EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6089
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Generalized Fractional Integral Extensions of Hermite-Hadamard Inequalities

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Abstract. In this article, we provide a number of Hermite-Hadamard type fractional integral inequalities for the Atangana-Baleanu and Prabhakar fractional operators, using extended generalized Mittag-Leffler functions as their kernel. Significant findings are provided for the integral inequalities involving fractional integrals of the type $(\Im_1+, \Im_2-)$ and $(\frac{\Im_1+\Im_2}{2})$. By employing certain functions to create visual graphs with matching numerical entries that depict the inequalities, we show the veracity of our findings.

2020 Mathematics Subject Classifications: 33E12, 26A33, 26D15

Key Words and Phrases: Atangana-Baleanu fractional calculus, Fractional calculus, Hermite-Hadamard inequality, Generalized fractional integral operators, Mittag-Leffler function

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1. Introduction

The expansion of differentiation and integration to non-integer orders is the work of fractional calculus, a subfield of mathematics, making it possible to represent complex processes more intricately [1, 2]. Fractional integral operators are utilized as mathematical tools in the study of fractional calculus because they are integrals of a certain order, which is not limited to integer values but can be any real or complex number. In the solution of fractional differential equations, fractional integral operators are crucial because they generalize several classical operators, including the integral and derivative [3], and they aid in describing processes that are not fully represented by traditional calculus, which is limited to integer orders. Fractional calculus uses definitions such as Riemann-Liouville to extend these operations to arbitrary real or complex numbers [4, 5]. Moreover, fractional

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6089

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calculus intersects with concepts like convex functions [6], leading to advancements in optimization and engineering [7, 8]. Overall, fractional calculus offers a robust framework for understanding and analyzing complex phenomena.
Giving complex functions boundaries and approximations is an essential part that inequalities play in providing important insights into their behavior. Integral inequalities are a key area in mathematical analysis, crucial for studying integral equations and differential equations [9, 10]. They play a crucial role in fractional differential equations, where fractional integral inequalities help to establish the uniqueness of solutions and provide bounds for fractional boundary value problems. Inequalities involving fractional derivatives are especially valuable in determining solutions for Cauchy problems as well as their upper limits [11, 12]. The goal of expanding the theory of integral inequalities through the use of fractional integral operators to generalize classical inequalities has been spurred by this significance [13], which improves theoretical comprehension and real-world applications [14, 15]. Sajid et al. have discussed some new Grüss type inequalities associated with generalized fractional derivatives in [16].
Special functions are closely related to fractional calculus in many ways [9, 17], like the Mittag-Leffler function, which extends the concepts of fractional operators [18, 19] and plays a crucial role in fractional calculus [20, 21]. Named after Gösta Mittag-Leffler, this function is essential for solving fractional differential equations. Usually accomplished by adding more parameters to its definition, the extended generalized Mittag-Leffler function is a further expanded form of the standard Mittag-Leffler function that provides more flexibility in modeling complex phenomena [22]. Two prominent models in this area that incorporate Mittag-Leffler functions are the Atangana-Baleanu [23] and Prabhakar models [24, 25]. These models advance fractional calculus by offering refined tools for describing systems with memory, non-singular and non-local effects [26, 27], making them valuable in various scientific and engineering fields [28, 29] while tackling practical issues in a variety of fields [30, 31]. The modified (k, s) fractional integral operator involving k-Mittag-Leffler function along with its properties is discussed in [32].
In this research, the Hermite-Hadamard $(\mathcal{H} - \mathcal{H})$ inequality [33] by applying generalized fractional integral operators through the extended generalized Mittag-Leffler function will be studied. The goal is to derive new inequalities that not only generalize but also enhance the classical $(\mathcal{H} - \mathcal{H})$ inequality [34], utilizing fractional calculus and special function methodologies.
As we continue our work, it is crucial to remember important definitions to ensure clarity and consistency. These concepts help to manage complex activities and initiatives by providing a strong foundation.

Definition 1. *[7] A function $\Upsilon : I \to \Re$ is known as a convex function if it satisfies the following inequality*

$$\Upsilon(c\Im_1 + (1-c)\Im_2) \leq c\Upsilon(\Im_1) + (1-c)\Upsilon(\Im_2),$$

where $c \in [0, 1]$ and $\Im_1, \Im_2 \in I$.

Definition 2. *[4] Let Υ be a function in $\mathcal{L}^1$ on the interval $[\Im_1, \Im_2]$. The G-th order left and right sided Reimann-Liouville integrals for any $c \in [\Im_1, \Im_2]$, applied to $\Upsilon(c)$ are defined as, provided that $\Re(G) > 0$*

$$ {}^{\mathcal{R}-\mathcal{L}}I^{G}_{\Im_1+}\Upsilon(c) = \frac{1}{\Gamma(G)}\int_{\Im_1}^{c}(c-\psi)^{G-1}\Upsilon(\psi)d\psi, \tag{1} $$

and

$$ {}^{\mathcal{R}-\mathcal{L}}I^{G}_{\Im_2-}\Upsilon(c) = \frac{1}{\Gamma(G)}\int_{c}^{\Im_2}(\psi-c)^{G-1}\Upsilon(\psi)d\psi. \tag{2} $$

Definition 3. *[23] Let Υ be a function in $\mathcal{L}^1$ on the interval $[\Im_1, \Im_2]$. The G-th order left and right sided Atangana-Baleanu integrals for any $c \in [\Im_1, \Im_2]$, applied to $\Upsilon(c)$ and for $1 > G > 0$, written as*

$$ {}^{\mathcal{A}-\mathcal{B}}I^{G}_{\Im_1+}\Upsilon(c) = \frac{G}{\mathcal{B}(G)}\left({}^{\mathcal{R}-\mathcal{L}}I^{G}_{\Im_1+}\Upsilon(c)\right) + \frac{(1-G)}{\mathcal{B}(G)}\Upsilon(c), \tag{3} $$

and

$$ {}^{\mathcal{A}-\mathcal{B}}I^{G}_{\Im_2-}\Upsilon(c) = \frac{G}{\mathcal{B}(G)}\left({}^{\mathcal{R}-\mathcal{L}}I^{G}_{\Im_2-}\Upsilon(c)\right) + \frac{(1-G)}{\mathcal{B}(G)}\Upsilon(c), \tag{4} $$

where $\mathcal{B}(G)$ is a normalization function that is both real and positive, having properties $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

Definition 4. *[24, 25] Given a function Υ that belongs to $\mathcal{L}^1$ on the interval $[\Im_1, \Im_2]$ and for any $c \in [\Im_1, \Im_2]$, then the left and right sided Prabhakar fractional integral operators applied to $\Upsilon(c)$ with $\Re(\alpha^*) > 0$, $\Re(\beta^*) > 0$ and $\gamma, \flat \in C$, given as*

$$ {}^{\mathcal{P}}I^{\alpha^*,\beta^*,\gamma,\flat}_{\Im_1+}\Upsilon(c) = \int_{\Im_1}^{c}(c-\psi)^{\beta^*-1}\mathcal{E}^{\gamma}_{\alpha^*,\beta^*}\left(\flat(c-\psi)^{\alpha^*}\right)\Upsilon(\psi)d\psi, \tag{5} $$

and

$$ {}^{\mathcal{P}}I^{\alpha^*,\beta^*,\gamma,\flat}_{\Im_2-}\Upsilon(c) = \int_{c}^{\Im_2}(\psi-c)^{\beta^*-1}\mathcal{E}^{\gamma}_{\alpha^*,\beta^*}\left(\flat(\psi-c)^{\alpha^*}\right)\Upsilon(\psi)d\psi, \tag{6} $$

where $\mathcal{E}^{\gamma}_{\alpha^*,\beta^*}(z)$ symbolizes the three parameters Mittag-Leffler function.

Definition 5. *[29] For any function $\Upsilon \in \mathcal{L}^1$ on the interval $[\Im_1, \Im_2] \subset R$ and $c \in [\Im_1, \Im_2]$, then the infinite series formula for left and right Prabhakar integrals applied to $\Upsilon(c)$, stated as*

$$ {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_1+}\Upsilon(c) = \sum_{\eth=0}^{\infty}\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\,{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(c), \tag{7} $$

and

$$ {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_2-}\Upsilon(c) = \sum_{\eth=0}^{\infty}\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\,{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(c). \tag{8} $$

Definition 6. *[18] The Mittag-Leffler function with one parameter can be defined as*

$$\mathcal{E}_{\aleph}(z) = \sum_{\eth=0}^{\infty} \frac{z^{\eth}}{\Gamma(\aleph\eth + 1)}, \quad (z \in \mathbb{C}, \Re(\aleph) > 0).$$

The first generalization of Mittag-Leffler function for two parameters, is given as

$$\mathcal{E}_{\aleph,\wp}(z) = \sum_{\eth=0}^{\infty} \frac{z^{\eth}}{\Gamma(\aleph\eth + \wp)}, \quad (z, \aleph, \wp \in \mathbb{C}, \Re(\aleph) > 0).$$

Prabhakar defined the Mittag-Leffler function of three parameters [25] as

$$\mathcal{E}^{\delta}_{\aleph,\wp}(z) = \sum_{\eth=0}^{\infty} \frac{(\delta)_{\eth}}{\Gamma(\aleph\eth + \wp)} \frac{z^{\eth}}{\eth!}, \quad (z, \aleph, \wp, \delta \in \mathbb{C}, \Re(\aleph) > 0).$$

Definition 7. *[22] Let $\aleph, \wp, \tau, \delta, c \in C$, with $\Re(\aleph), \Re(\wp), \Re(\tau) > 0$ and $\Re(b) > \Re(\delta) > 0$ with $0 \leq g$, $1 > 0$ and $l + \Re(\aleph) \geq s > 0$. Then the extended generalized Mittag-Leffler function $\mathcal{E}^{\delta,b,s,l}_{\aleph,\wp,\tau}(z; g)$ is defined by*

$$\mathcal{E}^{\delta,b,s,l}_{\aleph,\wp,\tau}(z; g) = \sum_{\eth=0}^{\infty} \frac{\mathcal{B}_g(\delta + \eth s, b - \delta)}{\mathcal{B}(\delta, b - \delta)} \frac{(b)_{\eth s}}{\Gamma(\aleph\eth + \wp)} \frac{z^{\eth}}{(\tau)_{\eth l}}, \tag{9}$$

where $(b)_{\eth s} = \frac{\Gamma(b+\eth s)}{\Gamma(b)}$, is the generalized Pochhammer symbol and $\mathrm{B}_g(i, j) = \int_0^1 t^{i-1}$ $(1-t)^{j-1} e^{-\frac{g}{t(1-t)}} dt$ with $\Re(i), \Re(j), \Re(g) > 0$, is an extended beta function.

