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Exponential Stability of Volterra Integro-Dynamic Sylvester Matrix System on Time Scales

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Abstract. This article investigates the exponential stability of the Volterra integro-dynamic (VID) Sylvester matrix system on time scales. The analysis begins by transforming the exponential stability VID Sylvester matrix system into an equivalent Kronecker product VID system using the vectorization operator. Subsequently, the boundedness properties of the solutions are rigorously established. Moreover, the theoretical results formulated on an arbitrary time scale unify and extend the stability conditions for both integral and discrete Volterra equations. Finally, the efficacy and validity of these findings are substantiated through diverse time scale scenarios, accompanied by a graphical comparative analysis.

2020 Mathematics Subject Classifications: 93D23, 45D05, 15A69, 34N05

Key Words and Phrases: Exponential stability, Volterra Integro-dynamical system, Kronecker Product, Time scales

1. Introduction

The examination of dynamic equations on time scales has garnered considerable interest due to its ability to unify discrete and continuous analysis. Bohner and Peterson [1, 2] laid the foundational groundwork for this discipline by establishing a coherent framework that integrates differential and difference equations. This approach has proven effective in modeling real-world systems exhibiting both continuous and discrete behavior, such as biological models, economic systems, and control theory applications.

A central area of study within time scale calculus is the stability analysis of dynamical systems. Researchers such as Berezansky et al. [3] have analyzed stability conditions for scalar Volterra difference equations, while Migda and Aldona [4] focused on the asymptotic

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behavior of second-order Volterra-type systems. Further contributions by Schmeidel et al. [5, 6] addressed exponential stability in multi-agent systems.

Stability and controllability of VID Sylvester matrix systems have also been extensively studied. Investigations by Sreenivasulu and Rao [7–9] introduced stability conditions for nonlinear Sylvester systems, including impulsive cases. Ramana and Deekshitulu [10, 11] contributed results on controllability and asymptotic behavior of Volterra-type matrix models, while Shivasivapriya et al. [12] explored delayed impulses and controller designs. Alexander's work on Kronecker products and matrix calculus in time scale settings and the methodology by Aulbach and Stefan [13] further advanced the analytical framework by merging discrete and continuous approaches.

The role of exponential stability in understanding long-term system behavior has been emphasized in studies by various authors [14–16]. Recent works by Suresh et al. [17–19] expanded this perspective by addressing fuzzy boundary value problems and Ψ-stability for nonlinear matrix difference equations, reflecting the growing theoretical depth and practical relevance of the field.

Complementary efforts by Adıvar and Raffoul [20] contributed foundational stability results, while Diamandescu et al. [21, 22] analyzed Ψ-conditional asymptotic stability in nonlinear Lyapunov matrix systems. Further contributions by Rao and Prasad [23], Girejko et al. [24], and Karpuz and Koyuncuoğlu [25] addressed stability in Sylvester-type and Volterra systems. The nonlinear time-varying stability analysis presented by Qiang et al. [26] adds to this evolving body of knowledge.

Additionally, modern control theory and fractional calculus have significantly influenced time scale dynamics. Sadek et al. [27, 28] examined controllability and observability in ϕ-conformable fractional systems and proposed numerical strategies for solving high-dimensional matrix equations [29, 30]. Their work on generalized fractional derivatives [31] supports broader modeling flexibility. Parallel developments by Kumar and co-authors [32–39] address impulsive systems, synchronization in neural networks, and fractional dynamic systems over irregular time domains.

The Sylvester matrix formulation offers a powerful tool for studying integro-dynamic systems. Its structured representation facilitates analytical tractability and efficient computation, particularly when combined with Kronecker product techniques. This matrix-based approach enhances the understanding of stability, qualitative properties in high-dimensional systems and serves as a versatile framework across engineering, physics, and applied sciences.

