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Some Fixed Point Theorems for $\Theta - \phi$-Multivalued Contraction Mappings in Rectangular b-Metric Spaces

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Abstract. In this paper, we give some fixed point theorems for $\theta-\phi-$multivalued contractions in $\alpha-$complete rectangular $b-$metric spaces. We establish some fixed point theorems including the $\alpha-$admissible $\theta-\phi-$multivalued Kannan type and Reich type. Our results improve and generalize some results from the literature.

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1. Introduction and preliminaries

Many generalizations of the concept of metric spaces have been defined and some fixed theorems were proven in these spaces [1–13]. Particularly, $b-$metric spaces were introduced by Bakhtin [4] and Branciari [5] introduced generalized metric spaces. Recently, George et al [7] announced the concept of rectangular $b-$metric spaces. In 2017, Zheng et al [14] established some fixed point results for $\theta-\phi-$contractions in complete metric spaces. Nadler [15] extented the contraction principle to multivalued mappings.
In this work, we introduce a notion of $\theta-\phi-$multivalued contraction mappings in rectangular $b-$metric spaces. We obtain some fixed point theorems for $\theta-\phi-$multivalued

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contractions in $\alpha-$complete rectangular $b-$metric spaces. We establish some fixed point theorems including the $\alpha-$admissible $\theta-\phi-$multivalued Kannan type [16] and Reich type [17] in rectangular $b-$metric spaces. Our results improve and generalize some results from the literature. We believe that our paper may be interesting to researchers in fixed point theory, because using the methods presented in this paper, at the end we give several open problems.

Definition 1. *[13] Let $\mathcal{U}$ be a non-empty set and $b \geq 1$. Suppose that the mapping $d : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ satisfies:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{U}$,

(iii) $d(x, y) \leq b[d(x, u) + d(u, v) + d(v, y)]$ *for all* $x, y \in \mathcal{U}$ *and for all distinct points* $u, v \in \mathcal{U} \setminus \{x, y\}$.

Then $(\mathcal{U}, d)$ is called a rectangular $b-$metric space with coefficient b.

Lemma 1. *[9] Let $(\mathcal{U}, d)$ be a rectangular $b-$metric space and $\{x_n\}$ be a sequence in $\mathcal{U}$ such that*

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = \lim_{n\to+\infty} d(x_n, x_{n+2}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$\varepsilon \leq \lim_{k\to+\infty} \inf d(x_{m_k}, x_{n_k}) \leq \lim_{k\to+\infty} \sup d(x_{m_k}, x_{n_k}) \leq b\varepsilon,$$

$$\varepsilon \leq \lim_{k\to+\infty} \inf d(x_{m_k}, x_{m_{k+1}}) \leq \lim_{k\to+\infty} \sup d(x_{n_k}, x_{m_{k+1}}) \leq b\varepsilon,$$

$$\varepsilon \leq \lim_{k\to+\infty} \inf d(x_{m_k}, x_{n_{k+1}}) \leq \lim_{k\to+\infty} \sup d(x_{m_k}, x_{n_{k+1}}) \leq b\varepsilon,$$

$$\frac{\varepsilon}{b} \leq \lim_{k\to+\infty} \inf d(x_{m_{k+1}}, x_{n_{k+1}}) \leq \lim_{k\to+\infty} \sup d(x_{m_{k+1}}, x_{n_{k+1}}) \leq b^2\varepsilon.$$

Zheng et al. [12] introduced a new type of contractions called $\theta-\phi-$contractions in metric spaces and proved a new fixed point theorems for such mapping.

Definition 2. *[6] We denote by Θ the set of functions $\theta : (0, +\infty) \to [1, +\infty)$ satisfying the following conditions:*

1) θ *is increasing,*

2) For each sequence $\{x_n\} \in (0, +\infty)$, $\lim_{n\to+\infty} \theta(x_n) = 1$ *if and only if* $\lim_{n\to+\infty} x_n = 0$,

3) θ *is continuous on* $(0, +\infty)$.

Definition 3. *[13] We denote by Φ the set of functions $\phi : [1, +\infty) \to [1, +\infty)$ satisfying the following conditions:*

1) ϕ is nondecreasing,

2) For each $\lambda > 1$, $\lim_{n\to+\infty} \phi^n(\lambda) = 1$,

3) ϕ is continuous on $[1, +\infty)$.

Lemma 2. *[13] If $\phi \in \Phi$, then $\phi(\lambda) < \lambda$ for all $\lambda \in (1, +\infty)$ and $\phi(1) = 1$.*

Definition 4. *[18] Let $(\mathcal{U}, d)$ be a rectangular $b-$metric space. Let $T : \mathcal{U} \to \mathcal{U}$ and $\alpha : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ be two mappings. A mapping T is said to be $\alpha-$admissible if*

$$\alpha(x, y) \geq 1 \textit{ implies } \alpha(Tx, Ty) \geq 1.$$

Definition 5. *[19] Let $T : \mathcal{U} \to \mathcal{U}$ and $\alpha : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ be two mappings such that T is $\alpha-$admissible. T is said to be triangular $\alpha-$admissible if*

$$\alpha(x, y) \geq 1 \textit{ and } \alpha(y, z) \geq 1 \textit{ implies } \alpha(x, z) \geq 1.$$