Definition 8. *[22] Let $\flat, \aleph, \wp, \tau, \delta, b \in C$ with $\Re(\aleph), \Re(\wp), \Re(\tau) > 0$ and $\Re(b) > \Re(\delta) > 0$ and let $g \geq 0$, $l > 0$ and $l + \Re(\aleph) \geq s > 0$. For a function $\Upsilon \in \mathcal{L}_1[\Im_1, \Im_2]$ and $c \in [\Im_1, \Im_2]$, the left and right sided generalized fractional integral operators $\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon$, $\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon$ given as*

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c; g) = \int_{\Im_1}^{c} (c - \psi)^{\wp-1} \mathcal{E}^{\delta,b,s,l}_{\aleph,\wp,\tau}\left(\flat(c - \psi)^{\aleph}; g\right) \Upsilon(\psi) d\psi, \tag{10}$$

and

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(c; g) = \int_{c}^{\Im_2} (\psi - c)^{\wp-1} \mathcal{E}^{\delta,b,s,l}_{\aleph,\wp,\tau}\left(\flat(\psi - c)^{\aleph}; g\right) \Upsilon(\psi) d\psi. \tag{11}$$

To derive the main results, we relied on theorems and lemmas provided in references [35], [36] and [37].

Theorem 1. *For an $\mathcal{L}^1$ continuous convex function $\Upsilon : [\Im_1, \Im_2] \to \Re$, with $\Im_2 > \Im_1$, the standard $(\mathcal{H} - \mathcal{H})$ inequality is stated as*

$$\Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \leq \frac{1}{\Im_2 - \Im_1} \int_{\Im_1}^{\Im_2} \Upsilon(c) dc \leq \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right).$$

Theorem 2. *For $\Upsilon : [\Im_1, \Im_2] \to \Re$ be a positive convex function with $0 \leq \Im_1 < \Im_2$ and $\Upsilon \in \mathcal{L}^1[\Im_1, \Im_2]$, then we have*

$$\begin{aligned}
&\Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \\
&\leq \frac{\Gamma(\alpha^* + 1)}{2(\Im_2 - \Im_1)^{\alpha^*}} \left({}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\Im_1+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\Im_2-}\Upsilon(\Im_1)\right) \\
&\leq \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Theorem 3. *If $\Upsilon : [\Im_1, \Im_2] \to \Re$ is $\mathcal{L}^1$ and convex, and $\alpha^* \in (0, 1)$, then we have the following inequality*

$$\begin{aligned}
&\Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \\
&\leq \frac{\mathcal{B}(\alpha^*)\Gamma(\alpha^*)}{2\left((\Im_2 - \Im_1)^{\alpha^*} + (1 - \alpha^*)\Gamma(\alpha^*)\right)} \times \left({}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\Im_1+}\Upsilon(\Im_2) + {}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\Im_2-}\Upsilon(\Im_1)\right) \\
&\leq \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Theorem 4. *For $\Upsilon : [\Im_1, \Im_2] \to \Re$ be a positive function with $0 \leq \Im_1 < \Im_2$ and $\Upsilon \in \mathcal{L}^1[\Im_1, \Im_2]$ be convex function, then we have*

$$\begin{aligned}
&\Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \\
&\leq \frac{2^{\alpha^*-1}\Gamma(\alpha^* + 1)}{(\Im_2 - \Im_1)^{\alpha^*}} \left({}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right) \\
&\leq \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Lemma 1. *For $\Upsilon : [\Im_1, \Im_2] \to \Re$ in L^1 and have a differentiable mapping on $(\Im_1, \Im_2)$ with $\Im_1 < \Im_2$ and if $\Upsilon' \in L^1[\Im_1, \Im_2]$ with $\alpha^* > 0$ then the following equality for fractional integrals holds*

$$\begin{aligned}
&\left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right) - \frac{\Gamma(\alpha^* + 1)}{2(\Im_2 - \Im_1)^{\alpha^*}} \left({}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\Im_1+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\Im_2-}\Upsilon(\Im_1)\right) \\
&= \frac{\Im_2 - \Im_1}{2} \int_0^1 \left((1 - t)^{\alpha^*} - t^{\alpha^*}\right) \Upsilon'(t\Im_1 + (1 - t)\Im_2)dt.
\end{aligned}$$

Lemma 2. *For $\Upsilon : [\Im_1, \Im_2] \to \Re$ in L^1 be a differentiable mapping on $(\Im_1, \Im_2)$ with $\Im_1 < \Im_2$ and if $\Upsilon' \in L^1[\Im_1, \Im_2]$ with $\alpha^* > 0$ then the following equality holds*

$$\begin{aligned}
&\frac{2^{\alpha^*-1}\Gamma(\alpha^* + 1)}{(\Im_2 - \Im_1)^{\alpha^*}} \left({}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(y)\right) - \Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \\
&= \frac{\Im_2 - \Im_1}{4} \int_0^1 t^{\alpha^*} \left(\Upsilon'\left(\frac{t}{2}\Im_1 + \frac{2 - t}{2}\Im_2\right) - \Upsilon'\left(\frac{2 - t}{2}\Im_1 + \frac{t}{2}\Im_2\right)\right) dt.
\end{aligned}$$

2. Generalized Fractional Integral Operators and Hermite-Hadamard Inequality in the Fractional Framework

In this section, we examine inequalities involving fractional integral of the type $(\Im_1+, \Im_2-)$ and $(\frac{\Im_1+\Im_2}{2})$. We also present some examples and their graphical representations to confirm our results.

2.1. Inequalities Involving Fractional Integral of the Type $(\Im_1+, \Im_2-)$

Proposition 1. *If* $\flat, \aleph, \wp, \tau, \delta, b \in C$ *with* $\Re(\aleph), \Re(\wp), \Re(\tau) > 0$ *and* $\Re(b) > \Re(\delta) > 0$ *and let* $g \geq 0$, $l > 0$ *and* $0 < s \leq l + \Re(\aleph)$. *For a function* $\Upsilon \in \mathcal{L}^1[\Im_1, \Im_2]$ *and* $c \in [\Im_1, \Im_2]$, *then the addition of left and right sided generalized fractional integral operators,* $\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon$, $\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon$ *are defined by*

$$\begin{aligned}&\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(\Im_2; g) + \varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(\Im_1; g)\\ &= \sum_{\eth=0}^{\infty} a_\eth \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right),\end{aligned}$$

where $a_\eth = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)}\frac{(b)_{\eth s}\flat^\eth}{(\tau)_{\eth l}}$.

Proof. Using left sided generalized fractional integral operator (10), we get

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c; g) = \int_{\Im_1}^{c} (c-\psi)^{\wp-1}\mathcal{E}^{\delta,b,s,l}_{\aleph,\wp,\tau}\left(\flat(c-\psi)^{\aleph}; g\right)\Upsilon(\psi)d\psi,$$

using extended generalized Mittag-Leffler function (9) in the above expression, we get

$$\begin{aligned}\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c; g) &= \int_{\Im_1}^{c} (c-\psi)^{\wp-1}\\ &\times \sum_{\eth=0}^{\infty}\left(\frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)}\frac{(b)_{\eth s}}{\Gamma(\aleph\eth+\wp)}\frac{\flat^\eth(y-\psi)^{\aleph\eth}}{(\tau)_{\eth l}}\right)\Upsilon(\psi)d\psi,\end{aligned}$$

after rearranging, the above equation can be expressed as

$$\begin{aligned}\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c; g) &= \sum_{\eth=0}^{\infty}\left(\frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)}\frac{(b)_{\eth s}\flat^\eth}{(\tau)_{\eth l}}\right)\\ &\times\left(\frac{1}{\Gamma(\aleph\eth+\wp)}\int_{\Im_1}^{c}(c-\psi)^{\wp-1+\aleph\eth}\Upsilon(\psi)d\psi\right),\end{aligned}$$

Using left sided Reimann-Liouville integral (1) in above equation, we acquire

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c; g) = \sum_{\eth=0}^{\infty}\left(\frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)}\frac{(b)_{\eth s}\flat^\eth}{(\tau)_{\eth l}}\right){}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(c). \tag{12}$$

Similarly for right sided generalized fractional integral operator (11), we have

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(\Im_1;g)=\sum_{\eth=0}^{\infty}\left(\frac{\mathcal{B}_g(\delta+\eth s,b-\delta)}{\mathcal{B}(\delta,b-\delta)}\frac{(b)_{\eth s}\flat^{\eth}}{(\tau)_{\eth l}}\right){}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(c). \tag{13}$$

Adding (12) and (13)

$$\begin{aligned}
&\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(\Im_2;g)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(\Im_1;g)\\
&=\sum_{\eth=0}^{\infty}\left(\frac{\mathcal{B}_g(\delta+\eth s,b-\delta)}{\mathcal{B}(\delta,b-\delta)}\frac{(b)_{\eth s}\flat^{\eth}}{(\tau)_{\eth l}}\right)\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)\\
&=\sum_{\eth=0}^{\infty}a_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right).
\end{aligned}$$

Here, the integral transform gives $(\aleph\eth+\wp)$th order left and right sided Reimann-Liouville fractional integrals of $\Upsilon(c)$, provided that $\Re(\aleph\eth+\wp)>0$.

