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A Note on k-Ideals in Ternary Semirings

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Abstract. In this paper, we give some examples of k-ideals of ternary semirings. We examine the results of k-ideals by distinguished classes in ternary semirings, which include k-maximal, k-prime and k-semiprime.

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1. Introduction

Lehmer [1] introduced ternary algebra in 1932 and investigated certain algebraic systems called triplexes, which are commutative ternary groups. Later, Banach also studied these algebraic structures and provided examples of a ternary semigroup that does not reduce to a semigroup. Additionally, Lister [2] introduced the concept of a ternary ring. The concept of ternary semirings was first introduced by Dutta and Kar [3] in 2003. Moreover, Dutta and Kar investigated some basic concepts of prime ideals and semiprime ideals of ternary semirings in [4] and [5], respectively. The concept of ternary semirings arises from the study of algebraic structures that extend semirings. Ternary semirings consist of two operations, typically the addition and the ternary multiplication. By using a ternary multiplication instead of a binary multiplication, every semiring can be turned to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring. Although ternary semirings generalize the notion of a semiring, they are not just a generalization because some certain notions, such as lateral ideals, lack an analog in a semiring. Many concepts from semiring theory were extended to the study of ternary semirings.

Ideal theory is the main area of research in the study of many algebraic structures. In 2005, Kar [6] introduced the notions of quasi-ideals and bi-ideals in ternary semirings and characterized regular ternary semirings in terms of quasi-ideals and bi-ideals. In 2010, Malee and Chinram studied fuzzifications of some ideals in ternary semirings [7] and [8].

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The next year, Dubey [9] investigated quasi k-ideals and bi k-ideals in ternary semirings. In the same year, Chaudhari and Ingale [10] introduced a partitioning ideal of a ternary semiring and Sunitha et al. [11] provided the characterization of full k-ideals in ternary semirings. Partitioning ideals are useful to develop the quotient structures of ternary semirings. Singular ideals were introduced in [12] in 2012. In 2016, the concept of the subtractive extension of an ideal of a ternary semiring was introduced in [13]. In 2024, Luangchaisri and Changphas [14] studied right weakly regular ternary semirings and fully prime right ternary semirings. In the same year, Goswami and Dube [15] investigated some aspects of k-ideals of semirings. This paper inspired by their work. Our goal is to explore various aspects of k-ideals in ternary semirings.

2. Preliminaries

In this section, we will recall some basic definitions of ternary semirings.

Definition 1 ([3])**.** A nonempty set R together with a binary operation, called the addition, and the ternary multiplication, denoted by juxtaposition, is said to be a *ternary semiring* if R is an additive commutative semigroup satisfying the following conditions:

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a+b)cd = acd + bcd$,

(iii) $a(b+c)d = abd + acd$,

(iv) $ab(c+d) = abc + abd$,

for all $a, b, c, d, e \in R$.

Example 1.

(1) Every semiring can be considered a ternary semiring under the ordinary addition and ternary multiplication of semirings.

(2) $\mathbb{Z}^-$ is a ternary semiring under the usual addition and ternary multiplication of integers, but it is not a semiring under the usual addition and binary multiplication of integers.

Definition 2 ([3])**.** Let R be a ternary semiring. If $0 \in R$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in R$, then 0 is called a *zero element*. In this case, R is called a *ternary semiring with zero*.

Definition 3. A ternary semiring R is said to be *commutative* if $abc = bca = cab = bac = cba = acb$ for all $a, b, c \in R$.

Definition 4 ([3])**.** An additive semigroup S of a ternary semiring R is called a *ternary subsemiring* of R if $s_1s_2s_3 \in S$ for all $s_1, s_2, s_3 \in S$.

Definition 5. An additive subsemigroup I of a ternary semiring R is called

(1) a *left ideal* of R if $r_1r_2a \in I$ for all $r_1, r_2 \in R$ and $a \in I$,

(2) a *right ideal* of R if $ar_1r_2 \in I$ for all $r_1, r_2 \in R$ and $a \in I$,

(3) a *lateral ideal* of R if $r_1ar_2 \in I$ for all $r_1, r_2 \in R$ and $a \in I$,

(4) an *ideal* of R if I is a left ideal, a right ideal, and a lateral ideal of R.

An ideal I of R is called a *proper ideal* if $I \neq R$.

