EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6095
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Interval Valued Spherical Fuzzy Matrix in Decision Making

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Abstract. Recent advancements have demonstrated the potential to augment matrix theory with fuzzy, intuitionistic fuzzy, picture fuzzy, interval-valued picture fuzzy matrix concepts for enhanced decision-making applications. We introduce the interval valued spherical fuzzy matrix, extending the spherical fuzzy matrix, to effectively represent and manipulate uncertain and vague information with enhanced flexibility. This paper establishes definitions and theorems for Interval-Valued spherical fuzzy matrices. We develop methods for computing determinant and adjoint, and develop algorithms using composition functions to determine the greatest and least eigenvalue interval valued spherical fuzzy sets and create a flow chart to depict the procedure. In this paper, a new distance measure has been proposed and is to be proved valid by satisfying all the conditions of the distance metric. In addition, an application of interval-valued spherical fuzzy matrices to deal with decision-making problems is presented.

2020 Mathematics Subject Classifications: 03E72, 15B15, 90B50

Key Words and Phrases: Interval valued spherical fuzzy sets, Interval valued spherical fuzzy matrix, Least eigenvalue, Greatest eigenvalue

Abbreviations

IVSFS	Interval valued spherical fuzzy set
IVSFSs	Interval valued spherical fuzzy sets
IVSFM	Interval valued spherical fuzzy matrix
IVSFMs	Interval valued spherical fuzzy matrices

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6095

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EIVSFS	Eigen interval valued spherical fuzzy set
GEIVSFS	Greatest Eigen Interval valued spherical fuzzy set
LEIVSFS	Least Eigen Interval valued spherical fuzzy set
IVSFR	Interval valued spherical fuzzy relation
AO	Administrative officer
AOs	Administrative officers
Government	Govt.
Governments	Govts.
DOC	Degree of closeness
DOCs	Degree of closenesses

1. Introduction

Uncertainty pervades our lives in our daily and scientific pursuits, affecting more or less most applications of interest. The medical, engineering, industrial, economic, and business domains present themselves as natural battlefields for making decisions in the presence of uncertainty when real-world applications are dealt with. In trying to overcome these, the existing methodologies are grossly inadequate. Addressing this imperative, Zadeh [1] introduced a theory called fuzzy set that is a milestone in studying certain varieties of uncertainty with which classical modes of study cannot cope or have utterly failed. Fuzzy set theory and its generalizations have resulted in great mathematical applications to almost all real-life problems involving uncertainty. Many extensions and modifications of fuzzy set theory have been developed to handle various forms of uncertainty. These include vague sets, rough sets, intuitionistic fuzzy sets, soft sets, Pythagorean fuzzy sets, picture fuzzy sets, and other advanced generalizations.

Kim and Roush [2] introduced the concept of fuzzy matrices, which play a vital role, in various fields of Science and Engineering, addressing problems involving various types of uncertainties [3]. Hashimoto [4] examined the convergence of power of transitive fuzzy matrix. Since then, extensive work has been done on fuzzy matrices. However, fuzzy matrix focus membership value only. Atanassov [5] extended the idea of fuzzy sets by introducing the of concept intuitionistic fuzzy sets, Which add a new dimension called the hesitancy degree. While fuzzy sets have membership and non membership degrees, intuitionistic fuzzy sets add hesitancy degree, which quantifies the level of uncertainty or ambiguity associated with an element's classification. This Additional dimension (hesitancy degree) allows intuitionistic fuzzy sets to solve complex problems involving imprecise or vague information. Since their introduction, intuitionistic fuzzy sets have been widely discussed and applied across various domains, including decision-making [6, 7], pattern recognition [8], and medical diagnosis [9, 10] where uncertainty plays a important role in analysis. El-Morsy [11] proposed an innovative method utilizing Pythagorean fuzzy numbers to evaluate the risked return rate, portfolio risk, and expected return rate. Atanassov and Gargov [12] expanded the concept of Intuitionistic Fuzzy Sets (IFS) to include interval-valued intuitionistic fuzzy sets, where interval numbers replace exact numbers, offering greater flexibility in defining membership degrees for elements. The versatility and effectiveness of interval-valued intuitionistic fuzzy sets have been demonstrated across various domains, as evidenced by numerous studies [13–16]. Additionally, pal [17] introduced intuitionistic fuzzy matrices, and several properties of these matrices were explored. Investigations performed by Bhowmik and Pal [18] focused on the convergence of max-product powers in intuitionistic fuzzy matrices. Similarly, Pradhan and Pal [19] examined the mean powers of convergence within these matrices. Additionally, the study of power convergence and the canonical structure of sss-transitive intuitionistic fuzzy matrices has been conducted [20]. Several researchers [21–27] have contributed to advancing the field of intuitionistic fuzzy matrices, achieving numerous significant results that prove useful for addressing uncertainty in real-life problems. However, despite their wide applicability, intuitionistic fuzzy sets fall short when it comes to handling contradictory scenarios in many practical applications. For instance, consider a voting system where outcomes can be categorized as voting for, abstaining, voting against, or refusing to vote. To address such complexities, Cuong and Kreinovich [28] introduced the concept of picture fuzzy sets. These sets encompass three degrees of membership: membership, neutral membership, and non-membership. This extension allows picture fuzzy sets to generalize intuitionistic fuzzy sets, enabling them to capture more nuanced, conflicting, and ambiguous information. Furthermore, Cuong and Kreinovich defined foundational operational laws for picture fuzzy sets and established several of their properties. Garg [29] introduced the weighted average and geometric aggregation operators for picture fuzzy numbers and applied them to decision-making problems. Multivariable Hermite and Apostol-type Frobenius–Genocchi polynomials enhance spherical fuzzy matrices by refining uncertainty modeling in decision-making processes [30]. The novel bivariate 2D-q Hermite polynomials can be used to define complex membership functions in spherical fuzzy matrices, enhancing multidimensional uncertainty representation in decision-making[31]. Ganie [32] proposed a new distance measure for picture fuzzy sets and demonstrated the application of the proposed distance measure in pattern recognition. Dogra and Pal [33] developed the concept of picture fuzzy matrices and explored their properties. Thereafter, many authors [34, 35] extended the study of picture fuzzy matrices. While picture fuzzy sets have a variety of dimensions, they are also bound by the same constraint as intuitionistic fuzzy sets the sum of their three parameters, membership, neutral membership, and non-membership grades must be ≤ 1. To break through this barrier, Mahmood et al. [36] extended the picture fuzzy set structure into a T-Spherical Fuzzy Set, which lets its membership parameters range over an enlarged range of values. For example, Kifayat et al.[37] did a geometrical comparison for fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, picture fuzzy sets, and T-Spherical fuzzy sets. Silamarasan [38] introduced the spherical fuzzy matrix and derived some of its properties with the help of matrix operations.

In this paper, we introduce interval valued spherical fuzzy matrix extending spherical fuzzy matrix (SFM), to effectively represent and manipulate uncertain and vague information with enhanced flexibility. We develop algorithms to find the greatest eigen vector and least eigen vector for an interval valued spherical fuzzy matrix using composition and propose a new distance measure, verified its properties. The application based on the decision making problem is presented with the help of an example.

The main research gap between SFMs and IVSFMs lies in their ability to model uncertainty and flexibility in decision making. Although SFMs offer a strong representative framework for membership, nonmembership, and hesitancy under spherical constraints, they are restricted to fixed scalar values, which may not represent variability or imprecision in real-world scenarios. IVSFM extends this to use interval values, hence giving a more comprehensive representation of uncertainty. The paper is organized as follows:
In Section 2, some basic definitions are introduced, and these definitions help readers to understand the paper easily. In Section 3, we introduce the new concept of IVSFM with the basic definitions and discuss some important properties and theorems. In Section 4, we find determinants and adjoints for IVSFM with the help of an example, and some fundamental properties are examined. In Section 5, we introduce the EIVSFS and provide algorithms for finding the GEIVSFS and LEIVSFS with the help of illustrations. In Section 6, we present a new distance measure for IVSFSs, investigate their properties, and explore their potential applications in decision making. The comparative study with existing work is conducted in Section 7. In Section 8, the conclusion of the paper is given with future directions.

2. Preliminaries

Definition 1. *[5] An Intuitionistic Fuzzy Set (IFS) A in universal set X is defined as a collection of the form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$*, where,* $\mu_A : X \to [0,1]$ *member ship function and* $\nu_A : X \to [0,1]$ *non-membership function of the element* $x \in X$ *and* $\forall$ $x \in X$ *such that* $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 2. *[17] Let* $X = \{x_1, x_2, ...x_m\}$ *represent a set of alternatives and* $Y = \{y_1, y_2, ...y_n\}$ *represents a set of attributes associated with each element of X. An intuitionistic fuzzy matrix is defined as* $A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j)\rangle)$ *for* $i = 1, 2...m$ *and* $j = 1, 2, ...n$*, where* $\mu_A : X \times Y \to [0,1]$ *and* $\nu_A : X \times Y \to [0,1]$ *satisfy the property* $0 \le \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \le 1$. *Intuitionistic fuzzy matrix denoted and defined as* $A = (\left\langle a_{ij}, a'_{ij} \right\rangle)$ *such that* $a_{ij} + a'_{ij} \le 1$ *for all* i, j. *Set of all intuitionistic fuzzy matrices of order of order* $m \times n$ *is denoted by* F_{mn} *and* F_n *denotes the set of intuitionistic fuzzy matrix, order n.*

Definition 3. *[17] Pythagorean fuzzy matrix of size* $m \times n$ *is expressed as* $A = (\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle)$*,* $\mu_{a_{ij}} \in [0,1]$ *is membership value and* $\nu_{a_{ij}} \in [0,1]$ *non-membership value of the* ij^{th} *element with property:*
$0 \le \mu_{a_{ij}} + \nu_{a_{ij}} \le 1$ *for all i,j.*

Definition 4. *[33] A picture fuzzy matrix A of the form,*
$A = (\langle \mu_{a_{ij}}, \eta_{a_{ij}}, \nu_{a_{ij}} \rangle)$*,* $\mu_{a_{ij}}, \eta_{a_{ij}}, \nu_{a_{ij}} \in [0,1]$ *with the condition*
$0 \le \mu_{a_{ij}} + \eta_{a_{ij}} + \nu_{a_{ij}} \le 1 \ \forall$ *i,j. Where,* $\mu_{a_{ij}} \in [0,1]$ *is the membership degree,* $\eta_{a_{ij}} \in [0,1]$ *is neutral membership degree, and* $\nu_{a_{ij}} \in [0,1]$ *is non-membership degree.*

Definition 5. *[38] A spherical fuzzy matrix A of the form,* $A = (\langle \mu_{a_{ij}}, \eta_{a_{ij}}, \nu_{a_{ij}} \rangle)$ *of a non negative real numbers* $\mu_{a_{ij}}, \eta_{a_{ij}}, \nu_{a_{ij}} \in [0,1]$ *satisfying the condition*

$0 \leq \mu^2_{a_{ij}} + \eta^2_{a_{ij}} + \nu^2_{a_{ij}} \leq 1$ *for all i,j* $\mu_{a_{ij}} \in [0,1]$ *is called the degree of membership,*$\eta_{a_{ij}} \in [0,1]$ *is called the degree of neutral membership and* $\nu_{a_{ij}} \in [0,1]$*is called the degree of non-membership*

Definition 6. *[39] An Interval valued picture fuzzy matrix A is defined as*
$A = (a_{ij}) = (\langle a_{ij\mu}, a_{ij\eta}, a_{ij\upsilon}\rangle), i = 1,2,...m, j = 1,2,...n$
where $a_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \in [0,1]$, $a_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \in [0,1]$ *and* $a_{ij\upsilon} = [a_{ij\upsilon L}, a_{ij\upsilon U}] \in [0,1]$
with the condition
$(a_{ij\mu U}) + (a_{ij\eta U}) + (a_{ij\upsilon U}) \leq 1$ *where* $a_{ij\mu}, a_{ij\eta}$ *and* $a_{ij\upsilon}$ *are the membership , neutral membership and non-membership degree of* a_{ij}

3. Interval valued spherical fuzzy matrix

We define an interval valued spherical fuzzy matrix (IVSFM) and its basic concepts by generalizing and extending the concept of spherical fuzzy matrix.