In this paper, we deals with exponential stability of VID Sylvester matrix system of the following form

$$\begin{cases} Y^{\Delta}(\iota) = A(\iota)Y(\iota) + Y(\iota)B(\iota) + \int_0^{\iota} H(\iota, \upsilon)Y(\upsilon)\Delta\upsilon + G(\iota), \iota \in [0, \infty)_{\mathbb{T}} \\ Y(0) = Y_0. \end{cases} \tag{1}$$

Where $A(\iota)$ and $B(\iota)$ are of matrix order $n \times n$. $G(\iota)$ and $H(\iota, \upsilon)$ are matrix functions of order $n \times n$. $Y^{\Delta}(\iota)$ is delta derivative of $Y(\iota)$.

The primary contributions and advantages of this manuscript are as follows:

- We develop a unified framework for analyzing the exponential stability of Volterra integro-dynamic Sylvester matrix systems on arbitrary time scales.
- We transform the original matrix system into an equivalent Kronecker product formulation using vectorization, enabling efficient analysis.
- We establish rigorous conditions for the boundedness and exponential stability of solutions, ensuring mathematical consistency across all time domains.
- We extend and generalize classical results for both discrete and continuous Volterra systems within a single theoretical structure.
- We validate the theoretical results through numerical examples and comparative graphical illustrations on various time scales.

2. Preliminaries

In this section, we will provide the essential definitions and key concepts of the Calculus of time scales. These notions serve as the foundation for the analysis of dynamic systems on time scales, as stated in the references [1, 2] respectively.

Definition 1. *[2] A closed subset of $\mathbb{R}$ that takes its topology from the normal topology of $\mathbb{R}$ is called a time scale $\mathbb{T}$. The backward jump operator is the mapping $\rho : \mathbb{T} \to \mathbb{T}$, which is defined as $\rho(\iota) = \sup\{\upsilon \in \mathbb{T} : \upsilon < \iota\}$, with $\sup \phi = \inf \mathbb{T}$.The condition that $\rho(\iota) = \iota$ and $\iota > \inf \mathbb{T}$ must be met for a point $\iota \in \mathbb{T}$ to be declared left-dense. On the other side, ι is considered left-scattered if and only if $\rho(\iota) < \iota$. In a similar vein, the ideas of right-scattered and right-dense points are defined, as is the forward leap operator, σ. The graininess function is defined as the function $\mu : \mathbb{T} \to [0, \infty)$ where $\mu(\iota) = \sigma(\iota) - \iota$.*

Definition 2. *[2] The number that satisfies the following criteria for any $\epsilon > 0$: there exists a neighborhood U about ι that is defined as the delta derivative of function g at a given point ι, $g^{\Delta}(\iota)$.*

$$| [g(\sigma(\iota)) - g(\upsilon)] - g^{\Delta}(\iota)[\sigma(\iota) - \upsilon] | \leq \epsilon | \sigma(\iota) - \upsilon |, \forall \upsilon \in U.$$

A function g is considered delta differentiable on $\mathbb{T}^k$ if $g^{\Delta}(\iota)$ exists for every $\iota \in \mathbb{T}^k$. consequently, the function $g^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is referred to as the derivative of f on $\mathbb{T}^k$.

At each right-dense point in $\mathbb{T}$, a function $g : \mathbb{T} \to \mathbb{R}$ is considered rd-continuous if and only if it has a finite left-sided limit at each left-dense point in $\mathbb{T}$. What we call $Crd(\mathbb{T}, \mathbb{R})$ is actually a collection of rd-continuous functions. Moreover, for any $t \in \mathbb{T}^k$, the condition $G^{\Delta} = g$ must hold for $G : \mathbb{T} \to \mathbb{R}$ to be defined as an anti-derivative of $f : \mathbb{T} \to \mathbb{R}$. The anti-derivative of a rd-continuous function must be remembered.