Proposition 1 ([3]). *Let R be a ternary semiring and $a \in R$. Then the following statements hold.*

(1) The principal left ideal generated by a is given by $\langle a\rangle_l = \mathbb{N}_0a + RRa$.

(2) The principal right ideal generated by a is given by $\langle a\rangle_r = \mathbb{N}_0a + aRR$.

(3) The principal lateral ideal generated by a is given by $\langle a\rangle_m = \mathbb{N}_0a + RaR + RRaRR$.

(4) The principal ideal generated by a is given by $\langle a\rangle = \mathbb{N}_0a + RRa + aRR + RaR + RRaRR$.

If R is commutative, we note that $\langle a\rangle = \mathbb{N}_0a + RRa$.

Definition 6. An ideal I of a ternary semiring R is called a *k-ideal* if, for all $x, y \in R$, $x \in I$ and $x + y \in I$ imply $y \in I$.

Definition 7. A proper ideal of a ternary semiring R is called *maximal* if it is not properly contained in any other proper ideal of R.

Definition 8. ([4]) A proper ideal P of a ternary semiring R is called *prime* if $IJK \subseteq P$ implies $I \subseteq P$, $J \subseteq P$, or $K \subseteq P$ for all ideals I, J, K of R.

Definition 9. ([5]) A proper ideal P of a ternary semiring R is called *semiprime* if $I^3 \subseteq P$ implies $I \subseteq P$ for every ideal I of R.

3. Main Results

Let $\mathcal{J}(R)$ denote the set of all ideals of a ternary semiring R and $\mathcal{J}_K(R)$ denote the set of all k-ideals of R.

Suppose R is a commutative ternary semiring with zero 0. If A is a nonempty subset of R, then the *annihilator* of A is defined by

$$\mathrm{Ann}_R(A) = \{r \in R \mid rxy = 0 \text{ for all } x, y \in A\}.$$

Proposition 2. *Let R be a commutative ternary semiring with zero 0. For any two nonempty subsets A and B of R, if $A \subseteq B$, then $Ann_R(B) \subseteq Ann_R(A)$.*

Proof. Let $r \in \mathrm{Ann}_R(B)$. So $rxy = 0$ for all $x, y \in B$. This implies $rxy = 0$ for all $x, y \in A$. Hence $r \in \mathrm{Ann}_R(A)$.

Example 2. Let $R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}\} \subseteq \mathbb{Z}_{16}$. We have that R is a commutative ternary semiring under the usual addition and ternary multiplication of integers modulo 16. The element $\overline{0}$ is a zero of R.

(1) $\mathrm{Ann}_R(R) = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}\}$.

(2) If $A = \{\overline{4}\}$, then $\mathrm{Ann}_R(A) = R$.

(3) If $A = \{\overline{2}\}$, then $\mathrm{Ann}_R(A) = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}\}$.

Proposition 3. *Let R be a commutative ternary semiring with zero 0 and $a \in R$. Then $Ann_R(\{a\}) = Ann_R(\langle a \rangle)$.*

Proof. By Proposition 2, we have $\mathrm{Ann}_R(\langle a \rangle) \subseteq \mathrm{Ann}_R(\{a\})$. Let $r \in \mathrm{Ann}_R(\{a\})$. Then $raa = 0$. Next, let $x, y \in \langle a \rangle$. So $x = ka + \sum s_i s_i' a$ and $y = k'a + \sum r_j r_j' a$ for some $k, k' \in \mathbb{N}_0$ and $s_i, s_i', r_j, r_j' \in R$. Since R is commutative, it is easy to see that $rxy = 0$. Hence $\mathrm{Ann}_R(\{a\}) = \mathrm{Ann}_R(\langle a \rangle)$.

Proposition 4. *Let R be a commutative ternary semiring with zero 0 and A be any nonempty subset of R. Then $Ann_R(A)$ is a k-ideal of R.*

Proof. Clearly, $0 \in \mathrm{Ann}_R(A)$, which implies that $\mathrm{Ann}_R(A) \neq \emptyset$. Let $r, s \in \mathrm{Ann}_R(A)$ and $x, y \in R$. Then $rab = 0$ and $sab = 0$ for all $a, b \in A$, so $(r + s)ab = 0$ and $(rxy)ab = 0$ for all $a, b \in A$. Then $r + s \in \mathrm{Ann}_R(A)$ and $rxy \in \mathrm{Ann}_R(A)$. This implies that $\mathrm{Ann}_R(A)$ is an ideal of R. Let $r, x \in R$ be such that $r + x \in \mathrm{Ann}_R(A)$ and $r \in \mathrm{Ann}_R(A)$. So $(r+x)ab = 0$ and $rab = 0$ for all $a, b \in A$. Then $xab = 0+xab = rab+xab = (r+x)ab = 0$. This implies that $x \in \mathrm{Ann}_R(A)$, and hence $\mathrm{Ann}_R(A)$ is a k-ideal of R.