Theorem 5. *Let* $\Upsilon:[\Im_1,\Im_2]\to\Re$ *be a convex function with* $\Upsilon\in\mathcal{L}^1[\Im_1,\Im_2]$ *and* $\flat,\aleph,\wp,\tau,\delta,b\in C$ *such that* $\Re(\aleph),\Re(\wp),\Re(\tau)>0$, $\Re(b)>\Re(\delta)>0$. *Let* $g\geq 0$, $l>0$ *and* $0<s\leq l+\Re(\aleph)$, *then for* $(\aleph\eth+\wp)>0$, *we can write*

$$\begin{aligned}
&\sum_{\eth=0}^{\infty}a_{\eth}v_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\
&\leq\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(\Im_2;g)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(\Im_1;g)\\
&\leq\sum_{\eth=0}^{\infty}a_{\eth}v_{\eth}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right),
\end{aligned}$$

where $a_{\eth}=\frac{\mathcal{B}_g(\delta+\eth s,b-\delta)}{\mathcal{B}(\delta,b-\delta)}\frac{(b)_{\eth s}\flat^{\eth}}{(\tau)_{\eth l}}$ *and* $v_{\eth}=\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma(\aleph\eth+\wp+1)}$.

Proof. Replacing α^* by $(\aleph\eth+\wp)$ in Theorem 2, we get

$$\begin{aligned}
&\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\
&\leq\frac{\Gamma(\aleph\eth+\wp+1)}{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)\\
&\leq\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Multiplying the above inequality with $\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma(\aleph,weobtain\eth+\wp+1)}$, we obtain

$$\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$

$$\leq \left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq \frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Again the above inequality is multiplied with $a_\eth$ to obtain

$$a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma(\aleph\eth+\wp+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty}a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma(\aleph\eth+\wp+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \sum_{\eth=0}^{\infty}a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq \sum_{\eth=0}^{\infty}a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Using Proposition 1, we get

$$\sum_{\eth=0}^{\infty}a_\eth v_\eth\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \varepsilon_{\Im_1+,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(\Im_2;g) + \varepsilon_{\Im_2-,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(\Im_1;g)$$
$$\leq \sum_{\eth=0}^{\infty}a_\eth v_\eth\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Hence the result is proved.

Proposition 2. *If* $\Upsilon : [\Im_1, \Im_2] \to \Re$ *is* $\mathcal{L}^1$ *(convex) and* $(\aleph\eth+\wp) \in (0,1)$*, we have* $(\mathcal{H}-\mathcal{H})$ *inequality for Atangana-Baleanu fractional integrals, where,* $\flat, \aleph, \wp, \tau, \delta, b \in C$ *with* $\Re(\aleph), \Re(\wp), \Re(\tau) > 0$ *and* $\Re(b) > \Re(\delta) > 0$ *and let* $g \geq 0$*,* $l > 0$ *and* $0 < s \leq l + \Re(\aleph)$

$$\sum_{\eth=0}^{\infty}a_\eth u_\eth\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right) \leq \varepsilon_{\Im_1+,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(\Im_2;g) + \varepsilon_{\Im_2-,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(\Im_1;g)$$
$$\leq \sum_{\eth=0}^{\infty}a_\eth u_\eth\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right),$$

where $a_{\eth} = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^{\eth}}{(\tau)_{\eth l}}$ *and*
$u_{\eth} = \frac{2\left((\Im_2-\Im_1)^{\aleph\eth+\wp}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma(\aleph\eth+\wp+1)} - \frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}$.

Proof. Replacing α^* by $(\aleph\eth+\wp)$ in Theorem 3, we obtain

$$\begin{aligned}
&\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\
&\leq \frac{\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}\\
&\times\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)\\
&\leq\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Multiplying the above inequality with $\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp), we have \Gamma(\aleph\eth+\wp)\right)}{\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}$, we get

$$\begin{aligned}
&\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\
&\leq\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right) \qquad (14)\\
&\leq\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Adding left and right sided Atangana-Baleanu integrals (3) and (4), we acquire

$$\begin{aligned}
&{}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\Im_2-}\Upsilon(\Im_1)\\
&=\frac{\alpha^*}{\mathcal{B}(\Im_1)}\left({}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\Im_2-}\Upsilon(\Im_1)\right)+\frac{1-\alpha^*}{\mathcal{B}(\Im_1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Using $\alpha^*=(\aleph\eth+\wp)$ in the above expression and then put the results in (14), we get

$$\begin{aligned}
&\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\
&\leq\frac{(\aleph\eth+\wp)}{\mathcal{B}(\aleph\eth+\wp)}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)\\
&+\frac{(1-\aleph\eth-\wp)}{\mathcal{B}(\aleph\eth+\wp)}\left(\Upsilon(\Im_1)+\Upsilon(\Im_2)\right)\\
&\leq\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Subtracting $\frac{1-\aleph\eth-\wp}{\mathcal{B}(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2))$ from the above inequality and then multiplying with $\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}$, we obtain

$$\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$

$$-\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}\left(\Upsilon(\Im_1)+\Upsilon(\Im_2)\right)\leq\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$$
$$-\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}\left(\Upsilon(\Im_1)+\Upsilon(\Im_2)\right).$$

Again we multiply the above inequality with $a_\eth$ to obtain

$$a_\eth\left(\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\right)$$
$$-a_\eth\left(\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2))\right)$$
$$\leq a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq a_\eth\left(\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)\right)$$
$$-a_\eth\left(\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}\left(\Upsilon(\Im_1)+\Upsilon(\Im_2)\right)\right).$$

By convexity of Υ we have, $\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\leq\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$

$$a_\eth\left(\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma(\aleph\eth\wp+1)}-\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\right)$$
$$\times\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\leq a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq a_\eth\left(\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma((\aleph\eth+\wp)+1)}-\frac{2(1-\aleph\eth-\wp)}{\aleph\eth\wp}\right)$$
$$\times\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty}a_\eth\left(\frac{2\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma((\aleph\eth+\wp)+1)}-\frac{2(1-\aleph\eth-\wp)}{\aleph\eth\wp}\right)$$
$$\times\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\leq\sum_{\eth=0}^{\infty}a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq\sum_{\eth=0}^{\infty}a_\eth\left(\frac{2\left((\Im_2-\Im_1)^{\aleph\eth+\wp}+(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{\Gamma((\aleph\eth+\wp)+1)}-\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\right)$$

$$\times\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Using Proposition 1, we get

$$\begin{aligned}&\sum_{\eth=0}^{\infty} a_{\eth}u_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\&\leq \varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(\Im_2;g)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(\Im_1;g)\\&\leq \sum_{\eth=0}^{\infty} a_{\eth}u_{\eth}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).\end{aligned}$$

It is our required result.

Proposition 3. *For a function* $\Upsilon\in\mathcal{L}^1[\Im_1,\Im_2]$ *and* $c\in[\Im_1,\Im_2]$*, the addition of the left and right sided Prabhakar and the left and right sided generalized fractional integral operators applied to* $\Upsilon(c)$ *are defined by the following integral transforms, where* $\Re(\aleph\eth+\wp)>0$ *also* $\flat,\gamma,\aleph,\wp,\tau,\delta,b\in C$ *with* $\Re(\flat),\Re(\gamma),\Re(\aleph),\Re(\wp),\Re(\tau)>0$ *and* $\Re(b)>\Re(\delta)>0$ *and let* $g\geq 0$*,* $l>0$ *and* $0<s\leq l+\Re(\aleph)$

$$\begin{aligned}&{}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_1+}\Upsilon(c)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c;g)+{}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_2-}\Upsilon(c)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(c;g)\\&=\sum_{\eth=0}^{\infty}h_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(c)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(c)\right),\end{aligned}$$

where $h_{\eth}=\frac{\mathcal{B}_g(\delta+\eth s,b-\delta)}{\mathcal{B}(\delta,b-\delta)}\frac{(b)_{\eth s}\flat^{\eth}}{(\tau)_{\eth l}}+\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}$.

Proof. Adding infinite series formula for left Prabhakar integral (7) and left sided generalized fractional integral operator (10), we get

$$\begin{aligned}&{}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_1+}\Upsilon(c)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c;g)\\&=\sum_{\eth=0}^{\infty}\left(\frac{\mathcal{B}_g(\delta+\eth s,b-\delta)}{\mathcal{B}(\delta,b-\delta)}\frac{(b)_{\eth s}\flat^{\eth}}{(\tau)_{\eth l}}+\sum_{\eth=0}^{\infty}\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\right){}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(c).\end{aligned}\tag{15}$$

Similarly adding infinite series formula for right Prabhakar integral (8) and right sided generalized fractional integral operator (11), we have

$$\begin{aligned}&{}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_2-}\Upsilon(c)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(c;g)\\&=\sum_{\eth=0}^{\infty}\left(\frac{\mathcal{B}_g(\delta+\eth s,b-\delta)}{\mathcal{B}(\delta,b-\delta)}\frac{(b)_{\eth s}\flat^{\eth}}{(\tau)_{\eth l}}+\sum_{\eth=0}^{\infty}\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\right){}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(c).\end{aligned}\tag{16}$$

Finally, we add (15) and (16) to get following required result

$${}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_1+}\Upsilon(c)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c;g)+{}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_2-}\Upsilon(c)+\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(c;g)$$

$$= \left(\sum_{\eth=0}^{\infty} \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^{\eth}}{(\tau)_{\eth l}} + \sum_{\eth=0}^{\infty} \frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\right)$$
$$\times \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(c)\right)$$
$$= \sum_{\eth=0}^{\infty} h_{\eth} \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(c)\right).$$

Theorem 6. *If $\Upsilon : [\Im_1, \Im_2] \to \Re$ is $\mathcal{L}^1$ and convex and the parameters, $\Re(\aleph\eth+\wp) > 0$ also $\Re(\flat), \Re(\gamma), \Re(\aleph), \Re(\wp), \Re(\tau) > 0$ and $\Re(b) > \Re(\delta) > 0$ and let $g \geq 0$, $l > 0$ and $0 < s \leq l + \Re(\aleph)$, then we have the following $(\mathcal{H}-\mathcal{H})$ inequality for Prabhakar fractional integrals and generalized fractional integral operators*

$$\sum_{\eth=0}^{\infty} h_{\eth} v_{\eth} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_1+}\Upsilon(c) + \varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(c;g) + {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\Im_2-}\Upsilon(c) + \varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(c;g) \tag{17}$$
$$\leq \sum_{\eth=0}^{\infty} h_{\eth} v_{\eth} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right),$$

where $h_{\eth} = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^{\eth}}{(\tau)_{\eth l}} + \frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}$ *and* $v_{\eth} = \frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$.

