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Comparative Results on Stability Analysis for Three Dimensional Functional Equations

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Abstract. This study investigates the stability of a three-dimensional cubic functional equation within several mathematical frameworks, including (n, β)-normed spaces, non-Archimedean (n, β)-normed spaces, and random normed spaces. The theoretical stability results are validated through experimental approaches, offering practical insight into the behavior of these functional equations. A comparative analysis is provided, highlighting differences in stability dynamics across the various spaces. Notably, the introduction of (n, β)-normed spaces and their non-Archimedean counterparts presents a novel framework for analyzing stability, while the inclusion of random normed spaces adds a stochastic dimension to the analysis. The experimental validation further strengthens the practical application of the stability results, distinguishing this study from traditional approaches.

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Key Words and Phrases: Functional equation, Stability, (n, β)-normed space, Random normed space

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1. Introduction

In nearly every branch of mathematical analysis, a fundamental question arises: under what conditions does an object that approximately satisfies a particular property also lie close to an object that exactly satisfies that same property? When this question is explicitly applied to functional equations, it leads to a crucial inquiry: if a solution to a slightly perturbed version of a functional equation is found, under what circumstances can we guarantee that this solution remains close to the exact solution of the original equation? This question of stability was first raised by Ulam [1] in the context of group homomorphisms in 1940. It has since been central to many developments in the theory of functional equations, particularly in the study of their stability. In 1941, Hyers provided an initial answer to Ulam's question regarding the stability of functional equations in Banach spaces [2]. Rassias extended this foundational work in 1978, introducing a generalized version of Hyers' theorem by considering additive mappings with unbounded control functions [3]. Rassias' extension significantly contributed to the development of what is now called Hyers-Ulam-Rassias stability for functional equations. Over time, several mathematicians have explored related problems in various spaces, further enriching the study of functional equations and their stability. Notable contributions include the investigation of intuitionistic fuzzy stability [4], functional equations in Menger-ϕ normed spaces [5], and functional equations in modular spaces [6]. Additionally, research has addressed quartic functional equations [7], fractional differential equations [8, 9], and fuzzy approximately cubic mappings [10]. Further studies have examined additive mappings in 2-Banach spaces and other related topics [11], broadening the understanding of functional equations in diverse mathematical frameworks.

On the other hand, some mathematicians have extended the concept of normed linear spaces. Gähler [12, 13] initiated the exploration of multi-norms within linear spaces. Significant contributions to the development of n-normed spaces have been made by Gunawan and Mashadi [14], Malčeski [15], Kim and Cho [16], and Misiak [17]. Park [11] studied approximate additive mappings, Jensen mappings, and related topics in 2-Banach spaces. In 2012, Xu and Rassias [18] investigated the stability of cubic and general mixed additive functional equations in n-Banach spaces. In 2015, Yang et al. introduced the concepts of non-Archimedean (n, β)-normed space (NA-(n, β)-NS) and (n, β)-normed space [19]. In 2022, Jyotsana et al. examined the stability of additive functional equations, quartic functional equations, and a-cubic and b-cubic functional equations in NA-(n, β)-normed spaces [20].

The theory of random normed spaces (RNS) is significant because it generalizes the deterministic results observed in linear normed spaces and has applications in the study of random operators and functional equations. Recent studies have extensively examined the stability and related properties of functional equations in various mathematical spaces. Researchers have explored fuzzy approximately cubic mappings [10], the σ-quadratic functional equation [21], and quadratic equations [22]. The Cauchy functional equation has also been investigated in random normed spaces [23], alongside analyses of cubic and quadratic mappings [24], as well as cubic and quartic mappings [25]. Additionally, Felbin's type non-Archimedean fuzzy normed spaces [26] have been studied, further enriching the field and expanding its theoretical foundations. In 2020, Govindan et al. [27] investigated the stability and solutions of the cubic functional equation.

$$\begin{aligned} g(2u+w+v) &= 3g(w+u+v)+g(w-u+v)+2g(w+u)+2g(u+v)-6g(u-w)\\ &\quad -6g(u-v)-3g(v+w)+2g(2u-v)+2g(2u-w)-6g(v)-6g(w)\\ &\quad -18g(u). \end{aligned} \tag{1}$$

1.1. Preliminaries and Definitions

In this subsection, we generalize the basic definitions, terminology, notations, and typical characteristics of (n,β)-NS, NA-(n,β)-NS, and RNS.

Definition 1. *[20] "Let $U(R)$ be a vector space with $\dim U \geq n$, and let $||\cdot,\dots,\cdot||_\beta : U^n \to R$ be a mapping that satisfies the following properties:*

(i) $||u_1,\dots,u_n||_\beta = 0$ *if and only if* $u_1,\dots,u_n$ *are linearly dependent,*

(ii) $||u_1,\dots,u_n||_\beta$ *remains unchanged under any permutation of the elements* $u_1,\dots,u_n$*,*

(iii) $||cu_1,\dots,u_n||_\beta = |c|^\beta ||u_1,\dots,u_n||_\beta$ *for any scalar* $c \in R$*,*

(iv) $||u_1,\dots,u_{n-1},u_n+u_{n+1}||_\beta \leq ||u_1,\dots,u_{n-1},u_n||_\beta + ||u_1,\dots,u_{n-1},u_{n+1}||_\beta$

for all $u_1,\dots,u_{n+1} \in U$*, and for* $0<\beta\leq 1$*. The mapping* $||\cdot,\dots,\cdot||_\beta$ *is referred to as the* (n,β)*-norm, and the pair* $(U, ||\cdot,\dots,\cdot||_\beta)$ *is called an* (n,β)*-normed space (NS)."*