Definition 7. *An IVSFM A is expressed as*
$A = (a_{ij}) = (\langle a_{ij\mu}, a_{ij\eta}, a_{ij\upsilon}\rangle), i = 1,2,...m, j = 1,2,...n$
where $a_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \in [0,1]$, $a_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \in [0,1]$ *and* $a_{ij\upsilon} = [a_{ij\upsilon L}, a_{ij\upsilon U}] \in [0,1]$
with the condition
$(a_{ij\mu U})^2 + (a_{ij\eta U})^2 + (a_{ij\upsilon U})^2 \leq 1$ *where,* $a_{ij\mu}$ *is membership,* $a_{ij\eta}$ *is neutral membership and* $a_{ij\upsilon}$ *is non-membership degree.*

Definition 8. *An IVSFM is called a square interval valued spherical fuzzy matrix (SIVSFM) if it has the same number of rows and number columns.*

Definition 9. *Let A and B be two IVSFM, such that*
$A = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\upsilon L}, a_{ij\upsilon U}])$ *and*
$B = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\upsilon L}, b_{ij\upsilon U}])$.
Then, $A \leq B$ *if* $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}$*;* $a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}$*;* $a_{ij\upsilon L} \geq b_{ij\upsilon L}, a_{ij\upsilon U} \geq b_{ij\upsilon U}$.

Definition 10. *An IVSFM A is said to be a null matrix if*
$[a_{ij\mu L}, a_{ij\mu U}] = [0,1], [a_{ij\eta L}, a_{ij\eta U}] = [0,1]$ *and* $[a_{ij\upsilon L}, a_{ij\upsilon U}] = [0,1]$ *for all* $i = 1,2,...m$ $j = 1,2,...n$

Definition 11. *An IVSFM A is said to be* μ *universal if* $[\mu_{ijL}, \mu_{ijU}] = [1,0], [\eta_{ijL}, \eta_{ijU}] = [0,1]$ *and* $[\upsilon_{ijL}, \upsilon_{ijU}] = [0,1]$ *for all* $i = 1,2,...m$ $j = 1,2,...n$

Definition 12. *An IVSFM A is said to be* η *universal if* $[\mu_{ijL}, \mu_{ijU}] = [0,1], [\eta_{ijL}, \eta_{ijU}] = [1,0]$ *and* $[\upsilon_{ijL}, \upsilon_{ijU}] = [0,1]$ *for all* $i = 1,2,...m$ $j = 1,2,...n$

Definition 13. *An IVSFM A is said to be* υ *universal if* $[\mu_{ijL}, \mu_{ijU}] = [0,1], [\eta_{ijL}, \eta_{ijU}] = [0,1]$ *and* $[\upsilon_{ijL}, \upsilon_{ijU}] = [1,0]$ *for all* $i = 1,2,...m$ $j = 1,2,...n$

3.1. Some Basic operations of IVSFM

Definition 14. *Let A and B be two IVSFM, of order $m \times n$,*
$A = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijv L}, a_{ijv U}])$ *and*
$B = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijv L}, b_{ijv U}])$*. Then,*
1. $A^c = ([a_{ijvL}, a_{ijvU}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\mu L}, a_{ij\mu U}])$
2. $A \vee B =$
$([max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})]$
$[min(a_{ij\eta L}, b_{ij\eta L})], [min(a_{ij\eta U}, b_{ij\eta U})][min(a_{ijvL}, b_{ijvL})][min(a_{ijvU}, b_{ijvU})])$
3. $A \wedge B =$
$([min(a_{ij\mu L}, b_{ij\mu L}), min(a_{ij\mu U}, b_{ij\mu U})]$
$[min(a_{ij\eta L}, b_{ij\eta L})], [min(a_{ij\eta U}, b_{ij\eta U})][max(a_{ijvL}, b_{ijvL})][max(a_{ijvU}, b_{ijvU})])$
4. $A^T = ([a_{jivL}, a_{jivU}], [a_{ji\eta L}, a_{ji\eta U}], [a_{ji\mu L}, a_{ji\mu U}])$
5. $A \oplus B =$
$([\sqrt{{a_{ij\mu L}}^2 + {b_{ij\mu L}}^2 - {a_{ij\mu L}}^2 {b_{ij\mu L}}^2}, \sqrt{{a_{ij\mu U}}^2 + {b_{ij\mu U}}^2 - {a_{ij\mu U}}^2 {b_{ij\mu U}}^2}]$
$[a_{ij\eta L} b_{ij\eta L}, a_{ij\eta U} b_{ij\eta U}][a_{ijvL} b_{ijvL}, a_{ijvU} b_{ijvU}])$ *6.* $A \otimes B =$
$([a_{ij\mu L} b_{ij\mu L}, a_{ij\mu U} b_{ij\mu U}][\sqrt{{a_{ij\eta L}}^2 + {b_{ij\eta L}}^2 - {a_{ij\eta L}}^2 {b_{ij\eta L}}^2}, \sqrt{{a_{ij\eta U}}^2 + {b_{ij\eta U}}^2 - {a_{ij\eta U}}^2 {b_{ij\eta U}}^2}]$
$[\sqrt{{a_{ijvL}}^2 + {b_{ijvL}}^2 - {a_{ijvL}}^2 {b_{ijvL}}^2}, \sqrt{{a_{ijvU}}^2 + {b_{ijvU}}^2 - {a_{ijvU}}^2 {b_{ijvU}}^2}])$
7. $A^n =$
$([[(a_{ij\mu L})^n, (a_{ij\mu U})^n][\sqrt{1 - (1 - (a_{ij\mu L})^2)^n}, \sqrt{1 - (1 - (a_{ij\mu U})^2)^n}]$
$[\sqrt{1 - (1 - (a_{ijvL})^2)^n}, \sqrt{1 - (1 - (a_{ijvU})^2)^n}]])$

Theorem 1. *Suppose* $A = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$,
$B = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ *and*
$C = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$ *are three IVSFM of same order order $m \times n$ then*
1. $A \vee B = B \vee A$
2. $A \wedge B = B \wedge A$
3. $(A^T)^T = A$
4. $(A^c)^T$
5. $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
6. $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
7. $A \oplus B = B \oplus A$
8. $A \otimes B = B \otimes A$
9. $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
10. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$
11. (a) $A \otimes (B \oplus C) \neq A \otimes B) \oplus (A \otimes C)$
(b) $(B \oplus C) \otimes C \neq (B \otimes A) \oplus (C \otimes A)$

Proof. 1. Let $A \vee B =$
$([max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})][min(a_{ij\eta L}, b_{ij\eta L})], [min(a_{ij\eta U}, b_{ij\eta U})]$
$[min(a_{ijvL}, b_{ijvL})][min(a_{ijvU}, b_{ijvU})])$
$= ([max(b_{ij\mu L}, a_{ij\mu L}), max(b_{ij\mu U}, a_{ij\mu U})][min(b_{ij\eta L}, a_{ij\eta L})], [min(b_{ij\eta U}, a_{ij\eta U})]$