Definition 3. *[2] Let G be an anti-derivative of g and $\upsilon, \iota \in \mathbb{T}$. the delta integral of g is defined as follows:*

$$\int_a^b g(\iota)\Delta\iota = G(b) - G(a).$$

Definition 4. *[2] Regressive functions are defined as those that map $\mathbb{T} \to \mathbb{R}$ and satisfy the condition that $1 + \mu(\iota)M(\iota) \neq 0, \forall \iota \in \mathbb{T}$. Regressive functions are aggregated by the right dense continuous function $\mathcal{R} = \mathbb{R}(\iota) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Similarly, all positively regressive functions are contained in $\mathbb{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{M \in \mathcal{R} : 1 + \mu(\iota)M(\iota) > 0, \forall \iota \in \mathbb{T}\}$.*

Remark 1. *Let $p \in \mathcal{R}$ and $P(\iota) < 0, \forall \iota \in \mathbb{T}$. All of the following assertions are supported by evidence: A declining behavior is observed in relation to the variable ι when the function $\Psi_P(\iota, \upsilon)$ is considered. With the context of the variable υ, the function $\Psi_P(\iota, \upsilon)$ displays a behavior that is growing in nature.*

Theorem 1. *[2] If the variable a and b belong the set $\mathbb{T}$ and the function $g : \mathbb{T} \to \mathbb{R}$ is a right-continuous function that satisfying the condition $g(\iota) \geq 0$, for all values of that fall within the interval $a \leq \iota \leq b$, then*

$$\int_a^b g(\iota)\Delta\iota \geq 0.$$

3. Exponential stability

In this section, we establish sufficient criteria for the exponential stability and boundedness of solutions to the Kronecker product VID system on time scales described by equation (1).

Theorem 2. *Let $z(\iota) = VecY(\iota)$ and $g(\iota) = VecG(\iota)$ with $z(0) = z_0$. Consequently, the VID Sylvester matrix system (1), is equivalent the system*

$$z^{\Delta}(\iota) = P(\iota)z(\iota) + \int_0^{\iota} (I \otimes H)(\iota, \upsilon)z(\upsilon)\Delta\upsilon + g(\iota), \tag{2}$$

where I be a identity matrix and $P(\iota) = [B^* \otimes I_n + I_n \otimes A]$.

Proof. After applying the Vec operator to the system (1), we are able to acquire the Kronecker product [40] by making use of the features that were discussed earlier.

$$z^{\Delta}(\iota) = P(\iota)z(\iota) + \int_0^{\iota} (I \otimes H)(\iota, \upsilon)z(\upsilon)\Delta\upsilon + g(\iota).$$

Result 3. *[23] When $Y(\iota)$ is the solution of (1) if and only if $z(\psi) = VecY(\iota)$ is the solution of (2).*

Definition 5. *If there are positive constants M and d, the following inequality is satisfied for every solution of the homogeneous system associated with $z(0) = z_0$, then the system given by the system (2) is said to be exponentially stable.*

$$| z(\iota) | \leq M | z_0 | \Psi^{-d}(0, \iota).$$

The initial order non-homogeneous linear dynamic system associated with system (2) can be expressed in the following manner:

$$z^{\Delta}(\iota) = P(\iota)z(\iota) + g(\iota), \tag{3}$$

this system is termed regressive. According to the formula of variation of constants [2], the only one solution to this equation, given $z(0) = z_0$ is provided as follows.

$$z(\iota) = \Psi_P(\iota, 0)z_0 + \int_0^{\iota} \Psi_P(\iota, \sigma(\tau))g(\tau)\Delta\tau. \tag{4}$$

It can be verified that the solution to the system (2) with $z(0) = z_0$ fulfills the following requirements:

$$\begin{aligned} z(\iota) = \Psi_P(\iota, 0)z_0 &+ \int_0^{\iota} \Psi_P(\iota, \sigma(\tau)) \int_0^{\iota} (I \otimes H)(\tau, \upsilon)z(\upsilon)\Delta\upsilon\Delta\tau \\ &+ \int_0^{t} \Psi_P(\iota, \sigma(\tau))g(\tau)\Delta\tau. \end{aligned} \tag{5}$$

Theorem 4. *Let $P(\iota) \leq 0, \forall \iota \in \mathbb{T}$. If $P \in \mathcal{R}^{n^2}$ then the subsequent inequalities are valid.*

$$-1 \leq \int_0^{\iota} \Psi_P(\iota, \sigma(\tau))P(\tau)\Delta\tau \leq 0, \tag{6}$$

and $\Psi_P(\iota, 0) \in [0, 1]$.