Let R be any ternary semiring. The *k-closure operation* on $\mathcal{J}(R)$ is defined for an ideal I of R by

$$\mathcal{C}_k(I) = \{r \in R \mid r + x \in I \text{ for some } x \in I\}.$$

Example 3. We consider a ternary semiring $\mathbb{Z}_0^-$ under the usual addition and ternary multiplication of integers.

(1) Let $I = 2\mathbb{Z}_0^- \setminus \{-2\} = \{0, -4, -6, -8, -10, \ldots\}$. It is easy to show that I is an ideal of $\mathbb{Z}_0^-$. We have $\mathcal{C}_k(I) = \{0, -2, -4, -6, -8, \ldots\} = 2\mathbb{Z}_0^-$.

(2) Let $I = \mathbb{Z}_0^- \setminus \{-1\} = \{0, -2, -3, -4, -5, \ldots\}$. It is easy to prove that I is an ideal of $\mathbb{Z}_0^-$. We have $\mathcal{C}_k(I) = \mathbb{Z}_0^-$.

Proposition 5. *Let R be a ternary semiring and I be an ideal of R. Then $\mathcal{C}_k(I)$ is the smallest k-ideal containing I.*

Proof. Let $x \in I$. So $x + x \in I$. Then $x \in \mathcal{C}_k(I)$. So $I \subseteq \mathcal{C}_k(I)$. Let $r, s \in \mathcal{C}_k(I)$. Then there exist $x, y \in I$ such that $r + x \in I$ and $s + y \in I$. Then $r + s + x + y \in I$. This implies that $r + s \in \mathcal{C}_k(I)$. Next, let $r \in \mathcal{C}_k(I)$ and $a, b \in R$. Thus there exists $x \in I$ such that $r + x \in I$. Then $rab + xab = (r + x)ab \in I$ and $xab \in I$. This shows that $rab \in \mathcal{C}_k(I)$. Similarly, $arb, abr \in \mathcal{C}_k(I)$. Then $\mathcal{C}_k(I)$ is an ideal of R. Next, assume that $r + x, x \in \mathcal{C}_k(I)$. There exist $a, b \in I$ such that $r + x + a, x + b \in I$. Then $x + a + b \in I$ and $r + x + a + b \in I$. Hence $r \in \mathcal{C}_k(I)$. We can conclude that $\mathcal{C}_k(I)$ is a k-ideal of R.

For the next step, let J be any k-ideal containing I. We need to show that $C_k(I) \subseteq J$. Let $r \in \mathcal{C}_k(I)$. Then there exists $x \in I$ such that $r + x \in I$. Since $I \subseteq J$, $r + x \in J$ and $x \in J$. Since J is a k-ideal such that $r + x \in J$ and $x \in J$, it follows that $r \in J$. Thus $C_k(I) \subseteq J$. Therefore $\mathcal{C}_k(I)$ is the smallest k-ideal containing I.

Proposition 6. *Let R be a ternary semiring. The following statements hold.*

(1) $\mathcal{C}_k(R) = R$.

(2) If R has a zero, then $\mathcal{C}_k(\{0\}) = \{0\}$.

Proof. (1) By Proposition 5, we have $R \subseteq \mathcal{C}_k(R)$. Then $\mathcal{C}_k(R) = R$.
(2) Let $x \in \mathcal{C}_k(\{0\})$. So $x + 0 \in \{0\}$. This implies that $x = 0$ and $\mathcal{C}_k(\{0\}) = \{0\}$.

Proposition 7. *Let R be a ternary semiring. If I and J are any ideals of R such that $I \subseteq J$, then $\mathcal{C}_k(I) \subseteq \mathcal{C}_k(J)$.*

Proof. Let $r \in \mathcal{C}_k(I)$. Then by definition of $C_k(I)$, we have $r + x \in I$ for some $x \in I$. Since $I \subseteq J$, $r + x \in J$ and $x \in J$. Thus $r \in \mathcal{C}_k(J)$. Therefore $\mathcal{C}_k(I) \subseteq \mathcal{C}_k(J)$.