Proof. Replacing α^* by $(\aleph\eth+\wp)$ in Theorem 2, we obtain

$$\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \frac{\Gamma((\aleph\eth+\wp)+1)}{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}} \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)$$
$$\leq \frac{\Upsilon(\Im_1)+\Upsilon(\Im_1)}{2}.$$

Multiplying the above inequality with $\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$, we get

$$\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)$$
$$\leq \frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Again we multiply the above inequality with $h_{\eth}$ to obtain

$$h_{\eth}\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$

$$\leq h_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq h_{\eth}\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty}h_{\eth}\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq\sum_{\eth=0}^{\infty}h_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$\leq\sum_{\eth=0}^{\infty}h_{\eth}\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Using Proposition 3 in the above expression, we get

$$\sum_{\eth=0}^{\infty}h_{\eth}v_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq{}^{\mathcal{P}}I_{\Im_1+}^{\aleph,\wp,\gamma,\flat}\Upsilon(c)+\varepsilon_{\Im_1+,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(c;g)+{}^{\mathcal{P}}I_{\Im_2-}^{\aleph,\wp,\gamma,\flat}\Upsilon(c)+\varepsilon_{\Im_2-,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(c;g)$$
$$\leq\sum_{\eth=0}^{\infty}h_{\eth}v_{\eth}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Hence the required result is proved.

Example 1. *We verify the result of Theorem 6 for convex function* $\Upsilon(c)=c^2$ *on the interval* $[0,1]$.
Using substitution $t=\frac{\psi}{c}$ *in left and right sided Reimann-Liouville integrals (1) and (2), we get*

$$^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}{\Im_2}^2=\frac{\Gamma(3)}{\Gamma((\aleph\eth+\wp)+3)},\tag{18}$$

$$^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}{\Im_1}^2=0.\tag{19}$$

Using (18) in infinite series formula for left Prabhakar integrals (7) and (19) in infinite series formula for right Prabhakar integrals (8), we have

$$^{\mathcal{P}}I_{\Im_1+}^{\aleph,\wp,\gamma,\flat}{\Im_2}^2=\sum_{\eth=0}^{\infty}\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\times\frac{\Gamma(3)}{\Gamma((\aleph\eth+\wp)+3)},\tag{20}$$

$$^{\mathcal{P}}I_{\Im_2-}^{\aleph,\wp,\gamma,\flat}{\Im_1}^2=\sum_{\eth=0}^{\infty}\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\times(0).\tag{21}$$

Also substitute (18) in left sided generalized fractional integral operator (10) and (19) in right sided generalized fractional integral operator (11), we acquire

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon\left(\Im_2{}^2;g\right)=\sum_{\eth=0}^{\infty}a_{\eth}\times\frac{\Gamma(3)}{\Gamma((\aleph\eth+\wp)+3)},\tag{22}$$

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon\left(\Im_1{}^2;g\right)=\sum_{\eth=0}^{\infty}a_{\eth}\times(0).\tag{23}$$

Substituting these expressions (20), (21), (22) and (23) in the inequality (17) and after some simplification, we get

$$\begin{aligned}&\sum_{\eth=0}^{\infty}h_{\eth}\frac{\Gamma((\aleph\eth+\wp)+1)}{2}\\&\leq\sum_{\eth=0}^{\infty}\left(\frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}\times\frac{\Gamma(3)}{\Gamma((\aleph\eth+\wp)+3)}\right)+\sum_{\eth=0}^{\infty}\left(a_{\eth}\times\frac{\Gamma(3)}{\Gamma((\aleph\eth+\wp)+3)}\right)\\&\leq\sum_{\eth=0}^{\infty}\frac{h_{\eth}}{\Gamma((\aleph\eth+\wp)+1)}.\end{aligned}$$

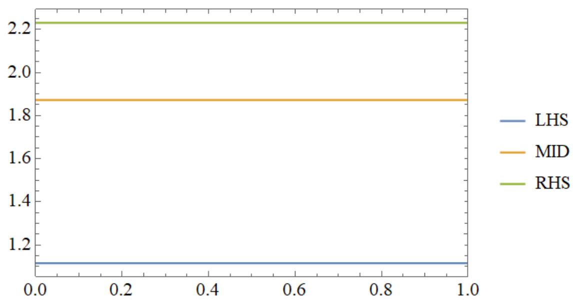

Figure 1: The 2D graph exhibiting the inequality (17) for $\eth = 1$.

2.2. Inequalities Involving Fractional Integral of the Type $\left(\frac{\Im_1+\Im_2}{2}\right)$

Theorem 7. *If* $\Upsilon:[\Im_1,\Im_2]\to\Re$ *is* $\mathcal{L}^1$ *and convex and the parameters,* $\Re(\aleph\eth+\wp)>0$ *also* $\Re(\flat),\Re(\gamma),\Re(\aleph),\Re(\wp),\Re(\tau)>0$ *and* $\Re(b)>\Re(\delta)>0$ *and let* $g\geq 0$*,* $l>0$ *and*

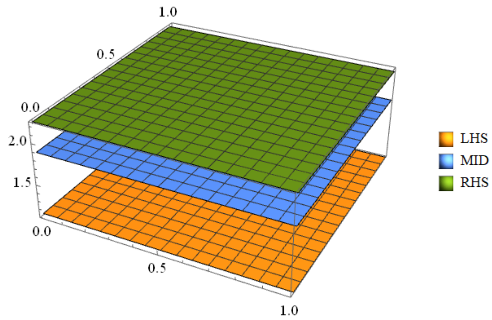

Figure 2: The 3D graph exhibiting the inequality (17) for convex function $\Upsilon(c) = c^2$ on the interval $[0, 1]$ and for $\eth = 1$.

$0 < s \leq l + \Re(\aleph)$, *then we have a distinct fractional development of the* $(\mathcal{H} - \mathcal{H})$ *inequality*

$$\begin{aligned}
&\sum_{\eth=0}^{\infty} a_{\eth} o_{\eth} \Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \\
&\leq \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau} \Upsilon(\Im_2; g) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau} \Upsilon(\Im_1; g) \\
&\leq \sum_{\eth=0}^{\infty} a_{\eth} o_{\eth} \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right),
\end{aligned}$$

where $a_{\eth} = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^{\eth}}{(\tau)_{\eth l}}$ *and* $o_{\eth} = \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$.

Proof. Replacing α^* by $(\aleph\eth + \wp)$ in Theorem 4, we get

$$\begin{aligned}
&\Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \\
&\leq \frac{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth + \wp) + 1)}{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}} \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+} \Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-} \Upsilon(\Im_1)\right) \\
&\leq \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Multiply the above expression with $\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$ to get

$$\begin{aligned}
&\frac{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth + \wp) + 1)} \Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right) \\
&\leq \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+} \Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-} \Upsilon(\Im_1)\right) \\
&\leq \frac{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth + \wp) + 1)} \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right).
\end{aligned}$$

Again the above inequality is multiplied with $a_{\eth}$ and then summing over all $\eth$

$$\sum_{\eth=0}^{\infty} a_{\eth} \frac{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \sum_{\eth=0}^{\infty} a_{\eth} \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+} \Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-} \Upsilon(\Im_1) \right)$$
$$\leq \sum_{\eth=0}^{\infty} a_{\eth} \frac{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Using Proposition 1 from the middle of interval $[\Im_1, \Im_2]$

$$\sum_{\eth=0}^{\infty} a_{\eth} o_{\eth} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau} \Upsilon(\Im_2; g) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau} \Upsilon(\Im_1; g)$$
$$\leq \sum_{\eth=0}^{\infty} a_{\eth} o_{\eth} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

This proves the desired result.

Proposition 4. *If $\Upsilon : [\Im_1, \Im_2] \to \Re$ is $\mathcal{L}^1$ and convex and the parameters, $\Re(\aleph\eth+\wp) > 0$ also $\Re(\flat), \Re(\gamma), \Re(\aleph), \Re(\wp), \Re(\tau) > 0$ and $\Re(b) > \Re(\delta) > 0$ and let $g \geq 0$, $l > 0$ and $0 < s \leq l + \Re(\aleph)$, then the $(\mathcal{H} - \mathcal{H})$ inequality for Atangana-Baleanu fractional integrals becomes*

$$\sum_{\eth=0}^{\infty} a_{\eth} s_{\eth} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau} \Upsilon(\Im_2; g) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau} \Upsilon(\Im_1; g)$$
$$\leq \sum_{\eth=0}^{\infty} a_{\eth} s_{\eth} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right),$$

where $a_{\eth} = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^{\eth}}{(\tau)_{\eth l}}$ *and* $s_{\eth} = \frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)} + 2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} - \frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}$.

Proof. Replacing α^* by $(\aleph\eth+\wp)$ in Theorem 3, we have

$$\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \frac{2^{(\aleph\eth+\wp)-1}\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)} + 2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}$$

$$\times\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$\leq\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Multiplying the above inequality with $\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}$, we get

$$\begin{aligned}&\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\&\leq\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)\\&\leq\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\times\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).\end{aligned}\tag{24}$$

Adding left and right sided Atangana-Baleanu integrals (3) and (4) from the middle of interval $[\Im_1,\Im_2]$, we have

$$\begin{aligned}&\left({}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)\\&=\frac{\alpha^*}{\mathcal{B}(\alpha^*)}\left({}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)\\&+\frac{1-\alpha^*}{\mathcal{B}(\alpha^*)}(\Upsilon(\Im_1)+\Upsilon(\Im_2)).\end{aligned}$$

Using $\alpha^*=(\aleph\eth+\wp)$ in the above equation and then put the results in (24), we get

$$\begin{aligned}&\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\&\leq\frac{(\aleph\eth+\wp)}{\mathcal{B}(\aleph\eth+\wp)}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)\\&+\frac{1-\aleph\eth-\wp}{\mathcal{B}(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2))\\&\leq\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\mathcal{B}(\aleph\eth+\wp)\Gamma(\aleph\eth+\wp)}\frac{\Upsilon(\Im_1)+(\Im_2)}{2}.\end{aligned}$$

Subtracting $\frac{1-\aleph\eth-\wp}{\mathcal{B}(\aleph\eth+\wp)}[\Upsilon(\Im_1)+\Upsilon(\Im_2)]$ from the above expression and then multiplying with $\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}$, we obtain

$$\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$

$$-\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2)) \leq \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{\aleph\eth+\wp}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}(\Im_1)\right)$$
$$\leq \frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{\hat{\aleph}\eth+\wp-1}\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$$
$$-\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2)).$$