Proof. Take into consideration the system that is described by equation (3), including a function that is defined as $f(\iota) = -P(\iota)$, and begin with $z_0 = -1$. When considering this particular scenario, it is crucial to take note of the fact that the function $z(\iota) \equiv 1$ functions as a solution to the initial value problem (IVP) that is represented by equations (3) and (2). In light of this, we are able to directly deduce that from equation (4).

$$1 = \Psi_P(\iota, 0) - \int_0^{\iota} \Psi_P(\iota, \sigma(\tau))P(\tau)\Delta\tau.$$

According to Theorem 1, we can express this as

$$\Psi_P(\iota, 0) = \Psi_P(\iota, \sigma(\tau))\Psi_P(\sigma(\tau), 0).$$

Given that $P \in \mathcal{R}^{n^2}$, it follows that $\Psi_P(\iota, 0) > 0, \forall \iota \in \mathbb{T}$ and $\Psi_P(\sigma(\tau), 0) > 0, \forall \iota \in \mathbb{T}$. Therefore, we can conclude that $\Psi_P(\iota, \sigma(\tau)) > 0, \forall r \in [0, \iota]_{\mathbb{T}}$. This completes the proof, as the function P assumes nonpositive values.

Now that we have taken the necessary steps, we are ready to provide sufficient criteria for the boundedness of solutions to the system (2).

Theorem 5. *Let us consider* $P \in \mathcal{R}^{n^2}$ *such that* $P(\iota) \leq -\phi < 0, \forall \iota \in \mathbb{T}$*, where* $\phi > 0$*. Furthermore, assume there exists a constant* $c \in (0,1)$ *such that*

$$\sup_{\iota \in \mathbb{T}} \frac{1}{\mid P(\iota) \mid} \int_{\iota_0}^{\iota} \mid (I \otimes H)(\iota, \upsilon) \mid \Delta\upsilon \leq c. \tag{7}$$

Given that the function g *is bounded, it follows that all of the solutions to the system* (2) *are also bounded under the same conditions.*

Proof. For a bounded function g, there exists a positive constant K that is such that $\mid g(\iota) \mid \leq K, \forall \iota \in \mathbb{T}$. Let $z : \mathbb{T} \to \mathbb{R}$ signify a solution of the IVP system (2). Given that z satisfies the criterion, it follows that any $\iota > \iota_0$, according to Theorem 4, we have the following:

$$\begin{aligned} \mid z(\iota) \mid \leq \mid z_0 \mid &+ \int_0^{\iota} \Psi_P(\iota, \sigma(\tau)) \int_0^{r} \mid (I \otimes H)(\tau, \upsilon) z(\upsilon) \mid \Delta\upsilon\Delta\tau \\ &+ \int_0^{\iota} \Psi_P(\iota, \sigma(\tau)) \mid g(\tau) \mid \Delta\tau \end{aligned}$$

and as a result,

$$\begin{aligned} \mid z(\iota) \mid \leq \mid z_0 \mid &+ \int_0^{\iota} \Psi_P(\iota, \sigma(\tau)) |P(\tau)| \int_0^{r} \frac{\mid (I \otimes H)(\tau, \upsilon) \mid}{\mid P(\tau) \mid} \sup_{\upsilon \in [0,\iota]_{\mathbb{T}}} \mid z(\upsilon) \mid \Delta\upsilon\Delta\tau \\ &+ K \int_0^{\iota} \frac{1}{\mid P(\tau) \mid} \Psi_P(\iota, \sigma(\tau)) \mid P(\tau) \mid \Delta\tau \\ &\leq \mid z_0 \mid + c \sup_{\upsilon \in [0,\iota]_{\mathbb{T}}} \mid z(\upsilon) \mid \int_0^{\iota} \Psi_P(\iota, \sigma(\tau)) |P(\tau)| \Delta\tau \\ &+ \frac{K}{\phi} \int_0^{\iota} \Psi_P(\iota, \sigma(\tau)) |P(\tau)| \Delta\tau. \end{aligned}$$