Theorem 1. *Let R be a ternary semiring and I be an ideal of R. Then I is a k-ideal of R if and only if $I = \mathcal{C}_k(I)$.*

Proof. Let I be a k-ideal of R. By Proposition 5, we have $I \subseteq \mathcal{C}_k(I)$. Let $r \in \mathcal{C}_k(I)$. Then there exists $x \in I$ such that $r + x \in I$. Since I is a k-ideal of R and $r + x, x \in I$, we obtain $r \in I$. Thus $\mathcal{C}_k(I) \subseteq I$. We conclude that $\mathcal{C}_k(I) = I$. Conversely, we assume that $\mathcal{C}_k(I) = I$. By Proposition 5, I is a k-ideal of R.

Proposition 8. *Let I be an ideal of a ternary semiring R. Then $\mathcal{C}_k(\mathcal{C}_k(I)) = \mathcal{C}_k(I)$.*

Proof. By Proposition 5, we know that $\mathcal{C}_k(I)$ is a k-ideal of R. By Theorem 1, we have that $\mathcal{C}_k(\mathcal{C}_k(I)) = \mathcal{C}_k(I)$.

Proposition 9. *Let I, J and K be ideals of a ternary semiring R. Then*

$$\mathcal{C}_k(I)\mathcal{C}_k(J)\mathcal{C}_k(K) \subseteq \mathcal{C}_k(IJK).$$

Proof. Let $x \in \mathcal{C}_k(I)$, $y \in \mathcal{C}_k(J)$ and $z \in \mathcal{C}_k(K)$. Then there exist $a \in I, b \in J$ and $c \in K$ such that $x + a \in I, y + b \in J$ and $z + c \in K$. So $abc \in IJK$ and each of the following is also a member of IJK:

$$xbc + abc = (x + a)bc \in IJK, \tag{1}$$
$$ayc + abc = a(y + b)c \in IJK, \tag{2}$$
$$abz + abc = ab(z + c) \in IJK, \tag{3}$$
$$xyc + ayc + xbc + abc = (x + a)(y + b)c \in IJK, \tag{4}$$
$$ayz + abz + ayc + abc = a(y + b)(z + c) \in IJK, \tag{5}$$
$$xbz + xbc + abz + abc = (x + a)b(z + c) \in IJK, \tag{6}$$

and

$$xyz + xyc + xbz + xbc + ayz + ayc + abz + abc = (x + a)(y + b)(z + c) \in IJK. \tag{7}$$

Since $abc \in IJK$ and $xbc + abc \in IJK$, we obtain $xbc \in \mathcal{C}_k(IJK)$. By the same argument, from (2) and (3), we also obtain $ayc \in \mathcal{C}_k(IJK)$ and $abz \in \mathcal{C}_k(IJK)$, respectively. Again, since $abc \in IJK$ and (4) holds, we get $xyc + ayc + xbc \in \mathcal{C}_k(IJK)$. By Proposition 5, we know that $\mathcal{C}_k(IJK)$ is a k-ideal of R, which implies that $xyc \in \mathcal{C}_k(IJK)$ because $xbc, ayc \in \mathcal{C}_k(IJK)$, Applying the same process to (5) and (6), we conclude that $ayz, xbz \in \mathcal{C}_k(IJK)$, respectively. Once again, we see that (7) implies

$$xyz + xyc + xbz + xbc + ayz + ayc + abz \in \mathcal{C}_k(IJK).$$

Since xbc, ayc, abz, xyc, ayz and xbz are all members of $\mathcal{C}_k(IJK)$, we eventually obtain $xyz \in \mathcal{C}_k(IJK)$. Therefore, $\mathcal{C}_k(I)\mathcal{C}_k(J)\mathcal{C}_k(K) \subseteq \mathcal{C}_k(IJK)$.

Suppose that I, J and K are ideals of a ternary semiring R. Then the *ideal quotient* of I over J and K is defined by

$$(I : J, K) = \{a \in R \mid aJK \subseteq I\}.$$

Example 4. Consider $R = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}\} \subseteq \mathbb{Z}_{16}$ which is a ternary semiring under the usual addition and ternary multiplication of integers modulo 16. Let $I = \{\overline{0}, \overline{8}\}$. We have that $(I : J, K) = R$ for all ideals J and K of R.