Again multiplying the above inequality with $a_\eth$

$$a_\eth\left(\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\right)$$
$$-a_\eth\left(\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)\right)$$
$$\leq a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$\leq a_\eth\left(\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\Gamma(\aleph\eth+\wp+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)\right)$$
$$-\left(\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)\right).$$

By convexity of Υ we have, $\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\leq\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$

$$a_\eth\left(\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{\aleph\eth+\wp-1}\Gamma((\aleph\eth+\wp)+1)}-\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\right)$$
$$\times\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\leq a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}+{}^{\mathcal{R}-\mathcal{L}}I^{\aleph\eth+\wp}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$\leq a_\eth\left(\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}-\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\right)$$
$$\times\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty}a_\eth\left(\frac{\left((\Im_2-\Im_1)^{(\aleph\eth+\wp)}+2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}-\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\right)$$
$$\times\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\leq\sum_{\eth=0}^{\infty}a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$

$$\leq \sum_{\eth=0}^{\infty} a_{\eth} \left(\frac{\left((\Im_2 - \Im_1)^{(\aleph\eth+\wp)} + 2^{(\aleph\eth+\wp)}(1-\aleph\eth-\wp)\Gamma(\aleph\eth+\wp)\right)}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} - \frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)} \right)$$
$$\times \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2} \right),$$

Using Proposition 1 from the middle of interval $[\Im_1, \Im_2]$, we acquire

$$\sum_{\eth=0}^{\infty} a_{\eth} s_{\eth} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right) \leq \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau} \Upsilon(\Im_2; g) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau} \Upsilon(\Im_1; g)$$
$$\leq \sum_{\eth=0}^{\infty} a_{\eth} s_{\eth} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Hence the result is established.

Theorem 8. *If $\Upsilon : [\Im_1, \Im_2] \to \Re$ is $\mathcal{L}^1$ convex and the parameters, $\Re(\aleph\eth+\wp) > 0$ also $\Re(\flat), \Re(\gamma), \Re(\aleph), \Re(\wp), \Re(\tau) > 0$ and $\Re(b) > \Re(\delta) > 0$ and let $g \geq 0$, $l > 0$ and $0 < s \leq l + \Re(\aleph)$, then we have the following $(\mathcal{H}-\mathcal{H})$ inequality for Prabhakar fractional integrals and generalized fractional integral operators*

$$\begin{aligned} &\sum_{\eth=0}^{\infty} h_{\eth} o_{\eth} f\left(\frac{\Im_1+\Im_2}{2}\right) \\ &\leq {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\left(\frac{\Im_1+\Im_2}{2}\right)+} \Upsilon(c) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau} \Upsilon(c; g) \\ &+ {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\left(\frac{\Im_1+\Im_2}{2}\right)-} \Upsilon(c) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau} \Upsilon(c; g) \leq \sum_{\eth=0}^{\infty} h_{\eth} o_{\eth} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right), \end{aligned} \tag{25}$$

where $h_{\eth} = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^{\eth}}{(\tau)_{\eth l}} + \frac{\Gamma(\gamma+\eth)\flat^{\eth}}{\Gamma(\gamma)\eth!}$ *and* $o_{\eth} = \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$.

Proof. Replacing α^* by $(\aleph\eth+\wp)$ in Theorem 4, we obtain

$$\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right) \leq \frac{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}$$
$$\times \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+} \Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-} \Upsilon(c) \right)$$
$$\leq \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).$$

Multiplying the above inequality with $\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$, we obtain

$$\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$

$$\leq \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(c) \right)$$
$$\leq \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2} \right).$$

Again we multiply the above expression with $h_\eth$, then it becomes

$$h_\eth \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq h_\eth \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{\aleph i+\wp}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(c) \right)$$
$$\leq h_\eth \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2} \right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty} h_\eth \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \sum_{\eth=0}^{\infty} h_\eth \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(c) \right) \quad (26)$$
$$\leq \sum_{\eth=0}^{\infty} h_\eth \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2} \right).$$

From the middle of interval $[\Im_1, \Im_2]$, Proposition 3 becomes

$$ {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(c) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau}\Upsilon(c;g) + {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(c)$$
$$+ \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau}\Upsilon(c;g) \quad (27)$$
$$= \sum_{\eth=0}^{\infty} h_\eth \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(c) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(c) \right).$$

Using (27) in (26), we obtain

$$\sum_{\eth=0}^{\infty} h_\eth o_\eth \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right) \leq {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(c) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau}\Upsilon(c;g)$$
$$+ {}^{\mathcal{P}}I^{\aleph,\wp,\gamma,\flat}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(c) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau}\Upsilon(c;g) \leq \sum_{\eth=0}^{\infty} h_\eth o_\eth \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2} \right).$$

That is our required result.

Example 2. *We verify the result of Theorem 8 for convex function $\Upsilon(c) = c^{4n}$, $n \in N$ on the interval $[-1, 1]$.*
Using substitution $t = \frac{\psi}{c}$ in left and right sided Reimann-Liouville integrals (1) and (2), we get

$$^{\mathcal{R}-\mathcal{L}}I_{0+}^{(\aleph\eth+\wp)}(1)^{4n} = \frac{(4n)!}{\Gamma(\aleph\eth + \wp + 4n + 1)}, \tag{28}$$

$$^{\mathcal{R}-\mathcal{L}}I_{0+}^{(\aleph\eth+\wp)}(-1)^{4n} = \frac{(4n)!}{\Gamma(\aleph\eth + \wp + 4n + 1)}. \tag{29}$$

Use (28) in infinite series formula for left Prabhakar integral (7) and (29) in infinite series formula for right Prabhakar integral (8), we have

$$^{\mathcal{P}}I_{0+}^{\aleph,\wp,\gamma,\flat}(1)^{4n} = \sum_{\eth=0}^{\infty} \frac{\Gamma(\gamma + \eth)\flat^{\eth}}{\Gamma(\gamma)\eth!} \frac{(4n)!}{\Gamma(\aleph\eth + \wp + 4n + 1)}, \tag{30}$$

$$^{\mathcal{P}}I_{0-}^{\aleph,\wp,\gamma,\flat}(-1)^{4n} = \sum_{\eth=0}^{\infty} \frac{\Gamma(\gamma + \eth)\flat^{\eth}}{\Gamma(\gamma)\eth!} \frac{(4n!}{\Gamma(\aleph\eth + \wp + 4n + 1)}. \tag{31}$$

Also substitute (28) in left sided generalized fractional integral operator (10) and (29) in right sided generalized fractional integral operator (11), we acquire

$$\varepsilon_{0^+,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon\left((1)^{4n}; g\right) = \sum_{\eth=0}^{\infty} a_{\eth} \frac{(4n)!}{\Gamma(\aleph\eth + \wp + 4n + 1)}, \tag{32}$$

$$\varepsilon_{0^-,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon\left((-1)^{4n}; g\right) = \sum_{\eth=0}^{\infty} a_{\eth} \frac{(4n)!}{\Gamma(\aleph\eth + \wp + 4n + 1)}. \tag{33}$$

Substituting these expressions (30), (31), (32) and (33) in the inequality (25) and after some simplification, we get

$$0^{4\eth} \leq \sum_{\eth=0}^{\infty} \left(\frac{\Gamma(\gamma + \eth)\flat^{\eth}}{\Gamma(\gamma)\eth!} + a_{\eth} \right) \frac{(4n)!}{\Gamma(\aleph\eth + \wp + 4n + 1)} \leq \sum_{\eth=0}^{\infty} \frac{h_{\eth}}{\Gamma(\aleph\eth + \wp + 1)}.$$

3. Applications of Key Results in Terms of Means

$(\mathcal{H} - \mathcal{H})$ inequality are often connected to additional integral inequalities, such as trapezoid-type (utilizing the interval's endpoints $\Im_1$ and $\Im_2$) and midpoint-type (utilizing the midpoint $(\frac{\Im_1+\Im_2}{2})$ of the interval). Many researchers have contributed to establishing these inequalities [15, 38].
In this section, we employed an equality of trapezoid type and an inequality of midpoint type for the $(\mathcal{H} - \mathcal{H})$ integrals.

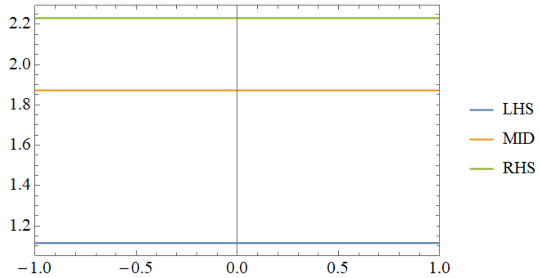

Figure 3: The 2D graph exhibiting the inequality (25) for $\eth = 1$.

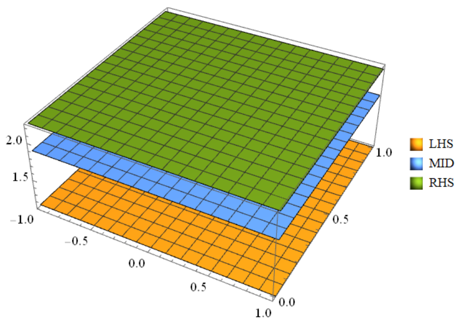

Figure 4: The 3D graph exhibiting the inequality (25) for convex function $\Upsilon(c) = (c)^{4\eth}$ on the interval $[-1, 1]$ and for $\eth = 1$.