Definition 6. *[11] Let $(\mathcal{U}, d)$ be a rectangular $b-$metric space with $b > 1$ and $T : \mathcal{U} \to \mathcal{U}$ be a mapping.*

(1) T is called $\theta - \phi-$contraction if there are $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$d(Tx, Ty) > 0 \textit{ implies } \theta[b^2 d(Tx, Ty)] \leq \phi[\theta(M(x, y))], \tag{1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}.$$

(2) T is called $\theta - \phi-$ Kannan-type contraction if there are $\theta \in \Theta$ and $\phi \in \Phi$ such that $d(Tx, Ty) > 0$ implies

$$\theta[b^2 d(Tx, Ty)] \leq \phi\left[\theta\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right)\right]. \tag{2}$$

(3) T is called $\theta - \phi-$Reich-type contraction if there are $\theta \in \Theta$ and $\phi \in \Phi$ such that $d(Tx, Ty) > 0$ implies

$$\theta[b^2 d(Tx, Ty)] \leq \phi\left[\theta\left(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}\right)\right]. \tag{3}$$

Kari et al. [11] recently obtained the following result.

Theorem 1. *[11] Let $(\mathcal{U}, d)$ be a complete rectangular $b-$metric space and $T : \mathcal{U} \to \mathcal{U}$ be a $\theta - \phi-$contraction. Then, T has a unique fixed point.*

In 2014, Hussain et al. [8] introduced a notion of $\alpha-$completeness for metric spaces.

Definition 7. *[8] Let $(\mathcal{U}, d)$ be a rectangular $b-$metric space and $\alpha : \mathcal{U} \times \mathcal{U} \to [0, +\infty[$ be a mapping. The space $\mathcal{U}$ is said to be $\alpha-$complete, if every Cauchy sequence $\{x_n\}$ in $\mathcal{U}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in $\mathcal{U}$.*

Remark 1. *(i) In this paper, using Definition 7, we generalize Theorem 1 in several directions.*
(ii) We also give a generalized version of Definition 7, which opens up new possibilities for further research.

In this section, in the end, we list some concepts regarding the multivalued mapping. Let $(\mathcal{U}, d)$ be a rectangular $b-$metric space, we will denote by $\mathcal{CB}(\mathcal{U})$ the set of non-empty bounded closed subsets of $\mathcal{U}$. For $M, N \in \mathcal{CB}(\mathcal{U})$ and $x \in \mathcal{U}$, we define

$$d(x, M) = \inf_{a \in M} d(x, a) \text{ and } d(M, N) = \sup_{a \in M} d(a, N).$$

The mapping

$$H : \mathcal{CB}(\mathcal{U}) \times \mathcal{CB}(\mathcal{U}) \to [0, +\infty),$$

given by

$$H(M, N) = \max\{\sup_{a \in M} d(a, N), \sup_{b \in N} d(b, M)\},$$

is the Hausdorff distance between M and N in $\mathcal{CB}(\mathcal{U})$. We define $\mathcal{B}(\mathcal{U})$ the set of non-empty compact subsets of $\mathcal{U}$. A point x is said to be a fixed point of multivalued mapping $T : \mathcal{U} \to \mathcal{CB}(\mathcal{U})$ provided $x \in T(x)$.

2. Main result

First, we introduce the concept of $\alpha-$admissible $\theta - \phi-$multivalued contraction in rectangular $b-$metric spaces.

Definition 8. *Let $(\mathcal{U}, d)$ be a rectangular $b-$metric space and $T : \mathcal{U} \to \mathcal{B}(\mathcal{U})$ be a mapping and*

$$W(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\}.$$

(i) T is called an $\alpha-$admissible $\theta-$multivalued contraction if exist $\theta \in \Theta$, $K \geq 0$ and $s \in (0, 1)$ such that

$$H(Tx, Ty) > 0 \text{ implies } \theta[\alpha(x, y)b^3 H(Tx, Ty)] \leq \theta[M(x, y)]^s + KW(x, y), \tag{4}$$

for all $x, y \in \mathcal{U}$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}.$$

(ii) T is called an $\alpha-$admissible $\theta - \phi-$multivalued contraction if exist $\theta \in \Theta$ and $K \geq 0$ such that

$$H(Tx, Ty) > 0 \text{ implies } \theta[\alpha(x, y)b^3 H(Tx, Ty)] \leq \phi[\theta(M(x, y))] + KW(x, y), \tag{5}$$

for all $x, y \in \mathcal{U}$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(x, Ty), d(y, Tx)\}.$$

(iii) *T is called an $\alpha-$admissible $\theta-\phi-$multivalued Kannan-type if there are $\theta \in \Theta$, $\phi \in \Phi$ and $K \geq 0$ such that $H(Tx, Ty) > 0$ implies*

$$\theta[\alpha(x,y)b^3 H(Tx,Ty)] \leq \phi\left[\theta\left(\frac{d(x,Tx)+d(y,Ty)}{2}\right)\right] + KW(x,y), \tag{6}$$

for all $x, y \in \mathcal{U}$.