$[min(b_{ij\upsilon L}, a_{ij\upsilon L})][min(b_{ij\upsilon U}, a_{ij\upsilon U})])$
$= B \vee A$
2. Let $A \wedge B =$
$([min(a_{ij\mu L}, b_{ij\mu L}), min(a_{ij\mu U}, b_{ij\mu U})][min(a_{ij\eta L}, b_{ij\eta L})], [min(a_{ij\eta U}, b_{ij\eta U})]$
$[max(a_{ij\upsilon L}, b_{ij\upsilon L})][max(a_{ij\upsilon U}, b_{ij\upsilon U})])$
$= ([min(b_{ij\mu L}, a_{ij\mu L}), min(b_{ij\mu U}, a_{ij\mu U})][min(b_{ij\eta L}, a_{ij\eta L})], [min(b_{ij\eta U}, a_{ij\eta U})]$
$[max(b_{ij\upsilon L}, a_{ij\upsilon L})][max(b_{ij\upsilon U}, a_{ij\upsilon U})])$
$= B \vee A$
3. $Let(A^T) = ([a_{ji\upsilon L}, a_{ji\upsilon U}], [a_{ji\eta L}, a_{ji\eta U}], [a_{ji\mu L}, a_{ji\mu U}])$
Now, $(A^T)^T = ([a_{ji\upsilon L}, a_{ji\upsilon U}], [a_{ji\eta L}, a_{ji\eta U}], [a_{ji\mu L}, a_{ji\mu U}])^T$
$= ([a_{ij\upsilon L}, a_{ij\upsilon U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\mu L}, a_{ij\mu U}])$.
The properties 4,5 and 6 are straightforward.
7. $A \oplus B = ([\sqrt{{a_{ij\mu L}}^2 + {b_{ij\mu L}}^2 - {a_{ij\mu L}}^2 {b_{ij\mu L}}^2}, \sqrt{{a_{ij\mu U}}^2 + {b_{ij\mu U}}^2 - {a_{ij\mu U}}^2 {b_{ij\mu U}}^2}]$
$[a_{ij\eta L} b_{ij\eta L}, a_{ij\eta U} b_{ij\eta U}][a_{ij\upsilon L} b_{ij\upsilon L}, a_{ij\upsilon U} b_{ij\upsilon U}])$
$= ([\sqrt{{b_{ij\mu L}}^2 + {a_{ij\mu L}}^2 - {b_{ij\mu L}}^2 {a_{ij\mu L}}^2}, \sqrt{{b_{ij\mu U}}^2 + {a_{ij\mu U}}^2 - {b_{ij\mu U}}^2 {a_{ij\mu U}}^2}]$
$[b_{ij\eta L} a_{ij\eta L}, b_{ij\eta U} a_{ij\eta U}][b_{ij\upsilon L} a_{ij\upsilon L}, b_{ij\upsilon U} a_{ij\upsilon U}])$
$= B \oplus A$
8. $A \otimes B = ([a_{ij\mu L} b_{ij\mu L}, a_{ij\mu U} b_{ij\mu U}]$
$[\sqrt{{a_{ij\eta L}}^2 + {b_{ij\eta L}}^2 - {a_{ij\eta L}}^2 {b_{ij\eta L}}^2}, \sqrt{{a_{ij\eta U}}^2 + {b_{ij\eta U}}^2 - {a_{ij\eta U}}^2 {b_{ij\eta U}}^2}]$
$[\sqrt{{a_{ij\upsilon L}}^2 + {b_{ij\upsilon L}}^2 - {a_{ij\upsilon L}}^2 {b_{ij\upsilon L}}^2}, \sqrt{{a_{ij\upsilon U}}^2 + {b_{ij\upsilon U}}^2 - {a_{ij\upsilon U}}^2 {b_{ij\upsilon U}}^2}])$
$= ([b_{ij\mu L} a_{ij\mu L}, b_{ij\mu U} a_{ij\mu U}]$
$[\sqrt{{b_{ij\eta L}}^2 + {a_{ij\eta L}}^2 - {b_{ij\eta L}}^2 {a_{ij\eta L}}^2}, \sqrt{{b_{ij\eta U}}^2 + {a_{ij\eta U}}^2 - {b_{ij\eta U}}^2 {a_{ij\eta U}}^2}]$
$[\sqrt{{b_{ij\upsilon L}}^2 + {a_{ij\upsilon L}}^2 - {b_{ij\upsilon L}}^2 {a_{ij\upsilon L}}^2}, \sqrt{{b_{ij\upsilon U}}^2 + {a_{ij\upsilon U}}^2 - {b_{ij\upsilon U}}^2 {a_{ij\upsilon U}}^2}])$
$= B \otimes A$
9. $A \oplus (B \oplus C) =$
$([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\upsilon L}, a_{ij\upsilon U}])\oplus$
$([\sqrt{{b_{ij\mu L}}^2 + {c_{ij\mu L}}^2 - {b_{ij\mu L}}^2 {c_{ij\mu L}}^2}, \sqrt{{b_{ij\mu U}}^2 + {c_{ij\mu U}}^2 - {b_{ij\mu U}}^2 {c_{ij\mu U}}^2}]$
$[b_{ij\eta L} c_{ij\eta L}, b_{ij\eta U} c_{ij\eta U}][b_{ij\upsilon L} c_{ij\upsilon L}, b_{ij\upsilon U} c_{ij\upsilon U}])$
$= ([\sqrt{{a_{ij\mu L}}^2 + {b_{ij\mu L}}^2 - {a_{ij\mu L}}^2 {b_{ij\mu L}}^2}, \sqrt{{a_{ij\mu U}}^2 + {b_{ij\mu U}}^2 - {a_{ij\mu U}}^2 {b_{ij\mu U}}^2}]$
$[a_{ij\eta L} b_{ij\eta L}, a_{ij\eta U} b_{ij\eta U}][a_{ij\upsilon L} b_{ij\upsilon L}, a_{ij\upsilon U} b_{ij\upsilon U}])\oplus$
$([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ij\upsilon L}, c_{ij\upsilon U}])$
$= (A \oplus B) \oplus C$
10. Similarly $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ can be proved.
11. (a) $A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C)$
Let $(B \oplus C) = ([\sqrt{{b_{ij\mu L}}^2 + {c_{ij\mu L}}^2 - {b_{ij\mu L}}^2 {c_{ij\mu L}}^2}, \sqrt{{b_{ij\mu U}}^2 + {c_{ij\mu U}}^2 - {b_{ij\mu U}}^2 {c_{ij\mu U}}^2}]$
$[b_{ij\eta L} c_{ij\eta L}, b_{ij\eta U} c_{ij\eta U}][b_{ij\upsilon L} c_{ij\upsilon L}, b_{ij\upsilon U} c_{ij\upsilon U}])$
Now, $A \otimes (B \oplus C) =$
$([a_{ij\mu L}(\sqrt{{b_{ij\mu L}}^2 + {c_{ij\mu L}}^2 - {b_{ij\mu L}}^2 {c_{ij\mu L}}^2}), a_{ij\mu U}(\sqrt{{b_{ij\mu U}}^2 + {c_{ij\mu U}}^2 - {b_{ij\mu U}}^2 {c_{ij\mu U}}^2})]$

$[\sqrt{a_{ij\eta L}{}^2 + b_{ij\eta L}c_{ij\eta L}{}^2 - a_{ij\eta L}{}^2}b_{ij\eta L}c_{ij\eta L}{}^2, \sqrt{a_{ij\eta U}{}^2 + b_{ij\eta U}c_{ij\eta U}{}^2 - a_{ij\eta U}{}^2}b_{ij\eta U}c_{ij\eta U}{}^2]$
$[\sqrt{a_{ij\upsilon L}{}^2 + b_{ij\eta L}c_{ij\eta L}{}^2 - a_{ij\upsilon L}{}^2}b_{ij\upsilon L}c_{ij\upsilon L}{}^2, \sqrt{a_{ij\upsilon U}{}^2 + b_{ij\upsilon U}c_{ij\upsilon U}{}^2 - a_{ij\upsilon U}{}^2}b_{ij\upsilon U}c_{ij\upsilon U}{}^2])$
$A \otimes B =$
$([a_{ij\mu L}b_{ij\mu L}, a_{ij\mu U}b_{ij\mu U}][\sqrt{a_{ij\eta L}{}^2 + b_{ij\eta L}{}^2 - a_{ij\eta L}{}^2 b_{ij\eta L}{}^2}, \sqrt{a_{ij\eta U}{}^2 + b_{ij\eta U}{}^2 - a_{ij\eta U}{}^2 b_{ij\eta U}{}^2}]$
$[\sqrt{a_{ij\upsilon L}{}^2 + b_{ij\upsilon L}{}^2 - a_{ij\upsilon L}{}^2 b_{ij\upsilon L}{}^2}, \sqrt{a_{ij\upsilon U}{}^2 + b_{ij\upsilon U}{}^2 - a_{ij\upsilon U}{}^2 b_{ij\upsilon U}{}^2}])$
$A \otimes C =$
$([a_{ij\mu L}c_{ij\mu L}, a_{ij\mu U}c_{ij\mu U}][\sqrt{a_{ij\eta L}{}^2 + c_{ij\eta L}{}^2 - a_{ij\eta L}{}^2 c_{ij\eta L}{}^2}, \sqrt{a_{ij\eta U}{}^2 + c_{ij\eta U}{}^2 - a_{ij\eta U}{}^2 c_{ij\eta U}{}^2}]$
$[\sqrt{a_{ij\upsilon L}{}^2 + c_{ij\upsilon L}{}^2 - a_{ij\upsilon L}{}^2 c_{ij\upsilon L}{}^2}, \sqrt{a_{ij\upsilon U}{}^2 + c_{ij\upsilon U}{}^2 - a_{ij\upsilon U}{}^2 c_{ij\upsilon U}{}^2}])$
$(A \otimes B) \oplus (A \otimes C) =$
$([a_{ij\mu L}\sqrt{b_{ij\mu L}{}^2 + c_{ij\mu L}{}^2 - b_{ij\mu L}{}^2 c_{ij\mu L}{}^2}, a_{ij\mu U}\sqrt{b_{ij\mu U}{}^2 + c_{ij\mu U}{}^2 - b_{ij\mu U}{}^2 c_{ij\mu U}{}^2}])$
$[(\sqrt{a_{ij\eta L}{}^2 + b_{ij\eta L}{}^2 - a_{ij\eta L}{}^2 b_{ij\eta L}{}^2})(\sqrt{a_{ij\eta L}{}^2 + c_{ij\eta L}{}^2 - a_{ij\eta L}{}^2 c_{ij\eta L}{}^2}),$
$(\sqrt{a_{ij\eta U}{}^2 + b_{ij\eta U}{}^2 - a_{ij\eta U}{}^2 b_{ij\eta U}{}^2})(\sqrt{a_{ij\eta U}{}^2 + c_{ij\eta}{}^2 - a_{ij\eta U}{}^2 c_{ij\eta U}{}^2})],$
$[(\sqrt{a_{ij\upsilon L}{}^2 + b_{ij\upsilon L}{}^2 - a_{ij\upsilon L}{}^2 b_{ij\upsilon L}{}^2})(\sqrt{a_{ij\upsilon L}{}^2 + c_{ij\upsilon L}{}^2 - a_{ij\upsilon L}{}^2 c_{ij\upsilon L}{}^2}),$
$(\sqrt{a_{ij\upsilon U}{}^2 + b_{ij\upsilon U}{}^2 - a_{ij\upsilon U}{}^2 b_{ij\upsilon U}{}^2})(\sqrt{a_{ij\upsilon U}{}^2 + c_{ij\upsilon}{}^2 - a_{ij\upsilon U}{}^2 c_{ij\upsilon U}{}^2})]$
So, $A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C)$
(b) Similar manner, we can prove $(B \oplus C) \otimes C \neq (B \otimes A) \oplus (C \otimes A)$

Theorem 2. *Let A and B be IVSFM of size n, then*
(1) $(A \vee B)^c = A^c \wedge B^c$
(2) $(A \wedge B)^c = A^c \vee B^c$

Proof. (1) Let $A = ([(a_{ij\mu L}, a_{ij\mu U})], [(a_{ij\eta L}, a_{ij\eta U})], [(a_{ij\upsilon L}, a_{ij\upsilon U})])$,
$B = ([(b_{ij\mu L}, b_{ij\mu U})], [(b_{ij\eta L}, b_{ij\eta U})], [(b_{ij\upsilon L}, b_{ij\upsilon U})])$ Then,
$A^c = ([(a_{ij\upsilon L}, a_{ij\upsilon U})], [(a_{ij\eta L}, a_{ij\eta L}), (a_{ij\mu L}, a_{ij\mu L})], [(a_{ij\mu L}, a_{ij\mu U})])$,
$B^c = ([(b_{ij\upsilon L}, b_{ij\upsilon U})], [(b_{ij\eta L}, b_{ij\eta L}), (b_{ij\mu L}, b_{ij\mu L})], [(b_{ij\mu L}, b_{ij\mu U})])$,
Let $A^c \wedge B^c = ([min(a_{ij\upsilon L}, b_{ij\upsilon L}), min(a_{ij\upsilon U}, b_{ij\upsilon U})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta U}, b_{ij\eta U})]$
$[max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})])$
$A \wedge B = ([max(a_{ij\upsilon L}, b_{ij\upsilon L}), max(a_{ij\upsilon U}, b_{ij\upsilon U})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta U}, b_{ij\eta U})]$
$[min(a_{ij\mu L}, b_{ij\mu L}), min(a_{ij\mu U}, b_{ij\mu U})])$ then,
$(A \wedge B)^c = ([min(a_{ij\upsilon L}, b_{ij\upsilon L}), min(a_{ij\upsilon U}, b_{ij\upsilon U})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta U}, b_{ij\eta U})]$
$[max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})])$
(2) Can be proved similarly.