By utilizing Theorem 4, we obtain

$$\mid z(\iota) \mid \leq \mid z_0 \mid + c \sup_{\upsilon \in [0,\iota]_{\mathbb{T}}} \mid z(\upsilon) \mid + \frac{K}{\phi},$$

We can now reach the conclusion that

$$\begin{aligned} \sup_{\upsilon \in [0,\iota]_{\mathbb{T}}} \mid z(\upsilon) \mid &\leq \mid z_0 \mid + c \sup_{\upsilon \in [0,\iota]_{\mathbb{T}}} \mid z(\upsilon) \mid + \frac{K}{\phi} \\ &\leq \frac{\mid z_0 \mid + \frac{K}{\phi}}{1 - c}. \end{aligned}$$

This proves that the proof is complete because it holds for any arbitrary value of $\iota > \iota_0$.

Theorem 6. *The subsequent conditions are assumed to be met.*

a. *In the real number system, there is a positive real number* η *that is included within the real number system* $\mathcal{R}^+$. *Additionally, there is a positive constant* L. *To the extent that*

$$|\ (I \otimes H)(\tau, \upsilon)\ | \leq L\Psi^{(-\eta)\oplus(-\eta)}(\sigma(\tau), 0), 0 \leq \upsilon \leq \tau,$$

b. *Suppose that* p *is an element of* $\mathcal{R}^+$, *and that there is a positive constant* $-\phi$ *that is also a member of* $\mathcal{R}^+$. *and*

$$P(\iota) \leq -\phi < 0, \tag{8}$$

c. $\phi < \eta$

d. g *is a bounded function.*

If this is the case, then the system (2) *demonstrates exponential stability.*

Proof. Let $z : \mathbb{T} \to \mathbb{R}$ express a solution to the homogeneous equation that is associated with the system that is described in (2). Consider the fact that if g is bounded, then z will also be bounded and will fulfill the conditions that relate to it. This is an important observation to make.

$$z(\iota) = \Psi_P(\iota, 0)z_0 + \int_0^\iota \Psi_P(\iota, \sigma(\tau)) \int_0^\iota (I \otimes H)(\tau, \upsilon)z(\upsilon)\Delta\upsilon\Delta\tau.$$

Consequently, for an arbitrary value of $\iota > \iota_0$ we derive

$$|\ z(\iota)\ | = T\Psi_P(\iota, 0) + T\int_0^\iota \Psi_P(\iota, \sigma(\tau)) \int_0^\iota (I \otimes H)(\tau, \upsilon)z(\upsilon)\Delta\upsilon\Delta\tau, \tag{9}$$

where $\sup_{\iota\in\mathbb{T}} |\ z(\iota)\ | \leq T$. It is important to note that, based on the characteristic of the exponential function and the observations made in Remark 1 for the range $0 \leq s \leq r$ we can conclude the following

$$|\ (I \otimes H)(\tau, \upsilon)\ | \leq L\Psi^{-\eta}(\sigma(\tau), 0)\Psi^{-\eta}(\tau, \sigma(\upsilon))$$

Utilizing the aforementioned estimation in relation to (9) results in.

$$|\ z(\iota)\ | = T\Psi_P(\iota, 0) + TL\int_0^\iota \Psi_P(\iota, \sigma(\tau)) \int_0^\iota \Psi^{-\eta}(\sigma(\tau), 0)\Psi^{-\eta}(\tau, \sigma(\upsilon))\Delta\upsilon\Delta\tau.$$

From equation (8) and Theorem 1, it can be concluded that

$$\Psi_P(\iota, 0) \leq \Psi^{-\phi}(\iota, 0), \text{ and } \Psi_P(\iota, \sigma(\tau)) \leq \Psi^{-\phi}(\iota, \sigma(\tau)), \forall\ 0, \sigma(\tau) \leq \iota.$$