(iv) *T is called an $\alpha-$admissible $\theta-\phi-$multivalued Reich-type if exist $\theta \in \Theta$, $\phi \in \Phi$ and $K \geq 0$ such that $H(Tx, Ty) > 0$ implies*

$$\theta[\alpha(x,y)b^3 H(Tx,Ty)] \leq \phi\left[\theta\left(\frac{d(x,y)+d(x,Tx)+d(y,Ty)}{3}\right)\right] + KW(x,y), \tag{7}$$

for all $x, y \in \mathcal{U}$.

(v) *T is called α-continuous multivalued mapping if, for all sequences $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} x_n = x \in \mathcal{U}$, we have $\lim_{n\to+\infty} Tx_n = Tx$ so that $\lim_{n\to+\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$ for every $n \in \mathbb{N}$, means that $\lim_{n\to+\infty} H(Tx_n, Tx) = 0$.*

Theorem 2. *Let $(\mathcal{U}, d)$ be a rectangular $b-$metric space and $T : \mathcal{U} \to \mathcal{B}(\mathcal{U})$ be an $\alpha-$admissible $\theta-$multivalued contraction satisfying:*

(i) *$(\mathcal{U}, d)$ is an $\alpha-$complete metric space,*

(ii) *$\alpha(x_0, x_1) \geq 1$ for $x_0 \in \mathcal{U}$ and $x_1 \in T(\mathcal{U})$,*

(iii) *T is triangular $\alpha-$admissible,*

(iv) *T is an $\alpha-$continuous multivalued mapping.*

Then, T has a fixed point.

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{U}$ such that $x_{n+1} \in Tx_n$ with $\alpha(x_n, x_{n+1}) \geq 1$, for all $k \in \mathbb{N} \cup \{0\}$. By *(iv)*, we have

$$\begin{aligned}\theta[H(Tx_{n-1}, Tx_n)] &\leq \theta[b^3 H(Tx_{n-1}, Tx_n)] \\ &\leq \theta[\alpha(x_{n-1}, x_n) b^3 H(Tx_{n-1}, Tx_n)] \\ &\leq \theta[M(x_{n-1}, x_n)]^s + KW(x_{n-1}, x_n),\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_n, Tx_{n-1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, Tx_n)\}\end{aligned}$$

and

$$\begin{aligned} W(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(Tx_{n-1}, x_n), d(x_{n-1}, Tx_n)\} \\ &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), 0, d(x_{n-1}, Tx_n)\} \\ &= 0. \end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, Tx_n)$, we have

$$d(x_{n+1}, x_n) \leq H(Tx_{n-1}, Tx_n).$$

Since $x_{n+1} \in Tx_n$ this implies that $d(x_n, Tx_n) \leq d(x_n, x_{n+1})$. Now, we obtain

$$\begin{aligned} \theta(d(x_{n+1}, x_n)) &\leq \theta(H(Tx_{n-1}, Tx_n)) \\ &\leq [\theta(M(x_{n-1}, x_n))]^s + KN(x_{n-1}, x_n) \\ &\leq [\theta(M(x_{n-1}, x_n))]^s \\ &< \theta(d(x_n, Tx_n)) \\ &= \theta(d(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction, so $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ and we have

$$\begin{aligned} \theta(d(x_{n+1}, x_n)) &\leq \theta(H(Tx_{n-1}, Tx_n)) \\ &\leq [\theta(M(x_{n-1}, x_n))]^s + KN(x_{n-1}, x_n) \\ &\leq [\theta(M(x_{n-1}, x_n))]^s \\ &= [\theta(d(x_{n-1}, x_n))]^s \\ &< \theta(d(x_{n-1}, x_n)). \end{aligned}$$

By the properties of θ we have,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

This implies that the sequence$\{d(x_n, x_{n+1})\}_n$ is strictly decreasing, this implies that there exists $\alpha > 0$ such that

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = \alpha.$$

Suppose that $\alpha > 0$, we can conclude that $d(x_n, x_{n+1}) \geq \alpha$, for all $n \in \mathbb{N}$. We get

$$\begin{aligned} \theta(d(x_{n+1}, x_n)) &\leq [\theta(d(x_{n-1}, x_n))]^s \\ &\leq [\theta(d(x_{n-2}, x_{n-1}))]^{s^2} \\ &\vdots \\ &\leq [\theta(d(x_0, x_1))]^{s^n}. \end{aligned}$$