Theorem 3. *Let A, B and C be IVSFM of size* $m \times n$ *and* $A \leq C$ *and* $B \leq C$*, then* $A \vee B \leq C$

Proof. Let $A = ([(a_{ij\mu L}, a_{ij\mu U})], [(a_{ij\eta L}, a_{ij\eta U})], [(a_{ij\upsilon L}, a_{ij\upsilon U})])$,
$B = ([(b_{ij\mu L}, b_{ij\mu U})], [(b_{ij\eta L}, b_{ij\eta U})], [(b_{ij\upsilon L}, b_{ij\upsilon U})])$
$C = ([(c_{ij\mu L}, c_{ij\mu U})], [(c_{ij\eta L}, c_{ij\eta U})], [(c_{ij\upsilon L}, c_{ij\upsilon U})])$

If $A \leq C$ then $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}, a_{ij\eta L} \leq C_{ij\eta L}, a_{ij\eta U} \leq a_{ij\eta U}, a_{ijvL}$ $\geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$ ∀ i,j, and
$B \leq C$ then $b_{ij\mu L} \leq c_{ij\mu L}, b_{ij\mu U} \leq c_{ij\mu U}, b_{ij\eta L}$ $\leq C_{ij\eta L}, b_{ij\eta U} \leq b_{ij\eta U}, b_{ijvL} \geq c_{ijvL}, b_{ijvU} \geq c_{ijvU}$ ∀ i,j. Now ,
$max(a_{ij\mu L}, b_{ij\mu L}) \leq c_{ij\mu L}, max(a_{ij\mu U}, b_{ij\mu U}) \leq c_{ij\mu U}, min(a_{ij\eta L}, b_{ij\eta L}) \leq c_{ij\eta L},$
$min(a_{ij\eta U}, b_{ij\mu U}) \leq c_{ij\eta U}, min(a_{ijvL}, b_{ijvL}) \geq c_{ijvL}, min(a_{ijvU}, b_{ijvU}) \geq c_{ijvU}.$
Hence $A \vee B \leq C$ (by def.3.3).

Theorem 4. *Let A, B and C be IVSFM of size* $m \times n$ *and* $A \leq B$, *then* $A \vee C \leq B \vee C$

Proof. Let $A = ([(a_{ij\mu L}, a_{ij\mu U})], [(a_{ij\eta L}, a_{ij\eta U})], [(a_{ijvL}, a_{ijvU})]),$
$B = ([(b_{ij\mu L}, b_{ij\mu U})], [(b_{ij\eta L}, b_{ij\eta U})], [(b_{ijvL}, b_{ijvU})])$
$C = ([(c_{ij\mu L}, c_{ij\mu U})], [(c_{ij\eta L}, c_{ij\eta U})], [(c_{ijvL}, c_{ijvU})]).$
If $A \leq B$ then, $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}, a_{ijvL}$ $\geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$, now $max(a_{ij\mu L}, C_{ij\mu L}) \leq max(b_{ij\mu L}, c_{ij\mu L}), max(a_{ij\mu U}, c_{ij\mu U})$ $\leq max(b_{ij\mu U}, c_{ij\mu U}), min(a_{ij\eta L}, c_{ij\eta L})$ $\leq min(a_{ij\eta L}, c_{ij\eta L}), min(a_{ij\eta U}, c_{ij\mu U}) \leq min(b_{ij\eta U}, c_{ij\eta U}), min(a_{ijvL}, c_{ijvL})$ $\geq min(b_{ijvL}, c_{ijvL}), min(a_{ijvU}, c_{ijvU}) \geq min(b_{ijvU}, c_{ijvU})$ ∀ i,j.
Hence $A \vee C \leq B \vee C$.

Theorem 5. *Let A, B and C be IVSFM of size* $m \times n$ *and* $C \leq A$ *and* $C \leq B$, *then* $C \leq A \wedge B$

Proof. Followed from Theorem 3.12.

Theorem 6. *Let A, B and C be IVSFM of size* $m \times n$ *and* $A \leq B$ *and* $A \leq C$ *and* $B \wedge C = 0$ *then, then* $A = 0$

Proof. If $A \leq B$ then, $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}, a_{ijvL} \geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$. $A \leq C$ then, $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}, a_{ij\eta L} \leq c_{ij\eta L}, a_{ij\eta U} \leq c_{ij\eta U}, a_{ijvL} \geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$. Hence by theorem 3.13 $A \leq B \wedge C, B \wedge C = 0$ such that $A = 0$.

Theorem 7. *Let A, B and C be IVSFM of size* $m \times n$ *and* $A \leq B$ *then* $A \leq C \leq B \wedge C$.

Proof. The proof follows from Definition 3.3.

Theorem 8. *Let A, B and C be IVSFM of size* $m \times n$ *and* $A \leq B$ *and* $B \leq C = 0$ *and then* $A \wedge C = 0$.

Proof. The proof follows from Theorem 3.15.

4. Determinant and adjoint of interval valued spherical fuzzy matrix

This section focus on the determinant and adjoint of the IVSFM, illustrated with the help of examples, and some fundamental properties are also examined.

Definition 15. *Determinant of IVSFM, assume* $A = ([(a_{ij\mu L}, a_{ij\mu U})], [(a_{ij\eta L}, a_{ij\eta U})], [(a_{ij\upsilon L}, a_{ij\upsilon U})])$, *is an IVSFM of size n. The determinant of A is denoted by* $|A|$ *and expressed as*
$|A| = (\vee_{h\in P_k}([a_1h(1)\mu L, a_1h(1)\mu U] \wedge [a_2h(2)\mu L, a_2h(2)\mu U] ... \wedge [a_kh(k)\mu L, a_kh(k)\mu U]),$
$\wedge_{h\in P_k}([a1h(1)\eta L, a_1h(1)\eta U] \wedge [a_2h(2)\eta L, a_2h(2)\eta U] ... \wedge [a_kh(k)\eta L, a_kh(k)\eta U]),$
$\wedge_{h\in P_k}([a_1h(1)\upsilon L, a_1h(1)\upsilon U] \wedge [a_2h(2)\upsilon L, a_2h(2)\upsilon U] ... \wedge [a_kh(k)\upsilon L, a_kh(k)\upsilon U]).$
Let, P_K *denote set of permutation of the set* $\{1, 2, 3, ...n\}$.

Example 1. *Consider an IVSFM of size 3*
$$|A| = \begin{pmatrix} [0.30,0.70][0.40,0.20][0.20,0.30] & [0.40,0.60][0.45,0.40][0.21,0.30] & [0.71,0.20][0.62,0.10][0.10,0.80] \\ [0.32,0.02][0.72,0.20][0.33,0.40] & [0.23,0.60][0.40,0.50][0.21,0.10] & [0.71,0.20][0.62,0.20][0.30,0.70] \\ [0.36,0.60][0.53,0.20][0.65,0.30] & [0.40,0.20][0.25,0.70][0.21,0.50] & [0.15,0.20][0.62,0.20][0.18,0.40] \end{pmatrix}$$
to calculate the determinant of A, we need to determine all the permutations of the $\{1, 2, 3\}$.
The permutations on $\{1, 2, 3\}$ *are*
$\phi_1 = I, \phi_2 = (23), \phi_3 = (12), \phi_4 = (13), \phi_5 = (123), \phi_6 = (132)$
The membership component of $|A| =$
$([a_1\phi_1(1)\mu L, a_1\phi_1(1)\mu U] \wedge [a_2\phi_1(2)\mu L, a_1\phi_1(2)\mu U]\wedge$
$[a_3\phi_1(3)\mu L, a_3\phi_1(3)\mu U]) \vee ([a_1\phi_2(1)\mu L, a_1\phi_2(1)\mu U] \wedge [a_2\phi_2(2)\mu L, a_1\phi_2(2)\mu U]\wedge$
$[a_3\phi_2(3)\mu L, a_3\phi_2(3)\mu U]) \vee ([a_1\phi_3(1)\mu L, a_3\phi_3(1)\mu U]\wedge$
$[a_2\phi_3(2)\mu L, a_1\phi_3(2)\mu U] \wedge [a_3\phi_3(3)\mu L, a_3\phi_3(3)\mu U]) \vee ([a_1\phi_4(1)\mu L, a_1\phi_4(1)\mu U]$
$\wedge[a_2\phi_4(2)\mu L, a_1\phi_4(2)\mu U] \wedge [a_3\phi_4(3)\mu L, a_3\phi_4(3)\mu U]) \vee ([a_1\phi_5(1)\mu L, a_1\phi_5(1)\mu U]$
$\wedge[a_2\phi_5(2)\mu L, a_1\phi_5(2)\mu U] \wedge [a_3\phi_5(3)\mu L, a_3\phi_5(3)\mu U]) \vee ([a_1\phi_6(1)\mu L, a_1\phi_6(1)\mu U]\wedge$
$[a_2\phi_6(2)\mu L, a_1\phi_6(2)\mu U] \wedge [a_3\phi_6(3)\mu L, a_3\phi_6(3)\mu U])$
$= ([a_{11}\mu L, a_{11}\mu U] \wedge [a_{22}\mu L, a_{22}\mu U][a_{33}\mu L, a_{33}\mu U]) \vee ([a_{11}\mu L, a_{11}\mu U] \wedge [a_{23}\mu L, a_{23}\mu U] \wedge$
$[a_{32}\mu L, a_{32}\mu U])$
$\vee([a_{12}\mu L, a_{12}\mu U] \wedge [a_{21}\mu L, a_{21}\mu U] \wedge [a_{33}\mu L, a_{33}\mu U]) \vee ([a_{12}\mu L, a_{12}\mu U] \wedge [a_{23}\mu L, a_{23}\mu U] \wedge$
$[a_{31}\mu L, a_{31}\mu U]) \vee ([a_{13}\mu L, a_{13}\mu U] \wedge [a_{21}\mu L, a_{21}\mu U] \wedge [a_{32}\mu L, a_{32}\mu U]) \vee [a_{13}\mu L, a_{13}\mu U] \wedge$
$([a_{22}\mu L, a_{22}\mu U] \wedge [a_{31}\mu L, a_{31}\mu U])$
$= ([0.30, 0.70]\wedge[0.23, 0.60]\wedge[0.15, 0.20])\vee([0.30, 0.70]\wedge[0.71, 0.20]\wedge[0.40, 0.20])\vee([0.40, 0.60]\wedge$
$[0.32, 0.02]\wedge[0.15, 0.20])\vee([0.40, 0.60]\wedge[0.71, 0.20]\wedge[0.36, 0.60])\vee([0.71, 0.20]\wedge[0.32, 0.02]\wedge$
$[0.40, 0.20]) \vee ([0.71, 0.20] \wedge [0.23, 0.60] \wedge [0.36, 0.60])$
$= [0.15, 0.20] \vee [0.30, 0.20] \vee [0.15, 0.02] \vee [0.36, 0.20] \vee [0.32, 0.02] \vee [0.23, 0.20]$
$= [0.36, 20]$
The neutral component of $|A|$
$= ([a_1\phi_1(1)\eta L, a_1\phi_1(1)\eta U] \wedge [a_2\phi_1(2)\eta L, a_1\phi_1(2)\eta U] \wedge [a_3\phi_1(3)\eta L, a_3\phi_1(3)\eta U])\vee$
$([a_1\phi_2(1)\eta L, a_1\phi_2(1)\eta U] \wedge [a_2\phi_2(2)\eta L, a_1\phi_2(2)\eta U] \wedge [a_3\phi_2(3)\eta L, a_3\phi_2(3)\eta U])\vee$
$([a_1\phi_3(1)\eta L, a_3\phi_3(1)\eta U] \wedge [a_2\phi_3(2)\eta L, a_1\phi_3(2)\eta U] \wedge [a_3\phi_3(3)\eta L, a_3\phi_3(3)\eta U])\vee$
$([a_1\phi_4(1)\eta L, a_1\phi_4(1)\eta U] \wedge [a_2\phi_4(2)\eta L, a_1\phi_4(2)\eta U] \wedge [a_3\phi_4(3)\eta L, a_3\phi_4(3)\eta U])\vee$