Proposition 10. *Let I, J and K be ideals of a ternary semiring R. Then $I \subseteq (I : J, K)$.*

Proof. Let $a \in I$. Since I is an ideal of R and $J, K \subseteq R$, we have that $aJK \subseteq I$. This implies that $I \subseteq (I : J, K)$.

Proposition 11. *Let R be a commutative ternary semiring. If I is a k-ideal and J, K are ideals of R, then $(I : J, K)$ is a k-ideal of R.*

Proof. Let $a, b \in (I : J, K)$ and $r, s \in R$. Then $aJK \subseteq I$ and $bJK \subseteq I$. Thus $(a + b)JK \subseteq I$ and $(ars)JK \subseteq a(rsJ)K \subseteq aJK \subseteq I$. So $a + b \in (I : J, K)$ and $ars \in (I : J, K)$. Hence $(I : J, K)$ is an ideal of R.

Next, let $a, b \in R$ be such that $a \in (I : J, K)$ and $a + b \in (I : J, K)$. This implies that $aJK + bJK = (a + b)JK \subseteq I$ and $aJK \subseteq I$. Let $x \in bJK$ and $y \in aJK$. So $x + y \in I$ and $y \in I$. Since I is a k-ideal of R, we have $x \in I$. This implies that $bJK \subseteq I$, so $b \in (I : J, K)$. Hence $(I : J, K)$ is a k-ideal of R.

A proper ideal of R is called *k-maximal* if it is not properly contained in any other proper k-ideal of R.

Example 5. We consider a ternary semiring $R = \mathbb{Z}_0^-$ under the usual addition and ternary multiplication of integers.

(1) Let $I = 2\mathbb{Z}_0^- = \{0, -2, -4, -6, -8, -10, \ldots\}$. Then I is a k-ideal of R. We have that I is k-maximal but not maximal.

(2) Let $I = \mathbb{Z}_0^- \setminus \{-1\} = \{0, -2, -3, -4, -5, -6, \ldots\}$. Then I is an ideal of R but not a k-ideal. We have that I is both maximal and k-maximal.

Theorem 2. *Let R be a ternary semiring and let I be a proper ideal of R.*

(1) If I is k-maximal, then I is a k-ideal or $\mathcal{C}_k(I) = R$.

(2) If I is a k-ideal and a maximal ideal, then I is k-maximal.

Proof. (1) Suppose I is k-maximal. By Proposition 5, $I \subseteq \mathcal{C}_k(I) \subseteq R$ and $\mathcal{C}_k(I)$ is a k-ideal of R. This implies that $\mathcal{C}_k(I) = I$ or $\mathcal{C}_k(I) = R$. Hence I is a k-ideal or $\mathcal{C}_k(I) = R$.

(2) Let I be a k-ideal and a maximal ideal. Suppose to the contrary, that I is not k-maximal. Then there exists a k-ideal J of R such that $I \subsetneq J \subsetneq R$. However, since I is a maximal ideal and J is an ideal of R, no such an ideal J exists unless $J = R$, which contradicts the fact that $J \subsetneq R$. Therefore I must be k-maximal.

A proper ideal P of a ternary semiring R is called *k-prime* if $IJK \subseteq P$ implies $I \subseteq P$, $J \subseteq P$, or $K \subseteq P$ for all k-ideals I, J, K of R.

Every prime ideal of R is obviously k-prime. We describe in the following theorem, where being prime and k-prime of an ideal always imply each other.

Theorem 3. *Let R be a ternary semiring and P be a k-ideal of R. Then P is k-prime if and only if P is prime.*

Proof. If P is prime, then it is clear that P is k-prime. Assume that P is a k-prime k-ideal of R. Let I, J and K be ideals of R such that $IJK \subseteq P$. By Proposition 9 and Theorem 1, we have

$$\mathcal{C}_k(I)\mathcal{C}_k(J)\mathcal{C}_k(K) \subseteq \mathcal{C}_k(IJK) \subseteq \mathcal{C}_k(P) = P.$$

This implies that $\mathcal{C}_k(I)\mathcal{C}_k(J)\mathcal{C}_k(K) \subseteq P$. Since P is k-prime and $\mathcal{C}_k(I)$, $\mathcal{C}_k(J)$ and $\mathcal{C}_k(K)$ are the smallest k-ideals containing I, J and K, respectively, we obtain that $I \subseteq \mathcal{C}_k(I) \subseteq P$, $J \subseteq \mathcal{C}_k(J) \subseteq P$, or $K \subseteq \mathcal{C}_k(K) \subseteq P$. Therefore, P is prime.