Using the property of θ, we obtain

$$1 < \theta(\alpha) \leq [\theta(d(x_0, x_1))]^{s^n}. \tag{8}$$

Letting $n \to +\infty$ in (8), we get

$$1 < \theta(\alpha) \leq 1.$$

This a contradiction. Now, we conclude that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$. Next, we show that $\{x_n\}_n$ is a Cauchy sequence in $\mathcal{U}$, there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that, for all positive integers k, $n_k > m_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{9}$$

$$d(x_{m_k}, x_{n_{k-1}}) < \varepsilon \tag{10}$$

we get

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq bd(x_{m_k}, x_{m_{k+1}}) + bd(x_{m_{k+1}}, x_{n_{k+1}}) + bd(x_{n_{k+1}}, x_{n_k}), \tag{11}$$

letting $k \to +\infty$, we get

$$\frac{\varepsilon}{b} \lim_{n\to+\infty} \sup d(x_{m_{k+1}}, x_{n_{k+1}}) \tag{12}$$

and

$$\lim_{n\to+\infty} \sup d(x_{m_k}, x_{n_k}) \leq b\varepsilon. \tag{13}$$

Since $\alpha(x_{m_k}, x_{n_k}) \geq 1$, we have

$$\begin{aligned} M(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{m_k})\} \\ &\leq \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\} \\ &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\} \end{aligned}$$

and

$$\begin{aligned} W(x_{m_k}, x_{n_k}) &= \min\{d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), d(Tx_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{n_k})\} \\ &\leq \min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_{k+1}}, x_{n_k}), d(x_{m_k}, x_{n_{k+1}})\}, \end{aligned}$$

letting $n \to +\infty$, we obtain

$$\begin{aligned} \lim_{k\to+\infty} M(x_{m_k}, x_{n_k}) &\leq \lim_{k\to+\infty} \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\} \\ &\leq \max\{b\varepsilon, 0, b^2\varepsilon\} \\ &= b^2\varepsilon \end{aligned}$$

and

$$\begin{aligned} \lim_{k\to+\infty} W(x_{m_k}, x_{n_k}) &\leq \lim_{k\to+\infty} \min\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\} \\ &\leq \min\{b\varepsilon, 0, b^2\varepsilon\} \end{aligned}$$

$$= 0.$$

So, we have

$$\begin{aligned}\theta[d(x_{m_{k+1}}, x_{n_{k+1}})] &\leq \theta[b^3 H(Tx_{m_k}, Tx_{n_k})] \\ &\leq \theta[\alpha(x_{m_k}, x_{n_k}) b^3 H(Tx_{m_k}, Tx_{n_k})] \\ &\leq [\theta(M(x_{m_k}, x_{n_k}))]^s + KW(x_{m_k}, x_{n_k}).\end{aligned}$$

Letting $k \to +\infty$, we obtain

$$\begin{aligned}\theta(\varepsilon b) &\leq \theta[b^3 \lim_{k\to+\infty} d(x_{m_{k+1}}, x_{n_{k+1}})] \\ &\leq [\theta(\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}))]^s + K \lim_{k\to+\infty} W(x_{m_k}, x_{n_k}) \\ &= [\theta(\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}))]^s \\ &\leq [\theta(b\varepsilon)]^s \\ &< \theta(b\varepsilon).\end{aligned}$$

This implies that $b\varepsilon < b\varepsilon$, which is a contradiction. Consequently, $\{x_n\}$ is a Cauchy sequence in $\mathcal{U}$, so there exists $z \in \mathcal{U}$ such that

$$\lim_{n\to+\infty} d(x_n, z) = 0.$$

Since T is $\alpha-$continuous multivalued mapping, we have

$$\lim_{n\to+\infty} H(Tx_n, Tz) = 0.$$

We now conclude that it is

$$\lim_{n\to+\infty} d(x_{n+1}, Tz) \leq \lim_{n\to+\infty} H(Tx_n, Tz) = 0.$$

Therefore, $z \in Tz$ i.e. T has a fixed point.

Example 1. *Let $\mathcal{U} = [-1, 1]$. Define $d : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ by $d(x, y) = (x - y)^2$. Then $(\mathcal{U}, d)$ is a rectangular $b-$metric space with parameter $b = 2$. Define a mapping $T : \mathcal{U} \to B(\mathcal{U})$ by*

$$Tx = \begin{cases} [0, \frac{x}{4}], & \textit{if } x, y \in [0, \frac{1}{4}] \\ [x, x^2], & \textit{otherwise} \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1, & \textit{if } x, y \in [0, \frac{1}{4}] \\ \\ 0, & \textit{otherwise} \end{cases}$$

and the function $\theta : [0,+\infty) \to [1,+\infty)$ by $\theta(x) = 1+x$. Then T is triangular $\alpha-$admissible and $H(Tx,Ty) = \frac{1}{4}(x-y)^2$.
Case 1 *If $x, y \in [0, \frac{1}{4}]$ we have $\alpha(x,y) \geq 1$ and*

$$\begin{aligned}\theta[\alpha(x,y)b^3H(Tx,Ty)] &\leq \frac{1}{2}(x-y)^2+1\\ &\leq d(x,y)+1\\ &\leq \theta[M(x,y)]+KW(x,y).\end{aligned}$$

Case 2 *If $x, y \in (\frac{1}{4},+\infty)$ we have $\alpha(x,y) = 0$ and*

$$\begin{aligned}\theta[\alpha(x,y)b^3H(Tx,Ty)] &= \theta(0)\\ &\leq \theta((x-y)^2)\\ &\leq d(x,y)+1\\ &\leq \theta[M(x,y)]+KW(x,y).\end{aligned}$$

Then, T has a fixed point.