$([a_1\phi_5(1)\eta L, a_1\phi_5(1)\eta U] \wedge [a_2\phi_5(2)\eta L, a_1\phi_5(2)\eta U] \wedge [a_3\phi_5(3)\eta L, a_3\phi_5(3)\eta U])\vee$
$([a_1\phi_6(1)\eta L, a_1\phi_6(1)\eta U] \wedge [a_2\phi_6(2)\eta L, a_1\phi_6(2)\eta U] \wedge [a_3\phi_6(3)\eta L, a_3\phi_6(3)\eta U])$
$= ([a_{11}\eta L, a_{11}\eta U]\wedge[a_{22}\eta L, a_{22}\eta U][a_{33}\eta L, a_{33}\eta U])\vee([a_{11}\eta L, a_{11}\eta U]\wedge[a_{23}\eta L, a_{23}\eta U]\wedge[a_{32}\eta L, a_{32}\eta U])\vee$
$([a_{12}\eta L, a_{12}\eta U]\wedge[a_{21}\eta L, a_{21}\eta U]\wedge[a_{33}\eta L, a_{33}\eta U])\vee([a_{12}\eta L, a_{12}\eta U]\wedge[a_{23}\eta L, a_{23}\eta U]\wedge[a_{31}\eta L, a_{31}\eta U])\vee$
$([a_{13}\eta L, a_{13}\eta U] \wedge [a_{21}\eta L, a_{21}\eta U] \wedge [a_{32}\eta L, a_{32}\eta U]) \vee ([a_{13}\eta L, a_{13}\eta U] \wedge [a_{22}\mu L, a_{22}\eta U] \wedge$
$[a_{31}\eta L, a_{31}\eta U])$
$= ([0.40, 0.20]\wedge[0.40, 0.50]\wedge[0.62, 0.20])\vee([0.40, 0.20]\wedge[0.62, 0.20]\wedge[0.62, 0.20])\vee([0.45, 0.40]\wedge$
$[0.72, 0.20]\wedge[0.62, 0.20])\vee([0.45, 0.40]\wedge[0.62, 0.20]\wedge[0.53, 0.20])\vee([0.62, 0.10]\wedge[0.72, 0.20]\wedge$
$[0.25, 0.70]) \vee ([0.62, 0.10] \wedge [0.40, 0.50] \wedge [0.53, 0.20)$
$= [0.40, 0.20] \vee [0.40, 0.20] \vee [0.45, 0.20] \vee [0.45, 0.20] \vee [0.25, 0.10] \vee [0.40, 0.10]$
$= [0.45, 20]$
The non membership component of $|A|$
$= ([a_1\phi_1(1)\upsilon L, a_1\phi_1(1)\upsilon U] \wedge [a_2\phi_1(2)\upsilon L, a_1\phi_1(2)\upsilon U] \wedge [a_3\phi_1(3)\upsilon L, a_3\phi_1(3)\upsilon U])\vee$
$([a_1\phi_2(1)\upsilon L, a_1\phi_2(1)\upsilon U] \wedge [a_2\phi_2(2)\upsilon L, a_1\phi_2(2)\upsilon U] \wedge [a_3\phi_2(3)\upsilon L, a_3\phi_2(3)\upsilon U])\vee$
$([a_1\phi_3(1)\upsilon L, a_3\phi_3(1)\upsilon U] \wedge [a_2\phi_3(2)\upsilon L, a_1\phi_3(2)\upsilon U] \wedge [a_3\phi_3(3)\upsilon L, a_3\phi_3(3)\upsilon U])\vee$
$([a_1\phi_4(1)\upsilon L, a_1\phi_4(1)\upsilon U] \wedge [a_2\phi_4(2)\upsilon L, a_1\phi_4(2)\upsilon U] \wedge [a_3\phi_4(3)\upsilon L, a_3\phi_4(3)\upsilon U])\vee$
$([a_1\phi_5(1)\upsilon L, a_1\phi_5(1)\upsilon U] \wedge [a_2\phi_5(2)\upsilon L, a_1\phi_5(2)\upsilon U] \wedge [a_3\phi_5(3)\upsilon L, a_3\phi_5(3)\upsilon U])\vee$
$([a_1\phi_6(1)\upsilon L, a_1\phi_6(1)\upsilon U] \wedge [a_2\phi_6(2)\upsilon L, a_1\phi_6(2)\upsilon U] \wedge [a_3\phi_6(3)\upsilon L, a_3\phi_6(3)\upsilon U])$
$= ([a_{11}\upsilon L, a_{11}\upsilon U] \wedge [a_{22}\upsilon L, a_{22}\upsilon U][a_{33}\upsilon L, a_{33}\upsilon U]) \vee ([a_{11}\upsilon L, a_{11}\upsilon U] \wedge [a_{23}\upsilon L, a_{23}\upsilon U] \wedge$
$[a_{32}\upsilon L, a_{32}\upsilon U]) \vee ([a_{12}\upsilon L, a_{12}\upsilon U] \wedge [a_{21}\upsilon L, a_{21}\upsilon U] \wedge [a_{33}\upsilon L, a_{33}\upsilon U]) \vee ([a_{12}\upsilon L, a_{12}\upsilon U] \wedge$
$[a_{23}\upsilon L, a_{23}\upsilon U] \wedge [a_{31}\upsilon L, a_{31}\upsilon U]) \vee ([a_{13}\upsilon L, a_{13}\upsilon U] \wedge [a_{21}\upsilon L, a_{21}\upsilon U] \wedge [a_{32}\upsilon L, a_{32}\upsilon U]) \vee$
$([a_{13}\upsilon L, a_{13}\upsilon U] \wedge [a_{22}\upsilon L, a_{22}\upsilon U] \wedge [a_{31}\upsilon L, a_{31}\upsilon U])$
$= ([0.20, 0.30]\wedge[0.21, 0.10]\wedge[0.18, 0.40)\vee([0.20, 0.30]\wedge[0.30, 0.70]\wedge[0.21, 0.50])\vee([0.21, 0.30]\wedge$
$[0.33, 0.40]\wedge[0.18, 0.40)\vee([0.21, 0.30]\wedge[0.30, 0.70]\wedge[0.65, 0.30])\vee([0.10, 0.80]\wedge[0.33, 0.40]\wedge$
$[0.21, 0.50]) \vee ([0.10, 0.80] \wedge [0.21, 0.10] \wedge [0.65, 0.30])$
$= [0.18, 0.10] \vee [0.20, 0.30] \vee [0.18, 0.30] \vee [0.21, 0.30] \vee [0.10, 0.40] \vee [0.10, 0.10]$
$= [0.21, 40]$

Definition 16. *Adjoint of IVSFM, Let* $A = ([(a_{ij\mu L}, a_{ij\mu U})], [(a_{ij\eta L}, a_{ij\eta U})], [(a_{ij\upsilon L}, a_{ij\upsilon U})])$, *be IVSFM of size n. So, the adjoint of A is denoted by* $adj([(a_{ij\mu L}, a_{ij\mu U})], [(a_{ij\eta L}, a_{ij\eta U})], [(a_{ij\upsilon L}, a_{ij\upsilon U})])$, *and defined as*
$adj([(a_{ij\mu L}, a_{ij\mu U})], [(a_{ij\eta L}, a_{ij\eta U})], [(a_{ij\upsilon L}, a_{ij\upsilon U})]) = S = ([s_{ij\mu}, s_{ij\eta}, s_{ij\upsilon}])$
$= ([(s_{ij\mu L}, s_{ij\mu U})], [(s_{ij\eta L}, s_{ij\eta U})], [(s_{ij\upsilon L}, s_{ij\upsilon U})]) = (\langle s_{ij}\mu, s_{ij}\eta, s_{ij}\upsilon\rangle)$ *where*
$s_{ij\mu} = \vee_{\delta\in Q_{m_j m_i}} \wedge_{u\in m_j} a_{u\delta(u)\mu}$
$s_{ij\eta} = \wedge_{\delta\in Q_{m_j m_i}} \wedge_{u\in m_j} a_{u\delta(u)\eta}$
$s_{ij\upsilon} = \wedge_{\delta\in Q_{m_j m_i}} \wedge_{u\in m_j} a_{u\delta(u)\upsilon}$

Example 2. *Consider an IVSFM of size 3.*

$$A = \begin{pmatrix} [0.30, 0.70][0.40, 0.20][0.20, 0.30] & [0.40, 0.60][0.45, 0.40][0.21, 0.30] & [0.71, 0.20][0.62, 0.10][0.10, 0.80] \\ [0.32, 0.02][0.72, 0.20][0.33, 0.40] & [0.23, 0.60][0.40, 0.50][0.21, 0.10] & [0.71, 0.20][0.62, 0.20][0.30, 0.70] \\ [0.36, 0.60][0.53, 0.20][0.65, 0.30] & [0.40, 0.20][0.25, 0.70][0.21, 0.50] & [0.15, 0.20][0.62, 0.20][0.18, 0.40] \end{pmatrix}$$

Let $j = 1$ *and* $i = 1$, *we have* $m_j = \{1, 2, 3\}-\{1\} = \{2, 3\}$ *and* $m_i = \{1, 2, 3\}-\{1\} = \{2, 3\}$.
The permutation of m_i *over* m_j *are* $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ *Now,*