Proposition 12. *Let R be a commutative ternary semiring and P be a proper ideal of R. If P is prime, then $P = (P : R, R)$.*

Proof. For any prime ideal P of R, by Proposition 10, we have that $P \subseteq (P : R, R)$. Assume that $P \neq (P : R, R)$. Then there exists $x \in (P : R, R)$ but $x \notin P$. So $xRR \subseteq P$. This implies that $\langle x \rangle RR \subseteq P$. However, since $x \notin P$, we have that $\langle x \rangle \nsubseteq P$ which is a contradiction because P is prime. This forces $P = (P : R, R)$.

Proposition 13. *Let R be a commutative ternary semiring and P be a proper ideal of R. If P is k-prime, then $P = (P : R, R)$.*

Proof. It is similar to Proposition 12.

A subset A of R is called a *multiplicatively closed set* if $abc \in A$ for all $a, b, c \in A$.

Theorem 4. *Let P be a proper k-ideal of a ternary semiring R. Then P is k-prime if and only if $R \setminus P$ is a multiplicatively closed set.*

Proof. Assume that P is k-prime. First of all, we will show that $R \setminus P$ is multiplicatively closed, we thus let $a, b, c \in R \setminus P$. To show that $abc \in R \setminus P$, we suppose for contradiction, that $abc \in P$. Since P is k-prime, this implies $a \in P$, $b \in P$, or $c \in P$. This contradicts the fact that $a, b, c \in R \setminus P$. Thus $abc \in R \setminus P$ and $R \setminus P$ is multiplicatively closed. Conversely, assume that $R \setminus P$ is multiplicatively closed. We need to show that P is k-prime. Let I, J, K be k-ideals of R such that $IJK \subseteq P$. We need to show that $I \subseteq P$, $J \subseteq P$, or $K \subseteq P$. Suppose to the contrary that there exist $a \in I$, $b \in J$ and $c \in K$ such that a, b and c are not members in P. Then we obtain that $abc \in IJK$ and $a, b, c \in R \setminus P$. Since $R \setminus P$ is multiplicatively closed, we then obtain $abc \in R \setminus P$. This contradicts $abc \in IJK \subseteq P$. Hence $a \in P$, $b \in P$, or $c \in P$. So P is k-prime.

A proper ideal Q of R is called *k-semiprime* if $I^3 \subseteq Q$ implies $I \subseteq Q$ for every k-ideal I of R. We have some remarks as the following.

(1) Every prime ideal is semiprime.

(2) Every k-prime ideal is k-semiprime.

(3) Every semiprime ideal is k-semiprime.

Theorem 5. *Let R be a ternary semiring and Q be a k-ideal of R. Then Q is k-semiprime if and only if Q is semiprime.*

Proof. If Q is semiprime, then Q is clearly k-semiprime. Let Q be a k-semiprime k-ideal of R and assume that $I^3 \subseteq Q$ for an ideal I of R. Then

$$\mathcal{C}_k(I)\mathcal{C}_k(I)\mathcal{C}_k(I) \subseteq \mathcal{C}_k(I^3) \subseteq \mathcal{C}_k(Q) = Q.$$

Then $\mathcal{C}_k(I)\mathcal{C}_k(I)\mathcal{C}_k(I) \subseteq Q$. Since Q is k-semiprime and $\mathcal{C}_k(I)$ is the smallest k-ideal containing I, we get $I \subseteq Q$.

4. Conclusion

In this paper, we focus on various properties of k-ideals of a ternary semiring R. We show that $\mathrm{Ann}_R(A)$ is a k-ideal of R for every nonempty subset A of R. For an ideal I of R, we prove that $\mathcal{C}_k(I)$ is the smallest k-ideal containing I. In particular, for a k-ideal I and ideals J, K of a commutative ternary semiring R, we demonstrate that $(I : J, K)$ is a k-ideal of R. Finally, we explore the relationship between k-ideals and k-maximal, k-prime and k-semiprime ideals.

In future work, we can study other types of ideals and their properties in ternary semirings.

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