Theorem 3. *Let $(\mathcal{U}, d)$ be a complete rectangular $b-$metric space and $T : \mathcal{U} \to \mathcal{B}(\mathcal{U})$ be an $\alpha-$admissible $\theta-$multivalued contraction satisfying:*

(i) *$(\mathcal{U}, d)$ is an $\alpha-$complete metric space,*

(ii) *$\alpha(x_0, x_1) \geq 1$ for $x_0 \in \mathcal{U}$ and $x_1 \in T(\mathcal{U})$,*

(iii) *T is triangular $\alpha-$admissible,*

(iv) *there exists a sequence $\{x_n\}$ in $\mathcal{U}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $\lim_{n\to+\infty} d(x_n, z) = 0$, for some $z \in \mathcal{U}$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$, for all $k \in \mathbb{N}\cup\{0\}$.*

Then, T has a fixed point.

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{U}$ such that $x_{n+1} \in Tx_n$ with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to z \in \mathcal{U}$. By (iv), we show that $z \in Tz$. Suppose that $z \notin Tz$, we have

$$\lim_{n\to+\infty} d(Tx_n, z) = 0$$

and

$$\begin{aligned}\frac{1}{b^2}d(z,Tz) &\leq \lim_{n\to+\infty}\inf H(Tx_n,Tz)\\ &\leq \lim_{n\to+\infty}\sup H(Tx_n,Tz)\\ &\leq b^2d(z,Tz).\end{aligned}$$

So, we have

$$\begin{aligned}\theta[b^3H(Tx_n,Tz)] &\leq \theta[\alpha(x_n,z)b^3H(Tx_n,Tz)]\\ &\leq \theta[M(x_n,z)]^s + KW(x_n,z) \text{ for all } n\in\mathbb{N},\end{aligned}$$

where

$$M(x_n,z)=\max\{d(x_n,z),d(x_n,Tx_n),d(z,Tz),d(z,Tx_n)\}$$

and

$$W(x_n,z)=\min\{d(x_n,Tx_n),d(z,Tz),d(Tx_n,z),d(x_n,Tz)\}.$$

Letting $n\to+\infty$ we obtain

$$\begin{aligned}\lim_{n\to+\infty}\sup M(x_n,z) &= \lim_{n\to+\infty}\sup\max\{d(x_n,z),d(x_n,Tx_n),d(z,Tz),d(z,Tx_n)\}\\ &\leq \lim_{n\to+\infty}\sup\max\{d(x_n,z),d(x_n,x_{n+1}),d(z,Tz),d(z,x_{n+1})\}\\ &\leq d(z,Tz)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to+\infty}\sup W(x_n,z) &= \lim_{n\to+\infty}\sup\min\{d(x_n,Tx_n),d(z,Tz),d(Tx_n,z),d(x_n,Tz)\}\\ &\leq \lim_{n\to+\infty}\sup\min\{d(x_n,x_{n+1}),d(z,Tz),d(x_{n+1},z),d(x_n,Tz)\}\\ &= 0.\end{aligned}$$

Then,

$$\begin{aligned}\theta(bd(z,Tz)) &\leq \theta[b^3\lim_{n\to+\infty}H(Tx_n,Tz)]\\ &\lim_{n\to+\infty}\theta[b^3H(Tx_n,Tz)]\\ &\leq \lim_{n\to+\infty}\theta[\alpha(x_n,z)b^3H(Tx_n,Tz)]\\ &\leq \theta[\lim_{n\to+\infty}M(x_n,z)]^s + K\lim_{n\to+\infty}W(x_n,z)\\ &\leq [\theta(d(z,Tz))]^s\\ &< \theta(d(z,Tz)).\end{aligned}$$

This implies that $bd(z,Tz)<d(z,Tz)$, this a contradiction, so $z\in Tz$.

Corollary 1. *Let $(\mathcal{U},d)$ be a complete rectangular $b-$metric space and $T:\mathcal{U}\to\mathcal{B}(\mathcal{U})$ be a mapping. If exist $\theta\in\Theta$ and $s\in(0,1)$ such that*

$$H(Tx,Ty)>0 \textit{ implies } \theta[b^3H(Tx,Ty)]\leq[\theta(d(x,y))]^s \textit{ for all } x,\ y\in\mathcal{U},$$

then, T has a fixed point.

Theorem 4. *Let $(\mathcal{U}, d)$ be a complete rectangular $b-$metric space and $T : \mathcal{U} \to \mathcal{B}(\mathcal{U})$ be an $\alpha-$admissible $\theta - \phi-$multivalued contraction satisfying:*

(i) *$(\mathcal{U}, d)$ is an $\alpha-$complete metric space,*

(ii) *$\alpha(x_0, x_1) \geq 1$ for $x_0 \in \mathcal{U}$ and $x_1 \in T(\mathcal{U})$,*

(iii) *T is triangular $\alpha-$admissible.*

(iv) *T is an $\alpha-$continuous multivalued mapping.*

Then, T has a fixed point.