3.1. An Equality of Trapezoid Type for the Hermite-Hadamard Integrals

Lemma 3. *Let* $\Upsilon : [\Im_1, \Im_2] \to \Re$ *is an* L^1 *function and* $(\aleph\eth + \wp) \in (0, 1)$ *also a differentiable function on* $(\Im_1, \Im_2)$ *with* $\Im_1 < \Im_2$*, and assume* $\Upsilon' \in L^1[\Im_1, \Im_2]$*. Consider* $\flat, \aleph, \wp, \tau, \delta, b \in C$ *with* $\Re(\aleph), \Re(\wp), \Re(\tau) > 0$ *and* $\Re(b) > \Re(\delta) > 0$*. Moreover, let* $g \geq 0$*,* $l > 0$ *and* $0 < s \leq l + \Re(\aleph)$ *with* $(\aleph\eth + \wp) > 0$*, then we have*

$$\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon(\Im_2; g) + \varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon(\Im_1; g)$$
$$= \sum_{\eth=0}^{\infty} a_\eth v_\eth \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right) - \sum_{\eth=0}^{\infty} a_\eth v_\eth$$
$$\times \left(\frac{\Im_2 - \Im_1}{2} \int_0^1 \left((1-t)^{(\aleph\eth+\wp)} - t^{(\aleph\eth+\wp)}\right) \Upsilon'(t\Im_1 + (1-t)\Im_2)dt\right),$$

where $a_\eth = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^\eth}{(\tau)_{\eth l}}$ *and* $v_\eth = \frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$.

Proof. Replacing α^* by $(\aleph\eth + \wp)$ in Lemma 1, we get

$$\left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2}\right) - \frac{\Gamma((\aleph\eth + \wp) + 1)}{2(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}} \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)$$
$$= \frac{\Im_2 - \Im_1}{2} \int_0^1 \left((1-t)^{(\aleph\eth+\wp)} - t^{(\aleph\eth+\wp)}\right) \Upsilon'(t\Im_1 + (1-t)\Im_2)dt.$$

Multiplying the above equation with $\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$, we get

$$\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)-\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$=\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$$
$$\times\left(\frac{\Im_2-\Im_1}{2}\int_0^1\left((1-t)^{(\aleph\eth+\wp)}-t^{\aleph\eth+\wp}\right)\Upsilon'(t\Im_1+(1-t)\Im_2)dt\right).$$

Again the above expression is multiplied with $a_\eth$ to get

$$a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$$
$$-a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)$$
$$=a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma(\aleph\eth+\wp+1)}$$
$$\times\left(\frac{\Im_2-\Im_1}{2}\int_0^1\left((1-t)^{(\aleph\eth+\wp)}-t^{(\aleph\eth+\wp)}\right)\Upsilon'(t\Im_1+(1-t)\Im_2)dt\right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty}a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$$
$$-\sum_{\eth=0}^{\infty}a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)}\Upsilon(\Im_1)\right)=\sum_{\eth=0}^{\infty}a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$$
$$\times\left(\frac{\Im_2-\Im_1}{2}\int_0^1\left((1-t)^{(\aleph\eth+\wp)}-t^{(\aleph\eth+\wp)}\right)\Upsilon'(t\Im_1+(1-t)\Im_2)dt\right).$$

Using Proposition 1

$$\sum_{\eth=0}^{\infty}a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$$
$$-\left(\varepsilon_{\Im_1+,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(\Im_2;g)+\left(\varepsilon_{\Im_2-,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}f\right)(\Im_1;g)\right)=\sum_{\eth=0}^{\infty}a_\eth\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$$
$$\times\left(\frac{\Im_2-\Im_1}{2}\int_0^1\left((1-t)^{(\aleph\eth+\wp)}-t^{(\aleph\eth+\wp)}\right)\Upsilon'(t\Im_1+(1-t)\Im_2)dt\right).$$

After rearranging the above expression, we get

$$\varepsilon_{\Im_1+,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(\Im_2;g)+\varepsilon_{\Im_2-,\aleph,\wp,\tau}^{\flat,\delta,b,s,l}\Upsilon(\Im_1;g)$$

$$= \sum_{\eth=0}^{\infty} a_\eth v_\eth \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2} \right) - \sum_{\eth=0}^{\infty} a_\eth v_\eth$$
$$\times \left(\frac{\Im_2 - \Im_1}{2} \int_0^1 \left((1-t)^{(\aleph\eth+\wp)} - t^{(\aleph\eth+\wp)} \right) \Upsilon'(t\Im_1 + (1-t)\Im_2) dt \right).$$

Thus the proof is completed.

Theorem 9. *Let $\Upsilon : [\Im_1, \Im_2] \to \Re$ is an L^1 function and $(\aleph\eth + \wp) \in (0,1)$ also a differentiable function on $(\Im_1, \Im_2)$ with $\Im_1 < \Im_2$ and assume $\Upsilon' \in L^1[\Im_1, \Im_2]$. Consider $\flat, \aleph, \wp, \tau, \delta, b \in C$ with $\Re(\aleph), \Re(\wp), \Re(\tau) > 0$ and $\Re(b) > \Re(\delta) > 0$. Moreover, let $g \geq 0$, $l > 0$ and $0 < s \leq l + \Re(\aleph)$ with $(\aleph\eth + \wp) > 0$*

$$\sum_{\eth=0}^{\infty} a_\eth \frac{\mathcal{B}(\aleph\eth + \wp)}{(\aleph\eth + \wp)} \left({}^{\mathcal{A}-\mathcal{B}}I_{\Im_1+}^{(\aleph\eth+\wp)} \Upsilon(\Im_2) + {}^{\mathcal{A}-\mathcal{B}}I_{\Im_2-}^{(\aleph\eth+\wp)} \Upsilon(\Im_1) \right)$$
$$- \sum_{\eth=0}^{\infty} a_\eth \frac{1 - (\aleph\eth + \wp)}{(\aleph\eth + \wp)} (\Upsilon(\Im_1) + \Upsilon(\Im_2))$$
$$= \sum_{\eth=0}^{\infty} a_\eth v_\eth \left(\frac{\Upsilon(\Im_1) + \Upsilon(\Im_2)}{2} \right) - \sum_{\eth=0}^{\infty} a_\eth v_\eth$$
$$\times \left(\frac{\Im_2 - \Im_1}{2} \int_0^1 \left((1-t)^{(\aleph\eth+\wp)} - t^{(\aleph\eth+\wp)} \right) \Upsilon'(t\Im_1 + (1-t)\Im_2) dt \right),$$

where $a_\eth = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)} \frac{(b)_{\eth s} \flat^\eth}{(\tau)_{\eth l}}$ and $v_\eth = \frac{2(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$.

Proof. Adding left and right sided Atangana-Baleanu integrals (3) and (4), we have

$${}^{\mathcal{A}-\mathcal{B}}I_{\Im_1+}^{\alpha^*} \Upsilon(\Im_2) + {}^{\mathcal{A}-\mathcal{B}}I_{\Im_2-}^{\alpha^*} \Upsilon(\Im_1)$$
$$= \frac{\alpha^*}{\mathcal{B}(\alpha^*)} \left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{\alpha^*} \Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{\alpha^*} \Upsilon(\Im_1) \right) + \frac{1-\alpha^*}{\mathcal{B}(\alpha^*)} (\Upsilon(\Im_1) + \Upsilon(\Im_2).$$

Replacing α^* by $(\aleph\eth + \wp)$ and then rearrange the above equation, we obtain

$$\frac{\mathcal{B}(\aleph\eth + \wp)}{(\aleph\eth + \wp)} \left({}^{\mathcal{A}-\mathcal{B}}I_{\Im_1+}^{(\aleph\eth+\wp)} \Upsilon(\Im_2) + {}^{\mathcal{A}-\mathcal{B}}I_{\Im_2-}^{(\aleph\eth+\wp)} \Upsilon(\Im_1) \right)$$
$$- \frac{1 - \aleph\eth - \wp}{(\aleph\eth + \wp)} (\Upsilon(\Im_1) + \Upsilon(\Im_2))$$
$$= \left({}^{\mathcal{R}-\mathcal{L}}I_{\Im_1+}^{(\aleph\eth+\wp)} \Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I_{\Im_2-}^{(\aleph\eth+\wp)} \Upsilon(\Im_1) \right).$$

Multiplying the above expression with $a_\eth$, we obtain

$$a_\eth \frac{\mathcal{B}(\aleph\eth + \wp)}{(\aleph\eth + \wp)} \left({}^{\mathcal{A}-\mathcal{B}}I_{\Im_1+}^{(\aleph\eth+\wp)} \Upsilon(\Im_2) + {}^{\mathcal{A}-\mathcal{B}}I_{\Im_2-}^{(\aleph\eth+\wp)} \Upsilon(\Im_1) \right)$$

$$- a_{\eth}\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2))$$
$$= a_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty} a_{\eth}\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)$$
$$-\sum_{\eth=0}^{\infty} a_{\eth}\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2))$$
$$=\sum_{\eth=0}^{\infty} a_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right).$$

Using Proposition 1, we obtain

$$\sum_{\eth=0}^{\infty} a_{\eth}\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)$$
$$-\sum_{\eth=0}^{\infty} a_{\eth}\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2))$$
$$=\left(\left(\varepsilon^{\flat,\delta,b,s,l}_{\Im_1+,\aleph,\wp,\tau}\Upsilon\right)(\Im_2;g)+\left(\varepsilon^{\flat,\delta,b,s,l}_{\Im_2-,\aleph,\wp,\tau}\Upsilon\right)(\Im_1;g)\right).$$

Comparing with Lemma 3, we get

$$\sum_{\eth=0}^{\infty} a_{\eth}\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_1+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\Im_2-}\Upsilon(\Im_1)\right)$$
$$-\sum_{\eth=0}^{\infty} a_{\eth}\frac{1-\aleph\eth-\wp}{(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2))$$
$$=\sum_{\eth=0}^{\infty} a_{\eth}\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma(\aleph\eth+\wp+1)}\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$$
$$-\sum_{\eth=0}^{\infty} a_{\eth}\frac{2(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{\Gamma((\aleph\eth+\wp)+1)}$$
$$\times\left(\frac{\Im_2-\Im_1}{2}\int_0^1\left((1-t)^{(\aleph\eth+\wp)}-t^{(\aleph\eth+\wp)}\right)\Upsilon'(t\Im_1+(1-t)\Im_2)dt\right).$$

This completes desired result.