As a result, we obtain

$$|\ z(\iota)\ | = T\Psi^{-\phi}(\iota, 0) + TL\int_0^\iota \Psi^{-\phi}(\iota, \sigma(\tau))\Psi^{-\eta}(\sigma(\tau), 0) \int_0^\iota \Psi^{-\eta}(\tau, \sigma(\upsilon))\Delta\upsilon\Delta\tau.$$

By applying the characteristics of the exponential function, we derive

$$\Psi^{-\phi}(\iota,\sigma(\tau))\Psi^{-\eta}(\sigma(\tau),0)=\Psi^{-\phi}(\iota,0)\Psi^{(-\eta)\ominus(-\phi)}(\sigma(\tau),0).$$

According to Theorem 1, we can conclude that

$$\int_0^{\iota}\Psi^{-\eta}(\tau,\sigma(\upsilon))\Delta\upsilon=\frac{1}{\eta}(1-\Psi^{-\eta}(\tau,0).$$

Therefore,

$$\begin{aligned}|\ z(\iota)\ | &= T\Psi^{-\phi}(\iota,0)+\frac{TL}{\eta}\Psi^{-\phi}(\iota,0)\int_0^{\iota}\Psi^{(-\eta)\ominus(-\phi)}(\sigma(\tau),0)(1-\Psi^{-\eta}(\tau,0))\Delta\tau\\ &\le T\Psi^{-\phi}(\iota,0)+\frac{TL}{\eta}\Psi^{-\phi}(\iota,0)\int_0^{\iota}\Psi^{(-\eta)\ominus(-\phi)}(\sigma(\tau),0)\Delta\tau.\end{aligned}$$

The expression $\Psi^{(-\eta)\ominus(-\phi)}(\sigma(\tau),0)\Psi^{-\eta}(\tau,0)$ yields positive values. Based on the assumptions regarding the constant η and ϕ it can be concluded that $(-\eta)\ominus(-\phi)(\iota)<0,\forall\iota\in\mathbb{T}$. utilizing Remark 1, we obtain the following

$$|\ z(\iota)\ |\le T\Psi^{-\phi}(\iota,0)+\frac{TL}{\eta}\Psi^{-\phi}(\iota,0)\int_0^{\iota}\Psi^{(-\eta)\ominus(-\phi)}(\tau,0)\Delta\tau,$$

obviously,

$$\int_0^{\iota}\Psi^{(-\eta)\ominus(-\phi)}(\tau,0)\Delta\tau=\frac{1}{(-\eta)\ominus(-\phi)}\left(\Psi^{(-\eta)\ominus(-\phi)}(\iota,0)-1\right),$$

This results in the subsequent estimation

$$|\ z(\iota)\ |\le T\left(1+\frac{L}{\eta\mid(-\eta)\ominus(-\phi)\mid}\right)\Psi^{-\phi}(\iota,0),$$

Therefore, by applying that $-\phi\in\mathcal{R}^+$ along with assumption (c), we derive

$$|\ z(\iota)\ |\le T\left(1+\frac{L}{\eta(\eta-\phi)}\right)\Psi^{-\phi}(\iota,0),$$

which ends the proof.

4. Example

Now, we have the efficacy and validity of these findings are substantiated through diverse time scale scenarios, accompanied by a graphical comparative analysis.

As bacterial populations grow exponentially, the availability of resources diminishes, leading to a mathematical model governed by the following linear VID system on time scales:

$$z^{\Delta}(\iota)=-\frac{z(\iota)}{\mu^2(\iota)+1}+\frac{1}{\iota^3+1}\int_2^{\infty}(\upsilon+\sigma(\upsilon))z(\upsilon)\Delta\upsilon+\frac{sin\iota}{\mu(\iota)+1},\tag{10}$$

where $0 \leq \mu(\iota) \leq 1, \forall \iota \in \mathbb{T}$. It can be observe that the function $P(\iota) = -\frac{z(\iota)}{\mu^2(\iota)+1}$ meets the criteria outlined in Theorem 4 with the constant $\phi = \frac{1}{2}$. additionally, the function $g(\iota) = \frac{sin\iota}{\mu(\iota)+1}$ is both bounded and rd-continuous. Furthermore, the expression $(I \otimes H)(\iota, \upsilon) = \frac{(\upsilon+\sigma(\upsilon))}{\iota^3+1}$ belongs to $C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$ and it can be verified that this holds true.