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{U}$ such that $x_{n+1} \in Tx_n$ with $\alpha(x_n, x_{n+1}) \geq 1$, By (iv), we have

$$\begin{aligned}\theta[H(Tx_{n-1}, Tx_n)] &\leq \theta[b^3 H(Tx_{n-1}, Tx_n)]\\ &\leq \theta[\alpha(x_{n-1}, x_n) b^3 H(Tx_{n-1}, Tx_n)]\\ &\leq \phi[\theta(M(x_{n-1}, x_n))] + KW(x_{n-1}, x_n)\ \forall n \in \mathbb{N}\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_n, Tx_{n-1})\}\\ &= \max\{d(x_{n-1}, x_n), d(x_n, Tx_n)\}\end{aligned}$$

and

$$\begin{aligned}W(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(Tx_{n-1}, x_n), d(x_{n-1}, Tx_n)\}\\ &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), 0, d(x_{n-1}, Tx_n)\}\\ &= 0.\end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, Tx_n)$, we have $d(x_{n+1}, x_n) \leq H(Tx_{n-1}, Tx_n)$. Since $x_{n+1} \in Tx_n$ we have $d(x_n, Tx_n) \leq d(x_n, x_{n+1})$. Now, we obtain

$$\begin{aligned}\theta(d(x_{n+1}, x_n)) &\leq \theta(H(Tx_{n-1}, Tx_n))\\ &\leq \phi[\theta(M(x_{n-1}, x_n))] + KN(x_{n-1}, x_n)\\ &\leq \phi[\theta(M(x_{n-1}, x_n))]\\ &< \phi[\theta(d(x_n, Tx_n))]\\ &< \theta(d(x_n, Tx_n)),\end{aligned}$$

which is a contradiction, so, $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ and

$$\begin{aligned}\theta(d(x_{n+1}, x_n)) &\leq \phi[\theta(d(x_{n-1}, x_n))]\\ &< \theta(d(x_{n-1}, x_n)).\end{aligned}$$

By the properties of θ we have, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$. This implies that the sequence $\{d(x_n, x_{n+1})\}_n$ is strictly decreasing, this implies that there exists $\alpha > 0$ such that

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = \alpha.$$

Suppose that $\alpha > 0$, we can conclude that $d(x_n, x_{n+1}) \geq \alpha$, for all $n \in \mathbb{N}$. We get

$$\begin{aligned} \theta(d(x_{n+1}, x_n)) &\leq \phi[\theta(d(x_{n-1}, x_n))] \\ &\leq \phi^2[\theta(d(x_{n-2}, x_{n-1}))] \\ &\vdots \\ &\leq \phi^n[\theta(d(x_0, x_1))]. \end{aligned}$$

Using the property of θ, we get

$$1 < \theta(\alpha) \leq \phi^n[\theta(d(x_0, x_1))]. \tag{14}$$

Letting $n \to +\infty$ in (14), we get

$$1 < \theta(\alpha) \leq 1.$$

This a contradiction, so we obtain

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0.$$

Next, we show that $\{x_n\}_n$ is a Cauchy sequence in $\mathcal{U}$, there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that, for all positive integers k, $n_k > m_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{15}$$

$$d(x_{m_k}, x_{n_{k-1}}) < \varepsilon \tag{16}$$

we get

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq bd(x_{m_k}, x_{m_{k+1}}) + bd(x_{m_{k+1}}, x_{n_{k+1}}) + bd(x_{n_{k+1}}, x_{n_k}) \tag{17}$$

Letting $k \to +\infty$, we get

$$\frac{\varepsilon}{b} \lim_{n \to +\infty} \sup d(x_{m_{k+1}}, x_{n_{k+1}}) \tag{18}$$

and

$$\lim_{n \to +\infty} \sup d(x_{m_k}, x_{n_k}) \leq b\varepsilon. \tag{19}$$

Since $\alpha(x_{m_k}, x_{n_k}) \geq 1$, we have

$$\begin{aligned} M(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{m_k})\} \\ &\leq \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\} \\ &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\} \end{aligned}$$

and

$$\begin{aligned}W(x_{m_k}, x_{n_k}) &= \min\{d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), d(Tx_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{n_k})\}\\ &\leq \min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_{k+1}}, x_{n_k}); d(x_{m_k}, x_{n_{k+1}})\}\end{aligned}$$

Letting $n \to +\infty$ we obtain

$$\begin{aligned}\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}) &\leq \lim_{k\to+\infty} \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}\\ &\leq \max\{b\varepsilon, 0, b^2\varepsilon\}\\ &= b^2\varepsilon.\end{aligned}$$

Now, we obtain

$$\begin{aligned}\lim_{k\to+\infty} W(x_{m_k}, x_{n_k}) &\leq \lim_{k\to+\infty} \min\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}\\ &\leq \min\{b\varepsilon, 0, b^2\varepsilon\}\\ &= 0.\end{aligned}$$

So, we have

$$\begin{aligned}\theta[d(x_{m_{k+1}}, x_{n_{k+1}})] &\leq \theta[b^3 H(Tx_{m_k}, Tx_{n_k})]\\ &\leq \theta[\alpha(x_{m_k}, x_{n_k}) b^3 H(Tx_{m_k}, Tx_{n_k})]\\ &\leq \phi[\theta(M(x_{m_k}, x_{n_k}))] + KW(x_{m_k}, x_{n_k}).\end{aligned}$$