3.2. An Inequality of Midpoint Type for the Hermite-Hadamard Integrals

Lemma 4. *Let* $\Upsilon : [\Im_1, \Im_2] \to \Re$ *is an* L^1 *function and* $(\aleph\eth + \wp) \in (0,1)$ *also a differentiable function on* $(\Im_1, \Im_2)$ *with* $\Im_1 < \Im_2$ *and assume* $\Upsilon' \in L^1[\Im_1, \Im_2]$*. Consider* $\flat, \aleph, \wp, \tau, \delta, b \in C$ *with* $\Re(\aleph), \Re(\wp), \Re(\tau) > 0$ *and* $\Re(b) > \Re(\delta) > 0$*. Moreover, let* $g \geq 0$*,* $l > 0$ *and* $0 < s \leq l + \Re(\aleph)$*. Under these conditions, the following equality for fractional integrals holds*

$$\begin{aligned}
&\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau}\Upsilon(\Im_2; g) + \varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau}\Upsilon(\Im_1; g)\\
&= \sum_{\eth=0}^{\infty} a_\eth o_\eth\\
&\times \left(\frac{\Im_2 - \Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\left(\Upsilon'\left(\frac{t}{2}\Im_1 + \frac{2-t}{2}\Im_2\right) - \Upsilon'\left(\frac{2-t}{2}\Im_1 + \frac{t}{2}\Im_2\right)\right)dt\right)\\
&+ \sum_{\eth=0}^{\infty} a_\eth o_\eth \Upsilon\left(\frac{\Im_1 + \Im_2}{2}\right),
\end{aligned}$$

where $a_\eth = \frac{\mathcal{B}_g(\delta+\eth s, b-\delta)}{\mathcal{B}(\delta, b-\delta)}\frac{(b)_{\eth s}\flat^\eth}{(\tau)_{\eth l}}$ *and* $o_\eth = \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$ *with* $(\aleph\eth + \wp) > 0$*.*

Proof. Replacing α^* by $(\aleph\eth + \wp)$ in Lemma 2, we get

$$\begin{aligned}
&\frac{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)\\
&- \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right) = \frac{\Im_2 - \Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\\
&\times \left(\Upsilon'\left(\frac{t}{2}\Im_1 + \frac{2-t}{2}\Im_2\right) - \Upsilon'\left(\frac{2-t}{2}\Im_1 + \frac{t}{2}\Im_2\right)\right)dt.
\end{aligned}$$

Multiplying the above expression with $\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$, we get

$$\begin{aligned}
&\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right) = \frac{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}\\
&\times \left(\frac{\Im_2 - \Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\left(\Upsilon'\left(\frac{t}{2}\Im_1 + \frac{2-t}{2}\Im_2\right) - \Upsilon'\left(\frac{2-t}{2}\Im_1 + \frac{t}{2}\Im_2\right)\right)dt\right)\\
&+ \frac{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).
\end{aligned}$$

Again we multiply the above equation with $a_\eth$

$$a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right) = a_\eth\frac{(\Im_2 - \Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$$

$$\times\left(\frac{\Im_2-\Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\left(\Upsilon'\left(\frac{t}{2}\Im_1+\frac{2-t}{2}\Im_2\right)-\Upsilon'\left(\frac{2-t}{2}\Im_1+\frac{t}{2}\Im_2\right)\right)dt\right)$$
$$+a_\eth\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

Summing over all $\eth$

$$\sum_{\eth=0}^{\infty}a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$=\sum_{\eth=0}^{\infty}a_\eth\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$$
$$\times\left(\frac{\Im_2-\Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\left(\Upsilon'\left(\frac{t}{2}\Im_1+\frac{2-t}{2}\Im_2\right)-\Upsilon'\left(\frac{2-t}{2}\Im_1+\frac{t}{2}\Im_2\right)\right)dt\right)$$
$$+\sum_{\eth=0}^{\infty}a_\eth\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

Using Proposition 1 from the middle of interval $[\Im_1,\Im_2]$, we obtain

$$\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau}\Upsilon(\Im_2;g)+\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau}\Upsilon(\Im_1;g)$$
$$=\sum_{\eth=0}^{\infty}a_\eth o_\eth$$
$$\times\left(\frac{\Im_2-\Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\left(\Upsilon'\left(\frac{t}{2}\Im_1+\frac{2-t}{2}\Im_2\right)-f'\left(\frac{2-t}{2}\Im_1+\frac{t}{2}\Im_2\right)\right)dt\right)$$
$$+\sum_{\eth=0}^{\infty}a_\eth o_\eth\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

This is our required result.

Theorem 10. *Let $\Upsilon:[\Im_1,\Im_2]\to\Re$ is an L^1 function and $(\aleph\eth+\wp)\in(0,1)$ also a differentiable function on $(\Im_1,\Im_2)$ with $\Im_1<\Im_2$ and assume $\Upsilon'\in L^1[\Im_1,\Im_2]$. Consider $\flat,\aleph,\wp,\tau,\delta,b\in C$ with $\Re(\aleph),\Re(\wp),\Re(\tau)>0$ and $\Re(b)>\Re(\delta)>0$. Moreover, let $g\geq 0$, $l>0$ and $0<s\leq l+\Re(\aleph)$*

$$\left(\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau}\Upsilon\right)(\Im_2;g)+\left(\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau}\Upsilon\right)(\Im_1;g)$$
$$-\sum_{\eth=0}^{\infty}a_\eth\left(o_\eth-1\right)\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\geq\sum_{\eth=0}^{\infty}a_\eth o_\eth$$

$$\times\left(\frac{\Im_2-\Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\left\{\Upsilon'\left(\frac{t}{2}\Im_1+\frac{2-t}{2}\Im_2\right)-\Upsilon'\left(\frac{2-t}{2}\Im_1+\frac{t}{2}\Im_2\right)\right\}dt\right)$$
$$+\sum_{\eth=0}^{\infty}a_\eth\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

where $a_\eth=\frac{\mathcal{B}_g(\delta+\eth s,b-\delta)}{\mathcal{B}(\delta,b-\delta)}\frac{(b)_{\eth s}b^{\eth}}{(\tau)_{\eth l}}$ *and* $o_\eth=\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$ *with* $(\aleph\eth+\wp)>0.$

Proof. Adding left and right sided Atangana-Baleanu integrals (3) and (4) from the middle of interval $[\Im_1,\Im_2]$, we have

$$\left({}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$=\frac{\alpha^*}{\mathcal{B}(\alpha^*)}\left({}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{\alpha^*}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$+\frac{1-\alpha^*}{\mathcal{B}(\alpha^*)}(\Upsilon(\Im_1)+\Upsilon(\Im_2)).$$

Using $\alpha^*=(\aleph\eth+\wp)$ in the above expression, we get

$$\frac{\aleph\eth+\wp}{\mathcal{B}(\aleph\eth+\wp)}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$=\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$-\frac{1-\aleph\eth-\wp}{\mathcal{B}(\aleph\eth+\wp)}(\Upsilon(\Im_1)+\Upsilon(\Im_2)).$$

Multiplying the above equality with $\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}$ and then subtracting $\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$, we obtain

$$\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{\aleph\eth+\wp}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)-\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$=\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{\aleph\eth+\wp}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$
$$-\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\frac{(\Upsilon(\Im_1)+\Upsilon(\Im_2))}{2}-\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

Again we multiply the above equality with $a_\eth$ to get

$$a_\eth\left({}^{\mathcal{R}-\mathcal{L}}I^{\aleph\eth+\wp}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)-a_\eth\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$=a_\eth\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{\aleph\eth+\wp}_{\left(\frac{\Im_1+\Im_2}{2}\right)-}\Upsilon(\Im_1)\right)$$

$$- a_{\eth}\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}\frac{(\Upsilon(\Im_1)+\Upsilon(\Im_2))}{2} - a_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

Summing over all $\eth$

$$\begin{aligned}&\sum_{\eth=0}^{\infty}a_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_+}\Upsilon(\Im_2)+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_-}\Upsilon(\Im_1)\right)-\sum_{\eth=0}^{\infty}a_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\ &=\sum_{\eth=0}^{\infty}a_{\eth}\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_-}\Upsilon(\Im_1)\right)\\ &-\sum_{\eth=0}^{\infty}a_{\eth}\frac{2(1-\aleph\eth-\wp)}{\aleph\eth+\wp}\frac{(\Upsilon(\Im_1)+\Upsilon(\Im_2))}{2}-\sum_{\eth=0}^{\infty}a_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).\end{aligned}\tag{34}$$

By convexity of Υ we have, $\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\leq\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right)$ with positive multiplier $a_{\eth}$

$$\begin{aligned}&\sum_{\eth=0}^{\infty}a_{\eth}\left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_+}+{}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_-}\Upsilon(\Im_1)\right)-\sum_{\eth=0}^{\infty}a_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\ &\leq\sum_{\eth=0}^{\infty}a_{\eth}\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_-}\Upsilon(\Im_1)\right)\\ &-\sum_{\eth=0}^{\infty}a_{\eth}\left(\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}+1\right)\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).\end{aligned}$$

Using Proposition 1 from the middle of interval $[\Im_1,\Im_2]$, we acquire

$$\begin{aligned}&\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)_+,\aleph,\wp,\tau}\Upsilon(\Im_2;g)+\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)_-,\aleph,\wp,\tau}\Upsilon(\Im_1;g)-\sum_{\eth=0}^{\infty}a_{\eth}\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\\ &\leq\sum_{\eth=0}^{\infty}a_{\eth}\frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)}\left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_+}\Upsilon(\Im_2)+{}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)_-}\Upsilon(\Im_1)\right)\\ &-\sum_{\eth=0}^{\infty}a_{\eth}\left(\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)}+1\right)\left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2}\right).\end{aligned}$$

Using Lemma 4 in the above inequality, we get

$$\begin{aligned}&\sum_{\eth=0}^{\infty}a_{\eth}\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}\\ &\times\left(\frac{\Im_2-\Im_1}{4}\int_0^1 t^{(\aleph\eth+\wp)}\left\{\Upsilon'\left(\frac{t}{2}\Im_1+\frac{2-t}{2}\Im_2\right)-\Upsilon'\left(\frac{2-t}{2}\Im_1+\frac{t}{2}\Im_2\right)\right\}dt\right)\\ &+\sum_{\eth=0}^{\infty}a_{\eth}\left(\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}-1\right)\Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)\end{aligned}$$

$$\leq \sum_{\eth=0}^{\infty} a_{\eth} \frac{\mathcal{B}(\aleph\eth+\wp)}{(\aleph\eth+\wp)} \left({}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+} \Upsilon(\Im_2) + {}^{\mathcal{A}-\mathcal{B}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-} \Upsilon(\Im_1) \right)$$
$$- \sum_{\eth=0}^{\infty} a_{\eth} \left(\frac{2(1-\aleph\eth-\wp)}{(\aleph\eth+\wp)} + 1 \right) \left(\frac{\Upsilon(\Im_1)+\Upsilon(\Im_2)}{2} \right).$$