$$\sup_{\iota \in \mathbb{T}} \frac{1}{P(\iota)} \int_2^\infty \left| \frac{(\upsilon + \sigma(\upsilon))}{\iota^3 + 1} \right| d\upsilon \leq \frac{8}{9}.$$

Therefore, all the conditions of Theorem 4 are satisfied, leading to the conclusion that all solutions of (10) are bounded.

Case(i): In the event that $\mathbb{T} = \mathbb{R}$, it therefore follows that every continuous function $p : \mathbb{R} \to \mathbb{R}$ is included in $\mathcal{R}^+$. In the event that $\mu(\iota) = 0$ A representation of the equation (10) can be represented in the following manner:

$$z'(\iota) = -z(\iota) + \frac{1}{\iota^3 + 1} \int_2^\infty \upsilon z(\upsilon) \Delta\upsilon + sint.$$

Case(ii): In the event that $\mathbb{T} = \mathbb{N}$, the function $p : \mathbb{N} \to \mathbb{R}$ is constrained such that $-1 < p(t) \leq 0$ and $\mu(\iota) = 1$. Under these conditions, equation (10) can be expressed in a specific form

$$\Delta z(\iota) = -\frac{1}{2} z(\iota) + \frac{1}{\iota^3 + 1} \sum_{i=2}^{\iota-1} (2i - 1) z(i) + \frac{sin\iota}{2}.$$

Case(iii): In the event that $\mathbb{T} = \frac{1}{3}\mathbb{N}$, $\mu(\iota) = 1$ then (10) takes the form

$$z^\Delta(\iota) = \frac{9}{10} + \frac{3}{\iota^3 + 27} \sum_{k=3}^{3\iota-1} (k + 1) z(k) + \frac{3}{4} sin\frac{\iota}{3}.$$

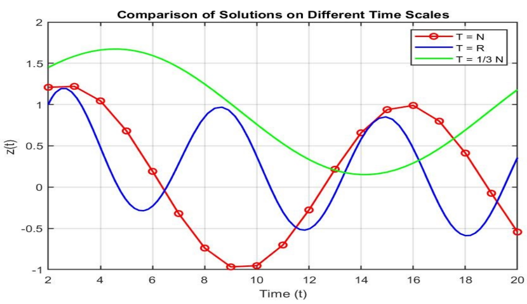

Figure 1: Comparison of solution on different time scales

The graph demonstrates exponential stability by comparing solutions on different time scales ($\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \frac{1}{3}\mathbb{N}$), where the discrete and continuous trajectories exhibit bounded oscillatory behavior.

5. Conclusion

Systems of the matrix Sylvester type, which are frequently found in mathematical physics and VID systems, are indispensable in a wide variety of engineering and optimization applications. Due to the fact that they are able to unite and extend both continuous and discrete models, timescale calculus and dynamic equations on time scales have recently garnered a significant amount of interest. Establishing a boundedness solution for the exponential stability of a VID Sylvester matrix system on time scales is the objective of this research. The analytical method uses the Kronecker product of matrices to determine the most important discoveries. Using the vectorization operator, the long-term stability of the VID Sylvester system is first changed into a similar vector dynamic system. The next step involves presenting sufficient conditions and criteria for constrained solutions associated with exponential stability. The findings include both continuous and discrete versions of VID Sylvester matrix systems, showing that they can be used in many different ways. Future studies will broaden this research to include oscillations, controllability, observability, and periodic solutions for Sylvester matrix fuzzy VID solutions on time scales.

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