Letting $k \to +\infty$, we obtain

$$\begin{aligned}\theta(\varepsilon b) &\leq \theta[b^3 \lim_{k\to+\infty} d(x_{m_{k+1}}, x_{n_{k+1}})]\\ &\leq \phi[\theta(\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}))] + K \lim_{k\to+\infty} W(x_{m_k}, x_{n_k})\\ &= \phi[\theta(\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}))]\\ &\leq \phi[\theta(b\varepsilon)].\end{aligned}$$

By Lemma 2 we have

$$\theta(b\varepsilon) \leq \phi[\theta(b\varepsilon)] < \theta(b\varepsilon).$$

This implies that

$$b\varepsilon < b\varepsilon,$$

which is a contradiction. Consequently, $\{x_n\}$ is a Cauchy sequence in $\mathcal{U}$. Therefore, there exists $z \in \mathcal{U}$ such that

$$\lim_{n\to+\infty} d(x_n, z) = 0.$$

Since T is $\alpha-$continuous multivalued mapping, we have

$$\lim_{n\to+\infty} H(Tx_n, Tz) = 0.$$

Now, we obtain,

$$\lim_{n\to+\infty} d(x_{n+1}, Tz) \leq \lim_{n\to+\infty} H(Tx_n, Tz) = 0.$$

Therefore, $z \in Tz$ i.e. T has a fixed point.

Example 2. *Let* $\mathcal{U} = A \cup B$, *where* $A = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$ *and* $B = [1, 2]$.
Define $d : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ *by*

$$d\left(0, \frac{1}{2}\right) = d\left(\frac{1}{2}, \frac{1}{3}\right) = 0.16, d\left(0, \frac{1}{3}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = 0.04, d\left(0, \frac{1}{4}\right) d\left(\frac{1}{2}, \frac{1}{4}\right) = 0.25,$$

$$d\left(x, y\right) = \left(x - y\right)^2, \text{ for } x, y \in [1, 2].$$

Then $(\mathcal{U}, d)$ *is a rectangular* $b-$*metric space with parameter* $b = 3$. *Let* $T : \mathcal{U} \to B(\mathcal{U})$ *defined by*

$$Tx = \begin{cases} A, & \text{if } x \in A \\ \left[0, \frac{x}{2}\right], & \text{if } x \in B, \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \left[0, \frac{1}{4}\right]. \\ 0, & \text{otherwise} \end{cases}$$

and the functions $\theta : [0, +\infty) \to [1, +\infty)$ *defined by*

$$\theta(z) = 1 + z$$

and $\phi : [1, +\infty) \to [1, +\infty)$ *defined by*

$$\phi(z) = \frac{1 + z}{2}.$$

Then a mapping T *is triangular* $\alpha-$*admissible and* $H(Tx, Ty) = \dfrac{1}{4}(x - y)^2$.

Corollary 2. *Let* $(\mathcal{U}, d)$ *be a complete rectangular* $b-$*metric space and* $T : \mathcal{U} \to \mathcal{B}(\mathcal{U})$ *be a mapping. If* $\theta \in \Theta$ *and* $\phi \in \Phi$ *we have*

$$H(Tx, Ty) > 0 \text{ implies } \theta[b^3 H(Tx, Ty)] \leq \phi[\theta(M(x, y))], \text{ for all } x, y \in \mathcal{U},$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(x, Ty), d(y, Tx)\}.$$

Then, T *has a fixed point.*

Theorem 5. *Let $(\mathcal{U}, d)$ be a complete rectangular $b-$metric space and $T : \mathcal{U} \to \mathcal{B}(\mathcal{U})$ be an $\alpha-$admissible $\theta - \phi-$multivalued contraction satisfying*

(i) $(\mathcal{U}, d)$ *is an $\alpha-$complete metric space*

(ii) $\alpha(x_0, x_1) \geq 1$ *for* $x_0 \in \mathcal{U}$ *and* $x_1 \in T(\mathcal{U})$

(iii) T *is triangular $\alpha-$admissible.*

(iv) exist $\{x_n\}$ *is a sequence in* $\mathcal{U}$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\lim_{n\to+\infty} d(x_n, z) = 0$*, for some* $z \in \mathcal{U}$ *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, z) \geq 1$ *for all* $k \in \mathbb{N} \cup \{0\}$.

Then, T has a fixed point.

