_2\cdots i_l)^{k-1}}$$

$$\times Bel_{n-j,\lambda}^{(r,k,Y)}\left(\frac{x}{2}\right)\frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty} Bel_{n,\lambda}^{(r,k,Y)}(x)\frac{t^n}{n!} = e^{\frac{2}{3}x\left(Ei_{k,\lambda}^{Y}(t)\right)} \sum_{k=0}^{\infty} Bel_{k,\lambda}^{(r,k,Y)}\left(\frac{x}{3}\right)\frac{t^n}{n!} \tag{20}$$

$$= \sum_{l=0}^{\infty}\left(\frac{2}{3}\right)^l \frac{x^l}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^l \sum_{m=0}^{\infty} Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{3}\right)\frac{t^n}{n!}$$

$$= \sum_{l=0}^{\infty}\left(\frac{2}{3}\right)^l\frac{x^l}{l!}\left(\sum_{i_1=1}^{\infty}\frac{E[E(Y)_{i_1,\lambda}]}{(i_1-1)!i_1^k}t^{i_1}\right)\left(\sum_{i_2=1}^{\infty}\frac{E[E(Y)_{i_2,\lambda}]}{(i_2-1)!i_2^k}t^{i_2}\right)\cdots\left(\sum_{i_l=1}^{\infty}\frac{E[E(Y)_{i_l,\lambda}]}{(i_1-1)!i_1^k}t^{i_l}\right)$$

$$\times \sum_{m=0}^{\infty} Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{3}\right)\frac{t^m}{m!}$$

$$= \sum_{l=0}^{\infty}\left(\frac{2}{3}\right)^{l}\frac{x^{l}}{l!}\sum_{j=l}^{\infty}\sum_{i_1+i_2+\cdots+i_l=j}\binom{j}{i_1 i_2\cdots i_l}\frac{E[(Y)_{i_1,\lambda}]E[(Y)_{i_2,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{i_1^{k-1}i_2^{k-2}\cdots i_l^{k-1}}\frac{t^j}{j!}\sum_{m=0}^{\infty}$$
$$\times Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{3}\right)\frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j}\sum_{i_1+i_2+\cdots+i_l=j}\binom{n}{j}\binom{j}{i_1 i_2\cdots i_l}\left(\frac{2}{3}\right)^{l}\frac{x^{l}}{l!}\frac{E[(Y)_{i_1,\lambda}]E[(Y)_{i_2,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{(i_1 i_2\cdots i_l)^{k-1}}$$
$$\times Bel_{n-j,\lambda}^{(r,k,Y)}\left(\frac{x}{3}\right)\frac{t^n}{n!}.$$

Repeating this process α times, we have

$$\sum_{n=0}^{\infty}Bel_{n,\lambda}^{(r,k,Y)}(x)\frac{t^n}{n!} = e^{\frac{\alpha-1}{\alpha}x\left(Ei_{k,\lambda}^{Y}(t)\right)}\sum_{k=0}^{\infty}Bel_{k,\lambda}^{(r,k,Y)}\left(\frac{x}{\alpha}\right)\frac{t^n}{n!} \tag{21}$$
$$= \sum_{l=0}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^{l}\frac{x^{l}}{l!}\left(Ei_{k,\lambda}^{Y}(t)\right)^{l}\sum_{m=0}^{\infty}Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{\alpha}\right)\frac{t^n}{n!}$$
$$= \sum_{l=0}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^{l}\frac{x^{l}}{l!}\left(\sum_{i_1=1}^{\infty}\frac{E[E(Y)_{i_1,\lambda}]}{(i_1-1)!i_1^k}t^{i_1}\right)\left(\sum_{i_2=1}^{\infty}\frac{E[E(Y)_{i_2,\lambda}]}{(i_2-1)!i_2^k}t^{i_2}\right)\cdots\left(\sum_{i_l=1}^{\infty}\frac{E[E(Y)_{i_l,\lambda}]}{(i_1-1)!i_1^k}t^{i_l}\right)$$
$$\times\sum_{m=0}^{\infty}Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{\alpha}\right)\frac{t^m}{m!}$$
$$= \sum_{l=0}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^{l}\frac{x^{l}}{l!}\sum_{j=l}^{\infty}\sum_{i_1+i_2+\cdots+i_l=j}\binom{j}{i_1 i_2\cdots i_l}\frac{E[(Y)_{i_1,\lambda}]E[(Y)_{i_2,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{i_1^{k-1}i_2^{k-2}\cdots i_l^{k-1}}\frac{t^j}{j!}\sum_{m=0}^{\infty}$$
$$\times Bel_{m,\lambda}^{(r,k,Y)}\left(\frac{x}{\alpha}\right)\frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j}\sum_{i_1+i_2+\cdots+i_l=j}\binom{n}{j}\binom{j}{i_1 i_2\cdots i_l}\left(\frac{\alpha-1}{\alpha}\right)^{l}\frac{x^{l}}{l!}\frac{E[(Y)_{i_1,\lambda}]E[(Y)_{i_2,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{(i_1 i_2\cdots i_l)^{k-1}}$$
$$\times Bel_{n-j,\lambda}^{(r,k,Y)}\left(\frac{x}{\alpha}\right)\frac{t^n}{n!}.$$

By comparing the coefficients on both sides in (21), we have the following theorem.

Theorem 6. *For $n,k\geq 0$, $\alpha\in\mathbb{N}$, we have*

$$Bel_{n,\lambda}^{(r,k,Y)}(x) = \sum_{j=0}^{n}\sum_{l=0}^{j}\sum_{i_1+i_2+\cdots+i_l=j}\binom{n}{j}\binom{j}{i_1 i_2\cdots i_l}\left(\frac{\alpha-1}{\alpha}\right)^{l}\frac{x^{l}}{l!} \tag{22}$$

$$\times \frac{E[(Y)_{i_1,\lambda}]E[(Y)_{i_2,\lambda}]\cdots E[(Y)_{i_l,\lambda}]}{(i_1 i_2 \cdots i_l)^{k-1}} Bel_{n-j,\lambda}^{(r,k,Y)}\left(\frac{x}{\alpha}\right).$$

3. Conclusion

In this paper, we considered probabilistic degenerate poly r-Stirling numbers of the second kind and r-Bell polynomials. We explored some identities of poly r-Stirling numbers of the second kind and r-Bell polynomials. Although not studied in this paper, there are still problems to solve in the case of continuous and discrete random variables. We will study these cases in the future.

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