Using (34) in the above expression

$$\sum_{\eth=0}^{\infty} a_{\eth} \frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)}$$
$$\times \left(\frac{\Im_2-\Im_1}{4} \int_0^1 t^{(\aleph\eth+\wp)} \left\{ \Upsilon'\left(\frac{t}{2}\Im_1 + \frac{2-t}{2}b\right) - \Upsilon'\left(\frac{2-t}{2}\Im_1 + \frac{t}{2}\Im_2\right) \right\} dt \right)$$
$$+ \sum_{\eth=0}^{\infty} a_{\eth} \left(\frac{(\Im_2-\Im_1)^{(\aleph\eth+\wp)}}{2^{(\aleph\eth+\wp)-1}\Gamma((\aleph\eth+\wp)+1)} - 1 \right) \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\leq \sum_{\eth=0}^{\infty} a_{\eth} \left({}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)+} \Upsilon(\Im_2) + {}^{\mathcal{R}-\mathcal{L}}I^{(\aleph\eth+\wp)}_{\left(\frac{\Im_1+\Im_2}{2}\right)-} \Upsilon(\Im_1) \right)$$
$$- \sum_{\eth=0}^{\infty} a_{\eth} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

Using Proposition 1 from the middle of interval $[\Im_1, \Im_2]$, we have

$$\left(\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)+,\aleph,\wp,\tau} \Upsilon \right)(\Im_2; g) + \left(\varepsilon^{\flat,\delta,b,s,l}_{\left(\frac{\Im_1+\Im_2}{2}\right)-,\aleph,\wp,\tau} \Upsilon \right)(\Im_1; g)$$
$$- \sum_{\eth=0}^{\infty} a_{\eth} (o_{\eth} - 1) \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right)$$
$$\geq \sum_{\eth=0}^{\infty} a_{\eth} o_{\eth}$$
$$\times \left(\frac{\Im_2-\Im_1}{4} \int_0^1 t^{(\aleph\eth+\wp)} \left\{ \Upsilon'\left(\frac{t}{2}\Im_1 + \frac{2-t}{2}\Im_2\right) - \Upsilon'\left(\frac{2-t}{2}\Im_1 + \frac{t}{2}\Im_2\right) \right\} dt \right)$$
$$+ \sum_{\eth=0}^{\infty} a_{\eth} \Upsilon\left(\frac{\Im_1+\Im_2}{2}\right).$$

Hence the required result is obtained.

4. Conclusion

Hermite-Hadamard inequalities are essential to many areas of mathematics, such as calculus, real analysis, and convex functions with practical applications in diverse areas such as physics, economics, optimization and engineering. In this article, we look into fractional integral inequalities to Atangana-Baleanu and Prabhakar fractional calculus operators. Using extended generalized Mittag-Leffler functions as their kernel, we present several Hermite-Hadamard type fractional integral inequalities for the Atangana-Baleanu and Prabhakar fractional operators. For the integral inequalities involving fractional integral of the kind $(\Im_1+, \Im_2-)$ and $(\frac{\Im_1+\Im_2}{2})$, important results are given. We demonstrate the validity of our results by using certain functions to generate visual graphs that illustrate the inequalities with corresponding numerical entries. This article seeks to provide more precise bounds and enhance the theoretical foundation for further studies to broaden the classical $(\mathcal{H} - \mathcal{H})$ inequality with generalized fractional integral operators. The latest inequalities will assist to boost comprehension in fractional calculus and convex analysis, with inference through several domains.

Acknowledgements

The authors I. Ayoob and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Declarations:

Availability of data and material

The data used to support the findings of this study are available from the corresponding author upon request.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final version.

Competing interests

The authors declare that they have no conflicts of interest.

References

[1] I. K. Guce. On fractional derivatives: the non-integer order of the derivative. *International Journal of Scientific & Engineering Research*, 4(3):1–5, 2013.

[2] S. A. David, J. L. Linares, and E. M. J. A. Pallone. Fractional order calculus: historical apologia, basic concepts and some applications. *Revista Brasileira de Ensino de Física*, 33(4):4302–4309, 2011.

[3] R. Gorenflo and F. Mainardi. Fractional calculus: integral and differential equations of fractional order. *Fractals*, 5(4):411–424, 1997.

[4] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.

[5] K. B. Oldham and J. Spanier. *The Fractional Calculus*. Academic Press, San Diego, 1974.

[6] H. M. Srivastava and A. K. Mishra. Applications of fractional calculus to parabolic starlike and uniformly convex functions. *Computers & Mathematics with Applications*, 39(3-4):57–69, 2000.

[7] D. Bertsekas. *Convex Optimization Theory*. Athena Scientific, Belmont, MA, 2009.

[8] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. Society for Industrial and Applied Mathematics, Philadelphia, 2001.

[9] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities, Theory and Applications*, volume 1. Academic Press, New York, 1969.

[10] W. Walter. *Differential and Integral Inequalities*, volume 55 of *Springer Tracts in Natural Philosophy*. Springer, Berlin, 2012. Original edition in German, 1964.

[11] P. Agarwal, J. Tariboon, and S. K. Ntouyas. Some generalized Riemann-Liouville k-fractional integral inequalities. *Journal of Inequalities and Applications*, 2016:122, 2016.

[12] G. A. Anastassiou. Opial-type inequalities involving Riemann-Liouville fractional derivatives of two functions with applications. *Mathematical and Computer Modelling*, 48(3-4):344–374, 2008.

[13] N. Mehmood, S. I. Butt, Ž. Pečarić, and J. Pečarić. Several new cyclic Jensen type inequalities and their applications. *Journal of Inequalities and Applications*, 2019:240, 2019.

[14] X. Li, S. Qaisar, J. Nasir, S. I. Butt, et al. Some results on integral inequalities via Riemann-Liouville fractional integrals. *Journal of Inequalities and Applications*, 2019:214, 2019.

[15] C. Zhu, M. Fečkan, and J. Wang. Fractional integral inequalities for differential convex mappings and applications to special means and a midpoint formula. *Journal of Applied Mathematics, Statistics and Informatics*, 8(2):21–28, 2012.

[16] S. Iqbal, M. Samraiz, G. Rahman, K. S. Nisar, and T. Abdeljawad. Some new Grüss inequalities associated with generalized fractional derivative. *AIMS Mathematics*, 8(1):213–227, 2022.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[18] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications*. Springer, Berlin, 2014.

[19] H. J. Haubold, A. M. Mathai, and R. K. Saxena. Mittag-leffler functions and their applications. *Journal of Applied Mathematics*, 2011:298628, 2011.

[20] A. M. Mathai and H. J. Haubold. *Mittag-Leffler functions and fractional calculus*. Springer, 2008.

[21] G. Rahman, I. Suwan, K. S. Nisar, T. Abdeljawad, M. Samraiz, and A. Ali. A basic study of a fractional integral operator with extended Mittag-Leffler kernel. *AIMS*

Mathematics, 6(11):12757–12770, 2021.

[22] M. Andrić, G. Farid, and J. Pečarić. A further extension of Mittag-Leffler function. *Fractional Calculus and Applied Analysis*, 21(5):1377–1395, 2018.

[23] A. Fernandez. A complex analysis approach to Atangana-Baleanu fractional calculus. *Mathematical Methods in the Applied Sciences*, 44(10):8070–8087, 2021.

[24] A. A. Kilbas, M. Saigo, and R. K. Saxena. Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integral Transforms and Special Functions*, 15(1):31–49, 2004.

[25] T. R. Prabhakar. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Mathematical Journal*, 19(1):7–15, 1971.

[26] T. Abdeljawad and D. Baleanu. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *Journal of Nonlinear Sciences and Applications*, 10(3):1098–1107, 2017.

[27] J.-D. Djida, A. Atangana, and I. Area. Numerical computation of a fractional derivative with non-local and non-singular kernel. *Mathematical Modelling of Natural Phenomena*, 12(3):4–13, 2017.

[28] A. Fernandez and D. Baleanu. The mean value theorem and Taylor's theorem for fractional derivatives with Mittag-Leffler kernel. *Advances in Difference Equations*, 2018:86, 2018.

[29] A. Fernandez, D. Baleanu, and H. M. Srivastava. Series representations for models of fractional calculus involving generalised Mittag-Leffler functions. *Communications in Nonlinear Science and Numerical Simulation*, 67:517–527, 2019.

[30] R. Garra and R. Garrappa. The Prabhakar or three parameter Mittag-Leffler function: theory and application. *Communications in Nonlinear Science and Numerical Simulation*, 56:314–329, 2018.

[31] S. Uçar, E. Uçar, N. Özdemir, and Z. Hammouch. Mathematical analysis and numerical simulation for a smoking model with Atangana-Baleanu derivative. *Chaos, Solitons & Fractals*, 118:300–306, 2019.

[32] M. Samraiz, Z. Perveen, T. Abdeljawad, S. Iqbal, and S. Naheed. On certain fractional calculus operators and applications in mathematical physics. *Physica Scripta*, 95(11):115210, 2020.

[33] S. Qaisar, J. Nasir, S. I. Butt, and S. Hussain. On some fractional integral inequalities of Hermite-Hadamard type through convexity. *Symmetry*, 11(2):137, 2019.

[34] S. I. Butt, Y. Saba, O. A. Ahmet, and A. D. Mustafa. New Hadamard-type integral inequalities via a general form of fractional integral operators. *Chaos, Solitons & Fractals*, 148:111025, 2021.

[35] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, 57(9-10):2403–2407, 2013.

[36] A. Fernandez and P. Mohammed. Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. *Mathematical Methods in the Applied Sciences*, 44(10):8414–8431, 2021.

[37] M. Z. Sarikaya and H. Yildirim. On Hermite-Hadamard type inequalities for Riemann-

Liouville fractional integrals. *Miskolc Mathematical Notes*, 17(2):1049–1059, 2017.

[38] K. Mehrez and P. Agarwal. New Hermite-Hadamard type integral inequalities for convex functions and their applications. *Journal of Computational and Applied Mathematics*, 350:274–285, 2019.