$(a_{22\mu} \wedge a_{33\mu}) \vee (a_{23\mu} \wedge a_{32\mu}$
$= ([0.23, 0.60] \wedge [0.15, 0.20]) \vee ([0.71, 0.20] \wedge [0.40, 0.20])$
$= [0.15, 0.20] \vee [0.40, 0.20] = [0.40, 0.20]$
$(a_{22\eta} \wedge a_{33\eta}) \vee (a_{23\eta} \wedge a_{32\eta}$
$= ([0.40, 0.50] \wedge [0.62, 0.20]) \vee ([0.62, 0.20] \wedge [0.25, 0.70])$
$= [0.40, 0.20] \vee [0.25, 0.20] = [0.40, 0.20]$
$(a_{22\upsilon} \wedge a_{33\upsilon}) \vee (a_{23\upsilon} \wedge a_{32\upsilon}$
$= ([0.21, 0.10] \wedge [0.18, 0.40]) \vee ([0.30, 0.70] \wedge [0.21, 0.50])$
$= [0.18, 0.10] \vee [0.21, 0.50] = [0.21, 0.50]$
For $j = 1$ *and* $i = 2$, $m_j = \{1, 2, 3\} - \{1\} = \{2, 3\}$ *and* $m_i = \{1, 2, 3\} - \{2\} = \{1, 3\}$.
The permutation of m_i *over* m_j *are*
$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ *Now,*
$(a_{12\mu} \wedge a_{33\mu}) \vee (a_{13\mu} \wedge a_{32\mu}$
$= ([0.40, 0.60] \wedge [0.15, 0.20]) \vee [0.71, 0.20] \wedge [0.40, 0.20])$
$= [0.15, 0.20] \vee [0.40, 0.20] = [0.40, 0.20]$
$(a_{12\eta} \wedge a_{33\eta}) \vee (a_{13\eta} \wedge a_{32\eta}$
$= ([0.45, 0.40] \wedge [0.62, 0.20]) \vee ([0.62, 0.10] \wedge [0.25, 0.70])$
$= [0.45, 0.20] \vee [0.25, 0.10] = [0.45, 0.20]$
$(a_{12\upsilon} \wedge a_{33\upsilon}) \vee (a_{13\upsilon} \wedge a_{32\upsilon}$
$= ([0.21, 0.30] \wedge [0.18, 0.40]) \vee ([0.30, 0.70] \wedge [0.10, 0.80])$
$= [0.18, 0.30] \vee [0.10, 0.70] = [0.18, 0.70]$
For $j = 1$ *and* $i = 3$, $m_j = \{1, 2, 3\} - \{1\} = \{2, 3\}$ *and* $m_i = \{1, 2, 3\} - \{3\} = \{1, 3\}$. *The permutation of* m_i *over* m_j *are*
$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ *Now,*
$(a_{12\mu} \wedge a_{23\mu}) \vee (a_{13\mu} \wedge a_{22\mu}$
$= ([0.40, 0.60] \wedge [0.71, 0.20]) \vee [0.71, 0.20] \wedge [0.23, 0.60])$
$= [0.40, 0.20] \vee [0.23, 0.20] = [0.40, 0.20]$
$(a_{12\eta} \wedge a_{23\eta}) \vee (a_{13\eta} \wedge a_{22\eta}$
$= ([0.45, 0.40] \wedge [0.62, 0.20]) \vee ([0.62, 0.10] \wedge [0.40, 0.50])$
$= [0.45, 0.20] \vee [0.40, 0.10] = [0.45, 0.20]$
$(a_{12\upsilon} \wedge a_{23\upsilon}) \vee (a_{13\upsilon} \wedge a_{22\upsilon}$
$= ([0.21, 0.30] \wedge [0.30, 0.70]) \vee ([0.30, 0.70] \wedge [0.21, 0.10])$
$= [0.21, 0.30] \vee [0.21, 0.10] = [0.21, 0.30]$
Similar manner, we can find other values as well, Adjoint(A) is obtained as
Adjoint(A)=

$$\begin{pmatrix} [0.40, 0.20][0.40, 0.20][0.21, 0.50] & [0.40, 0.20][0.40, 0.20][0.18, 0.70] & [0.40, 0.20][0.40, 0.20][0.21, 0.30] \\ [0.40, 0.20][0.62, 0.20][0.21, 0.50] & [0.40, 0.20][0.40, 0.20][0.18, 0.50] & [0.30, 0.20][0.40, 0.20][0.20, 0.30] \\ [0.40, 0.20][0.25, 0.20][0.33, 0.50] & [0.40, 0.20][0.25, 0.20][0.20, 0.50] & [0.23, 0.60][0.40, 0.20][0.20, 0.10] \end{pmatrix}$$

In the next section, we introduce eigen interval valued spherical fuzzy sets and propose the algorithms to determine least and greatest eigen interval valued spherical fuzzy sets.

5. Algorithms for eigen interval valued spherical fuzzy set

In this section , we propose the EIVSFS and provide algorithms to determine the GEIVSFS and LEIVSFS with the help of illustration.

Definition 17. *An interval valued spherical fuzzy relation (IVSFR) between IVSFS X and Y expressed as:* $R = \{((x,y).\mu_R(x,y), \eta_R(x,y), \upsilon_R(x,y)) : x \in X, y \in Y\}$, *where* $\mu_R = [\mu_R^L, \mu_R^U], \eta_R = [\eta_R^L, \eta_R^U], \upsilon_R = [\upsilon_R^L, \upsilon_R^U]$*, such that* $0 \le (\mu_R^U)^2 + (\eta_R^U)^2 + (\upsilon_R^U)^2 \le 1 \;\forall\; (x,y) \in (X \times Y)$.

Consider $R_1 \in (X \times y)$ and $R_2 \in (Y \times Z)$ be IVSFR.

Max-Min Operation

Let the max-min composition operator for two IVSFR $R_1 \in (X \times y)$ and $R_2 \in (Y \times Z)$ are defined as
$R_1 \circ R_2 = \{((x_{ij}, z_{ij}), \mu_{R_1 \circ R_2}(x_{ij}, z_{ij}), \eta_{R_1 \circ R_2}(x_{ij}, z_{ij}), \upsilon_{R_1 \circ R_2}(x_{ij}, z_{ij})) : x_{ij} \in X, z_{ij} \in Z\}$,
where, $\mu_{R_1 \circ R_2}(x_{ij}, z_{ij}) = [\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$
$\eta_{R_1 \circ R_2}(x_{ij}, z_{ij}) = [\eta^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \eta^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$ and
$\upsilon_{R_1 \circ R_2}(x_{ij}, z_{ij}) = [\upsilon^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \upsilon^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$
In addition,
$\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = max_{y \in Y}\left\{min_{x \in X}(\mu^L_{R_1}(x_{ij}, z_{ij}), \mu^L_{R_2}(x_{ij}, z_{ij}))\right\}, \mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})$
$\mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = max_{y \in Y}\left\{min_{x \in X}(\mu^U_{R_1}(x_{ij}, z_{ij}), \mu^U_{R_2}(x_{ij}, z_{ij}))\right\}, \mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij})$
$\eta^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{min_{x \in X}(\eta^L_{R_1}(x_{ij}, z_{ij}), \eta^L_{R_2}(x_{ij}, z_{ij}))\right\}, \eta^U_{R_1 \circ R_2}(x_{ij}, z_{ij})$
$\eta^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{min_{x \in X}(\eta^U_{R_1}(x_{ij}, z_{ij}), \eta^U_{R_2}(x_{ij}, z_{ij}))\right\}, \eta^L_{R_1 \circ R_2}(x_{ij}, z_{ij})$
$\upsilon^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{max_{x \in X}(\upsilon^L_{R_1}(x_{ij}, z_{ij}), \upsilon^L_{R_2}(x_{ij}, z_{ij}))\right\}, \upsilon^U_{R_1 \circ R_2}(x_{ij}, z_{ij})$
$\upsilon^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{max_{x \in X}(\upsilon^U_{R_1}(x_{ij}, z_{ij}), \upsilon^U_{R_2}(x_{ij}, z_{ij}))\right\}, \upsilon^L_{R_1 \circ R_2}(x_{ij}, z_{ij})$

Min-Max Operation

Let the min-max composition operator for two IVSFR be defined as
$R_1 \bullet R_2 = \{((x_{ij}, z_{ij}), \mu_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \eta_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \upsilon_{R_1 \bullet R_2}(x_{ij}, z_{ij})) : x_{ij} \in X, z_{ij} \in Z\}$,
where, $\mu_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = [\mu^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \mu^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})]$
$\eta_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = [\eta^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \eta^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})]$ and
$\upsilon_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = [\upsilon^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \upsilon^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})]$
In addition,
$\mu^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{max_{x \in X}(\mu^L_{R_1}(x_{ij}, z_{ij}), \mu^L_{R_2}(x_{ij}, z_{ij}))\right\}, \mu^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})$
$\mu^U_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{max_{x \in X}(\mu^U_{R_1}(x_{ij}, z_{ij}), \mu^U_{R_2}(x_{ij}, z_{ij}))\right\}, \mu^L_{R_1 \bullet R_2}(x_{ij}, z_{ij})$
$\eta^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{min_{x \in X}(\eta^L_{R_1}(x_{ij}, z_{ij}), \eta^L_{R_2}(x_{ij}, z_{ij}))\right\}, \eta^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})$
$\eta^U_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = min_{y \in Y}\left\{min_{x \in X}(\eta^U_{R_1}(x_{ij}, z_{ij}), \eta^U_{R_2}(x_{ij}, z_{ij}))\right\}, \eta^L_{R_1 \bullet R_2}(x_{ij}, z_{ij})$
$\upsilon^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = max_{y \in Y}\left\{min_{x \in X}(\upsilon^L_{R_1}(x_{ij}, z_{ij}), \upsilon^L_{R_2}(x_{ij}, z_{ij}))\right\}, \upsilon^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})$
$\upsilon^U_{R_1 \bullet R_2}(x_{ij}, z_{ij}) = max_{y \in Y}\left\{min_{x \in X}(\upsilon^U_{R_1}(x_{ij}, z_{ij}), \upsilon^U_{R_2}(x_{ij}, z_{ij}))\right\}, \upsilon^L_{R_1 \bullet R_2}(x_{ij}, z_{ij})$

Definition 18. *Let R be an IVSFR defined on IVSFS of X. An IVSFS N is called an eigen interval valued spherical fuzzy set associated with the given relation R if $N * R = N$, $*$ is either the min-max or max-min operator.*

5.1. Greatest Eigen interval valued spherical fuzzy set

For finding GEIVSFS with the IVSFR R using the max-min composition operator. Let N_1 represent the IVSFS, where the degree of membership is highest and the degrees of neutral membership and degree of non-membership is the lowest of all elements of the column of R:
$\mu_{N_1}(u) = max_{x \in X} \mu_R(x, u) \forall u \in Y,$
$\eta_{N_1}(u) = min_{x \in X} \eta_R(x, u) \forall u \in y,$
$\upsilon_{N_1}(u) = min_{x \in X} \upsilon_R(x, u) \forall u \in y,$ (1)
It is simple to check that N_1 is an eigen inter-valued spherical fuzzy set, but not always be the GEIVSFS. To find GEIVSFS, the following steps are evaluated using max- min operation
$N_1 \circ R = N_2,$
$N_2 \circ R = N_1 \circ R^2 = N_3,$
$N_3 \circ R = N_1 \circ R^3 = N_4,$
$\vdots\ N_n \circ R = N_1 \circ R^n = N_{n+1},$
Now, we provide an algorithm to calculate GEIVSFS.

Algorithm A

Step 1: Find the set N_1 from R by using equation 1
Step 2: Choose the index $n = 1$ and calculate $N_{n+1} = N_n \circ R$.
Step 3: If $N_{n+1} \neq N_n$ the go to step 2. Step 4: If $N_{n+1} = N_n$ then N_n is the GEIVSFS.