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{U}$ such that $x_{n+1} \in Tx_n$ with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to z \in \mathcal{U}$. By (iv), we show that $z \in Tz$.
Suppose that $z \notin Tz$, we have

$$\lim_{n\to+\infty} d(Tx_n, z) = 0$$

and

$$\begin{aligned}
\frac{1}{b^2} d(z, Tz) &\leq \lim_{n\to+\infty} \inf H(Tx_n, Tz) \\
&\leq \lim_{n\to+\infty} \sup H(Tx_n, Tz) \\
&\leq b^2 d(z, Tz).
\end{aligned}$$

So, we have

$$\begin{aligned}
\theta[b^3 H(Tx_n, Tz)] &\leq \theta[\alpha(x_n, z) b^3 H(Tx_n, Tz)] \\
&\leq \phi\theta[M(x_n, z)] + KW(x_n, z) \text{ for all } n \in \mathbb{N},
\end{aligned}$$

where

$$M(x_n, z) = \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\}$$

and

$$W(x_n, z) = \min\{d(x_n, Tx_n), d(z, Tz), d(Tx_n, z), d(x_n, Tz)\}.$$

Letting $n \to +\infty$ we obtain

$$\begin{aligned}
\lim_{n\to+\infty} \sup M(x_n, z) &= \lim_{n\to+\infty} \sup \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(z, Tx_n)\} \\
&\leq \lim_{n\to+\infty} \sup \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), d(z, x_{n+1})\}
\end{aligned}$$

$$\leq d(z,Tz)$$

and

$$\begin{aligned}\lim_{n\to+\infty}\sup W(x_n,z) &= \lim_{n\to+\infty}\sup\min\{d(x_n,Tx_n),d(z,Tz),d(Tx_n,z),d(x_n,Tz)\}\\ &\leq \lim_{n\to+\infty}\sup\min\{d(x_n,x_{n+1}),d(z,Tz),d(x_{n+1},z),d(x_n,Tz)\}\\ &= 0.\end{aligned}$$

Now, we obtain

$$\begin{aligned}\theta(bd(z,Tz)) &\leq \theta[b^3\lim_{n\to+\infty}H(Tx_n,Tz)]\\ &\leq \lim_{n\to+\infty}\theta[b^3H(Tx_n,Tz)]\\ &\leq \lim_{n\to+\infty}\theta[\alpha(x_n,z)b^3H(Tx_n,Tz)]\\ &\leq \phi(\theta[\lim_{n\to+\infty}M(x_n,z)])+K\lim_{n\to+\infty}W(x_n,z)\\ &\leq \phi[\theta(d(z,Tz))]\\ &< \theta(d(z,Tz)).\end{aligned}$$

This implies that

$$bd(z,Tz) < d(z,Tz),$$

this a contradiction, then $z\in Tz$.

The following corollaries are immediate results of Theorem 4 and Theorem 5.

Corollary 3. *Let $(\mathcal{U},d)$ be a complete rectangular $b-$metric space and $T:\mathcal{U}\to\mathcal{B}(\mathcal{U})$ be an $\alpha-$admissible $\theta-\phi-$multivalued contraction Kannan type satisfying:*

(i) $(\mathcal{U},d)$ *is an* $\alpha-$*complete metric space,*

(ii) $\alpha(x_0,x_1)\geq 1$ *for* $x_0\in\mathcal{U}$ *and* $x_1\in T(\mathcal{U})$,

(iii) T *is triangular* $\alpha-$*admissible,*

(iv) T *is an* $\alpha-$*continuous multivalued mapping or exists a sequence* $\{x_n\}$ *in* $\mathcal{U}$ *such that* $\alpha(x_n,x_{n+1})\geq 1$ *for all* $n\in\mathbb{N}\cup\{0\}$ *and* $\lim_{n\to+\infty}d(x_n,z)=0$, *for some* $z\in\mathcal{U}$ *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)},z)\geq 1$ *for all* $k\in\mathbb{N}\cup\{0\}$.

Then, T has a fixed point.

Corollary 4. *Let $(\mathcal{U},d)$ be a complete rectangular $b-$metric space and $T:\mathcal{U}\to\mathcal{B}(\mathcal{U})$ be an $\alpha-$admissible $\theta-\phi-$multivalued contraction Reich-type satisfying:*

(i) $(\mathcal{U},d)$ *is an* $\alpha-$*complete metric space,*

(ii) $\alpha(x_0, x_1) \geq 1$ *for* $x_0 \in \mathcal{U}$ *and* $x_1 \in T(\mathcal{U})$*,*

(iii) T *is triangular* $\alpha-$*admissible,*

(iv) *there exists* $\{x_n\}$ *is a sequence in* $\mathcal{U}$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\lim\limits_{n\to+\infty} d(x_n, z) = 0$*, then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, z) \geq 1$ *for all* $k \in \mathbb{N} \cup \{0\}$*.*

Then, T *has a fixed point.*

Conclusion

We obtain some fixed point theorems for $\theta-\phi-$multivalued contractions in $\alpha-$complete rectangular $b-$metric spaces. We establish some fixed point theorems including the $\alpha-$admissible $\theta - \phi-$multivalued Kannan type and Reich type. Our results improve and generalize some results from the literature. We believe that our paper may be interesting to researchers in fixed point theory, because using the methods presented in this paper, the following problems remain open:

1. Prove the Hardy-Rogers result for $\theta - \phi$-multivalued contractions in α-complete rectangular b-metric spaces.
2. Prove the Ćirić result for $\theta - \phi$-multivalued contractions in α-complete rectangular b-metric spaces.

Of course, other questions such as Kirk theorem of fixed point, Suzuki fixed point theorem, etc.

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Author contributions

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Data Availability

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

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