Example 3. $A =$

$$\begin{pmatrix} [0.30, 0.70][0.40, 0.20][0.20, 0.30] & [0.40, 0.60][0.45, 0.40][0.21, 0.30] & [0.71, 0.20][0.62, 0.10][0.10, 0.80] \\ [0.32, 0.02][0.72, 0.20][0.33, 0.40] & [0.23, 0.60][0.40, 0.50][0.21, 0.10] & [0.71, 0.20][0.62, 0.20][0.30, 0.70] \\ [0.36, 0.60][0.53, 0.20][0.65, 0.30] & [0.40, 0.20][0.25, 0.70][0.21, 0.50] & [0.15, 0.20][0.62, 0.20][0.18, 0.40] \end{pmatrix}$$

Step 1:
$N_1 =$
$([0.36, 0.70][0.40, 0.20][0.20, 0.30])$
$([0.40, 0.60][0.25, 0.40][0.21, 0.10])$
$([0.71, 0.20][0.62, 0.10][0.10, 0.40])$
Step 2:
Setting $n = 1, N_2 = N_1 \circ R$
$N_2 = ([0.36, 0.70][0.25, 0.10][0.20, 0.30])$
$([0.40, 0.60][0.25, 0.40][0.21, 0.10])$
$([0.40, 0.20][0.25, 0.10][0.18, 0.40])$
Step 3: $N_2 \neq N_1$ *then choose n=2 in step 2 and find* N_3 *that is* $N_3 = N_2 \circ R$*:*

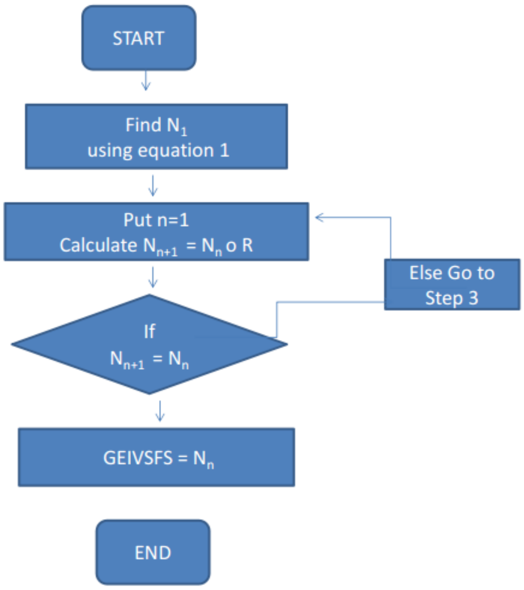

Figure 1: Flow chart For Algorithm A

$N_3 =$
$([0.36, 0.70][0.25, 0.10][0.20, 0.30])$
$([0.40, 0.60][0.25, 0.10][0.21, 0.10])$
$([0.40, 0.20][0.25, 0.10][0.18, 0.40])$

Step 4: Since $N_3 = N_2$, is the GEIVSFS associated with R.

5.2. Least Eigen interval valued spherical fuzzy set

For finding LEIVSFS with the IVSFR R using the max-min composition operator. Let N_1 be the IVSFS, in which the degree of membership and the degree of neutral membership are the lowest of all the members of the column of R and the degree of non-membership is the highest of all the members of the column of R.
$\mu_{N_1}(u) = min_{x \in X} \mu_R(x, u) \forall u \in Y,$
$\eta_{N_1}(u) = min_{x \in X} \eta_R(x, u) \forall u \in y,$
$\upsilon_{N_1}(u) = max_{x \in X} \upsilon_R(x, u) \forall u \in y, \quad (2)$
It is easy to check that N_1 is an eigen interval valued spherical fuzzy set, but it should not always be the LEIVSFS. To find LEIVSFS, the following steps are evaluated using max-min operation
$N_1 \circ R = N_2,$
$N_2 \circ R = N_1 \circ R^2 = N_3,$
$N_3 \circ R = N_1 \circ R^3 = N_4,$
$\vdots\ N_n \circ R = N_1 \circ R^n = N_{n+1},$

Now, we give an algorithm to find LEIVSFS.

Algorithm B

Step 1: Obtain the set N_1 from R by equation 2
Step 2: Choosing the index $n = 1$ and calculating $N_{n+1} = N_n \circ R$.
Step 3: If $N_{n+1} \neq N_n$, go to Step 2. Step 4: If $N_{n+1} = N_n$, then N_n is the LEIVSFS.

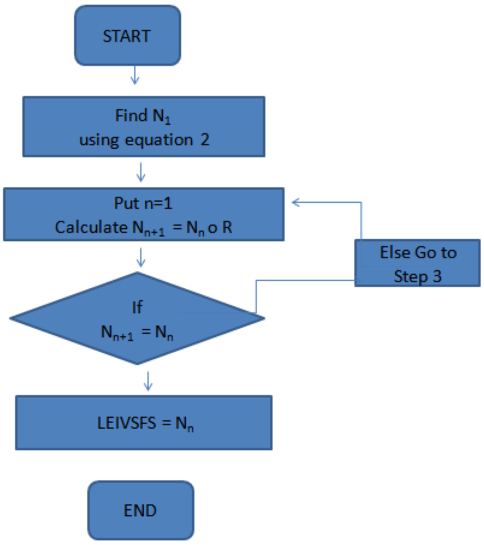

Figure 2: Flow chart for Algorith B

Example 4. $A =$

$$\begin{pmatrix} [0.30, 0.70][0.40, 0.20][0.20, 0.30] & [0.40, 0.60][0.45, 0.40][0.21, 0.30] & [0.71, 0.20][0.62, 0.10][0.10, 0.80] \\ [0.32, 0.02][0.72, 0.20][0.33, 0.40] & [0.23, 0.60][0.40, 0.50][0.21, 0.10] & [0.71, 0.20][0.62, 0.20][0.30, 0.70] \\ [0.36, 0.60][0.53, 0.20+][0.65, 0.30] & [0.40, 0.20][0.25, 0.70][0.21, 0.50] & [0.15, 0.20][0.62, 0.20][0.18, 0.40] \end{pmatrix}$$

Step 1:
$N_1 =$
([0.30,0.02][0.40,0.20][0.65,0.40])
([0.23,0.20][0.25,0.40][0.21,0.50])
([0.15,0.20][0.62,0.10][0.30,0.80])
Step 2:
Set $n = 1,\ N_2 = N_1 \circ R$
$N_2 =$
([0.30, 0.20][0.25, 0.10][0.30, 0.40])
([0.23, 0.20][0.25, 0.10][0.21, 0.50])
([0.15, 0.20][0.25, 0.10][0.21, 0.50])
Step 3:
$N_2 \neq N_1$ *then choose* $n = 2$ *in step 2 and find* N_3 *that is* $N_3 = N_2 \circ R$*:*

$N_3 = ([0.30, 0.20][0.25, 0.10][0.30, 0.40])$
$([0.23, 0.20][0.25, 0.10][0.21, 0.50])$
$([0.15, 0.20][0.25, 0.10][0.21, 0.50])$
Hence, $N_3 = N_2$*,so* N_3 *is the LEIVSFS associated with R.*

6. Distance measure in decision-making problem

This section presents a distance measure for IVSFS, investigates its properties, and explores its potential applications in decision making using IVSFM.

6.1. Distance measure of IVSFSs

Definition 19. *let consider two IVSFSs*
$A = \{[a_1(x_i), b_1(x_i)], [c_1(x_i), d_1(x_i)], [e_1(x_i), f_1(x_i)]\}$
$B = \{[a_2(x_i), b_2(x_i)], [c_2(x_i), d_2(x_i)], [e_2(x_i), f_2(x_i)]\}$
the distance measure between A and B is defined as follows:
$d(A, B) = \frac{1}{3}(|a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| + |c_1(x_i) - c_2(x_i)|$
$+|d_1(x_i) - d_2(x_i)| + |e_1(x_i) - e_2(x_i)| + |f_1(x_i) - f_2(x_i)|)$
where $i = 1, 2, 3...n$

Theorem 9. *The distance measure between IVSFSs A and B is a function* $d : IVSFSs \to IVSFSs$*, which satisfies the following conditions*
1. $0 \leq d(A, B) \leq 1$
2. $d(A, B) = 0$ *iff* $A = B$
3. $d(A, B) = d(B, A)$ (d_1)
4. Let $A, B, B \in IVSFSs$ *then* $d(A, C) \leq d(A, B) + d(B, C)$

Proof. 1. It is obvious $d(A, B) \in [0, 1]$
2. As $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2, e_1 = e_2, f_1 = f_2$ such that $d(A, B) = 0$ iff if $A = B$
3. $d(A, B) = \frac{1}{3}(|a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| + |c_1(x_i) - c_2(x_i)|$
$+|d_1(x_i) - d_2(x_i)| + |e_1(x_i) - e_2(x_i)| + |f_1(x_i) - f_2(x_i)|)$
$= \frac{1}{3}(\sum |a_2(x_i) - a_1(x_i)| + |b_2(x_i) - b_1(x_i)| + |c_2(x_i) - c_1(x_i)|$
$+|d_2(x_i) - d_1(x_i)| + |e_2(x_i) - e_1(x_i)| + |f_2(x_i) - f_1(x_i)|)$ = d(B,A)
4. $A = \{[a_1(x_i), b_1(x_i)], [c_1(x_i), d_1(x_i)], [e_1(x_i), f_1(x_i)]\}$
$B = \{[a_2(x_i), b_2(x_i)], [c_2(x_i), d_2(x_i)], [e_2(x_i), f_2(x_i)]\}$
$C = \{[a_3(x_i), b_3(x_i)], [c_3(x_i), d_3(x_i)], [e_3(x_i), f_3(x_i)]\}$
Consider
$d(A, C) = \frac{1}{3}(|a_1(x_i) - a_3(x_i)| + |b_1(x_i) - b_3(x_i)| + |c_1(x_i) - c_3(x_i)|$
$+|d_1(x_i) - d_3(x_i)| + |e_1(x_i) - e_3(x_i)| + |f_1(x_i) - f_3(x_i)|)$
$= \frac{1}{3}(|a_1(x_i) - a_2(x_i) + a_2(x_i) - a_3(x_i)| + |b_1(x_i) - b_2(x_i) + b_2(x_i) - b_3(x_i)| + |c_1(x_i) - c_2(x_i) + c_2(x_i) - c_3(x_i)|$
$+|d_1(x_i) - d_2(x_i) + d_2(x_i) - d_3(x_i)| + |e_1(x_i) - e_2(x_i) + e_2(x_i) - e_3(x_i)| + |f_1(x_i) - f_2(x_i) + f_2(x_i) - f_3(x_i)|)$
$\leq \frac{1}{3}(|a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| + |c_1(x_i) - c_2(x_i)|$

$+|d_1(x_i) - d_2(x_i)| + |e_1(x_i) - e_2(x_i)| + |f_1(x_i) - f_2(x_i)|)$
$+\frac{1}{3}(|a_2(x_i) - a_3(x_i)| + |b_2(x_i) - b_3(x_i)| + |c_2(x_i) - c_3(x_i)|$
$+|d_2(x_i) - d_3(x_i)| + |e_2(x_i) - e_3(x_i)| + |f_2(x_i) - f_3(x_i)|)$
Hence $d(A, C) \leq d(A, B) + d(B, C)$

6.2. Application in decision making problem

To tackle the decision making problem of selecting administrative officers (AOs) for the promotion of Govt secretaries, we use the IVSFM. This approach is useful when dealing with uncertainty.

Let us consider three Govts. G_1, G_2, G_3 based on three political parties PP_1, PP_2 and PP_3 . Five AOs, A_1, A_2, A_3, A_4 and A_5 are shortlisted for promotion to the position of Govt Secretary.

Consider IVSFM A of dimension 5×3 which presents the view of An AO towards a Govt. supported by a political party.

Consider the other IVSFM B of order 3×3 which shows the work carried by Govt.

$$A = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccc} G_1 & G_2 & G_3 \end{array} \\ \begin{pmatrix} [0.40,0.20][0.40,0.20][0.21,0.50] & [0.40,0.20][0.40,0.20][0.18,0.70] & [0.40,0.20][0.40,0.10][0.21,0.30] \\ [0.30,0.20][0.62,0.20][0.21,0.50] & [0.70,0.20][0.50,0.20][0.18,0.50] & [0.30,0.20][0.20,0.20][0.60,0.30] \\ [0.60,0.20][0.25,0.20][0.33,0.50] & [0.20,0.20][0.25,0.20][0.20,0.50] & [0.23,0.60][0.10,0.20][0.20,0.10] \\ [0.60,0.20][0.25,0.20][0.33,0.50] & [0.20,0.20][0.25,0.20][0.20,0.50] & [0.23,0.60][0.40,0.20][0.30,0.10] \\ [0.60,0.20][0.25,0.20][0.33,0.50] & [0.20,0.20][0.25,0.20][0.20,0.50] & [0.23,0.60][0.30,0.10][0.25,0.70] \end{pmatrix} \end{array}$$

followed by a political party during the election period.

Step A. In the IVSFM A, the information of each AOs is provided in relation with set

$$B = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \end{array} \begin{array}{c} \begin{array}{ccc} PP_1 & PP_2 & PP_3 \end{array} \\ \begin{pmatrix} [0.20,0.10][0.40,0.10][0.30,0.50] & [0.40,0.20][0.30,0.20][0.18,0.70] & [0.20,0.20][0.30,0.10][0.21,0.30] \\ [0.20,0.30][0.62,0.20][0.20,0.50] & [0.60,0.20][0.50,0.20][0.18,0.50] & [0.30,0.20][0.20,0.20][0.60,0.20] \\ [0.60,0.20][0.25,0.20][0.33,0.50] & [0.20,0.20][0.25,0.20][0.20,0.50] & [0.23,0.60][0.10,0.20][0.40,0.20] \end{pmatrix} \end{array}$$

of political parties $\{PP_1, PP_2, PP_3\}$, while as in the IVSFM B, the information of each Govts. is given with respect to the same set of political parties $\{PP_1, PP_2, PP_3\}$. On the basis of that concept, eight IVSFSs are obtained over the set $\{PP_1, PP_2, PP_3\}$ as follows:

$A_1 = (PP_1[0.40, 0.20][0.40, 0.20][0.21, 0.50]), (PP_2[0.40, 0.20][0.40, 0.20][0.18, 0.70]),$
$(PP_3[0.40, 0.20][0.40, 0.10][0.21, 0.30])$
$A_2 = (PP_1[0.30, 0.20][0.62, 0.20][0.21, 0.50])(PP_2[0.70, 0.20][0.50, 0.20][0.18, 0.50]),$
$(PP_3[0.30, 0.20][0.20, 0.20][0.60, 0.30])$
$A_3 = (PP_1[0.60, 0.20][0.25, 0.20][0.33, 0.50])(PP_2[0.20, 0.20][0.25, 0.20][0.20, 0.50]),$
$(PP_3[0.23, 0.60][0.10, 0.20][0.20, 0.10])$
$A_4 = (PP_1[0.60, 0.20][0.25, 0.20][0.33, 0.50])(PP_2[0.20, 0.20][0.25, 0.20][0.20, 0.50]),$
$(PP_3[0.23, 0.60][0.40, 0.20][0.30, 0.10])$
$A_5 = (PP_1[0.60, 0.20][0.25, 0.20][0.33, 0.50])(PP_2[0.20, 0.20][0.25, 0.20][0.20, 0.50]),$

$(PP_3[0.23, 0.60][0.30, 0.10][0.25, 0.70])$
$G_1 = (PP_1[0.20, 0.10][0.40, 0.10][0.30, 0.50])(PP_2[0.40, 0.20][0.30, 0.20][0.18, 0.70]),$
$(PP_3[0.20, 0.20][0.30, 0.10][0.21, 0.30])$
$G_2 = (PP_1[0.20, 0.30][0.62, 0.20][0.20, 0.50])(PP_2[0.60, 0.20][0.50, 0.20][0.18, 0.50]),$
$(PP_3[0.30, 0.20][0.20, 0.20][0.60, 0.20])$
$G_3 = (PP_1[0.60, 0.20][0.25, 0.20][0.33, 0.50])(PP_2[0.20, 0.20][0.25, 0.20][0.20, 0.50]),$
$(PP_3[0.23, 0.60][0.10, 0.20][0.40, 0.20])$
Step B. Using the distance measure between two IVSFSs the following distance matrix D of dimensions 5×3, is derived and d_{im} represents the distance between A_i and G_m for $i = 1, 2, 3, 4, 5$ and $m = 1, 2, 3$

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{matrix} G_1 & G_2 & G_3 \end{matrix} \\ \begin{pmatrix} 0.2966 & 0.5933 & 0.7666 \\ 0.6333 & 0.1333 & 0.8100 \\ 0.7300 & 0.9133 & 0.1000 \\ 0.7233 & 0.9124 & 0.1666 \\ 0.7066 & 0.9983 & 0.9500 \end{pmatrix} \end{array}$$

3. The following observation are made using the distance measure.
1. In Case of the Govt. G_1, degree of closeness DOC between the AO A_1 and the performance of Govt. G_1 is maximum because $DOC(A_1, G_1) > DOC(A_2, G_1) > DOC(A_5, G_1) > DOC(A_4, G_1) > DOC(A_3, G_1)$
2. In Case of the Govt. G_2, degree of closeness DOC between the AO A_2 and the performance of Govt. G_2 is maximum because DOC$(A_2, G_2) > DOC(A_1, G_2) > DOC(A_4, G_2) > DOC(A_3, G_2) > DOC(A_5, G_2)$
3. In case of the Govt. G_3, degree of closeness (DOC) between the AO A_3 and the performance of Govt. G_3 is maximum because $DOC(A_3, G_3) > DOC(A_4, G_3) > DOC(A_1, G_3) > DOC(A_2, G_3) > DOC(A_5, G_3)$
where $>$ denotes the degree of closeness, the lower the value indicates a higher degree of closeness. As per calculation , the final selected list of AOs for different governments is determined as follows:

Govt:	AOs
(G_1)	A_1, A_2
(G_2)	A_2, A_1
(G_3)	A_3, A_4

AOs A_1, A_2 are selected for all Govts. Fig. 3 illustrates a flow chart that states the procedure to be followed for AO selection.

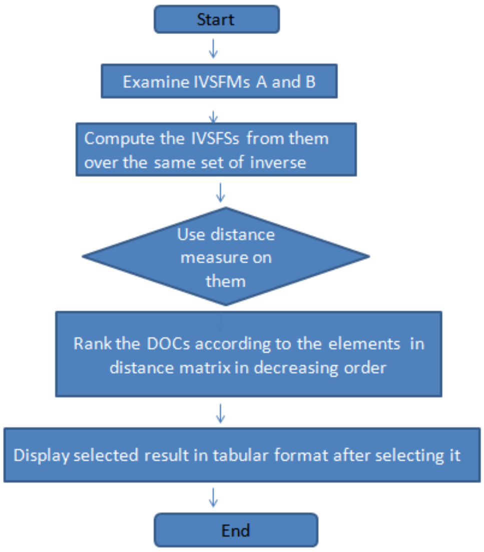

Figure 3: Selection Procedure for AOs

7. Comparative analysis

Whereas in previous studies on picture-fuzzy decision making, the information was accepted in picture-fuzzy form. But if we are to address various types of uncertainty in the information, conventional methods are not in a position to address such a situation. In such circumstances, we have to gather or represent the information in an interval-valued spherical fuzzy sense. In such instances, the process formulated so far becomes crucial in arriving at an effective and useful conclusion.

Dogra and Pal [33] suggested a model for the selection of a selected group of administrative officers for various governments on the basis of the distance measure between two picture fuzzy matrix. Their approach considered membership, neutral membership, and non-membership degrees in the picture fuzzy matrix context. In contrast, current research generalizes this work by considering membership, neutral membership, and nonmembership degrees as interval numbers, which has more relevance to real-world issues in the selection of administrative officers using the new distance measure.

In addition, Ejegwa et al.[26] developed a model for computing students' career paths based on the distance formula between two intuitionistic fuzzy sets. In this context, intuitionistic fuzzy sets considered only membership and non-membership degrees. Moreover, Khalaf [27] solved medical diagnosis problems using the interval valued intuitionistic fuzzy set with max–min and min-max composition. They expressed these problems as uncertain decision matrices and provided decisions based on fuzzy scores for each attribute.

However, our current method is distinct with matrices with interval valued spherical fuzzy values. We obtain interval valued spherical fuzzy sets from these matrices on a given universe. Using the new formula for the distance between two IVSFSs, we obtain a distance matrix which leads to a decision. The use of this method is incredibly straightforward since it does not include various complex calculations, hence avoiding any complication in its application. Therefore, developing an algorithm and computer programming for this method is straightforward. Besides, the data points utilized in this method have an exceptional capability to handle a greater level of vagueness in information. Recalling that the interval-valued spherical fuzzy concept is a generalization of the picture fuzzy concept, this study can be viewed as a generalization of higher order fuzzy logic.

In summary, the study of IVSFM has immense advantages in the solution of real-world problems, particularly in the selection of administrative officers, manufacturing companies, and health insurance providers. These concepts are practical tools and subjects of research for various applications and, as such, are significant aspects in modern-day research and real-world problem solving scenarios.

8. Conclusion

In this paper, we have introduced the concept of IVSFM along with some definitions and theorems. We proposed the definition of determinant and adjoint of IVSFM along with some related results. Further, we propose a formal definition of an EIVSFS for the interval-valued spherical fuzzy relations, along with the algorithms to find GEIVSFS and LEIVSFS by using min-max and max-min composition operators, respectively. Numerical examples have also been presented to illustrate the algorithms. In addition, the implementation of the GEIVPFS and LEIVSFS in decision-making problems has been demonstrated quite effectively. Furthermore, the new distance measure is proposed to solve the problems of decision making quite effectively with the help of IVSFM. Interval-valued spherical fuzzy matrices offer enhanced uncertainty modeling for multi-criteria decision-making, enabling more accurate and flexible assessments. They are highly applicable in real-world scenarios like medical diagnosis, supply chain management, and AI-based decision support systems where ambiguity and expert variability are critical. This study provides a platform for researchers to expand and generalize our findings across various types of data sets. It can be extended in fields such as image information retrieval, genetic algorithms for image reconstruction, and the concept of interval-valued eigen spherical fuzzy soft sets or soft matrices, which have been briefly stated for future research.

Acknowledgements

The authors would like to thank the Editor and reviewer's for their valuable suggestions and comments to improve this article in the present form.

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