Lemma 1. *[20] "Suppose* $(U, ||.,...,.||_\beta)$ *is an* (n,β)*-NS,* $n\geq 2, 0<\beta\leq 1$*. If* $v\in U$ *and* $||v,u_1,...,u_{n-1}||_\beta = 0$ *for all linearly independent vectors* $u_1,...,u_{n-1}\in U$*, then* $v=0$*."*

Definition 2. *[20] "A sequence* $\{v_m\}$ *in a* (n,β)*-NS* U *is called convergent sequence if*

$$lim_m \to_\infty ||v_m - v, u_1, ..., u_{n-1}||_\beta = 0$$

and it is called the Cauchy sequence if

$$lim_{m,k} \to_\infty ||v_m - v_k, u_1, ..., u_{n-1}||_\beta = 0,$$

for all $u_1,...,u_{n-1}\in U$*. If every Cauchy sequence converges in linear* (n,β)*-NS, it is called complete* (n,β)*-NS."*

Definition 3. *[20] "Let* $V(R)$ *be a vector space with* $\dim(V)\geq n$*, equipped with a non-Archimedean (NA) nonzero valuation* $|\cdot|$*, and let* β *be a constant such that* $0<\beta\leq 1$*. A mapping* $||\cdot,\dots,\cdot||_\beta : V^n \to R$ *is referred to as a NA-*(n,β)*-norm on* V *if it satisfies the following conditions:*

(i) $||u_1,\dots,u_n||_\beta = 0$ *if and only if* $u_1,\dots,u_n$ *are linearly dependent,*

(ii) $||u_1, \ldots, u_n||_\beta$ *is invariant under any permutation of the elements* $u_1, \ldots, u_n$*,*

(iii) $||cu_1, \ldots, u_n||_\beta = |c|^\beta ||u_1, \ldots, u_n||_\beta$ *for all* $c \in R$*,*

(iv) $||v_1 + v_2, u_1, \ldots, u_{n-1}||_\beta \leq \max\{||v_1, u_1, \ldots, u_{n-1}||_\beta, ||v_2, u_1, \ldots, u_{n-1}||_\beta\}$

for all $v_1, v_2, u_1, \ldots, u_n \in V$*. The pair* $(V, ||\cdot, \ldots, \cdot||_\beta)$ *is called a NA-*(n,β)*-normed space (NS)."*

Definition 4. *[28] "A function* $T : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if it satisfies the following properties:*

(i) T *is commutative, associative, and continuous.*

(ii) $T(\alpha_1, 1) = \alpha_1 \quad \forall \quad \alpha_1 \in [0,1]$.

(iii) $T(\alpha_1, \alpha_2) \leq T(\alpha_3, \alpha_4)$ *whenever* $\alpha_1 \leq \alpha_3$ *and* $\alpha_2 \leq \alpha_4$*, for all* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$.

Remark 1. *Let* T *be a t-norm and* $\{u_m\}$ *be a sequence then* $T_{i=1}^m u_i$ *is defined as*

$$T_{i=1}^m u_i = \begin{cases} u_1, & \text{if } m = 1. \\ T(T_{i=1}^{m-1} u_i, u_m), & \text{if } m \geq 2. \end{cases}$$

Here, Δ^+ *denotes the set of distribution functions, which are functions* $F : \mathbb{R} \to [0,1]$ *with the following properties: they are left-continuous, non-decreasing over the real numbers, and satisfy* $F(0) = 0$*. A subset of* Δ^+*, denoted* D^+*, consists of all functions* F *such that* $l^- F(+\infty) = 1$*, where* $l^- F(u) = \lim_{t \to u^-} F(t)$*. The function* ϵ_0 *is a particular distribution function defined as follows:*

$$\epsilon_0(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Definition 5. *[29] "A triplet* (Z, μ, T) *is called RNS if it satisfies the below conditions:*

(i) $\mu_u(t) = \epsilon_0(t) \quad \forall \quad t > 0 \iff u = 0$;

(ii) $\mu_{\alpha u}(t) = \mu_u(\frac{t}{|\alpha|}) \quad \forall \quad u \in Z, \alpha \neq 0$;

(iii) $\mu_{u_1 + u_2}(t + s) \geq T(\mu_{u_1}(t), \mu_{u_2}(s)) \quad \forall \quad u_1, u_2 \in Z$ *and all* $t, s \geq 0$*."*

Definition 6. *[30] Consider a RNS* (Z, μ, T)*. We define the following:*

(i) A sequence $\{u_\alpha\}$ *in* Z *converges to an element* $u \in Z$ *if, for any* $s > 0$ *and* $\lambda > 0$*, there exists a positive integer* n_0 *such that* $\mu_{u_n - u}(s) > 1 - \lambda$ *for all* $n \geq n_0$.

(ii) A sequence $\{u_\alpha\}$ *in* Z *is termed Cauchy if, for any* $s > 0$ *and* $\lambda > 0$*, there exists a positive integer* n_0 *such that* $\mu_{u_n - u_m}(s) > 1 - \lambda$ *for all* $n \geq m \geq n_0$.

A RNS (Z, μ, T) is called complete if every Cauchy sequence in Z converges.

The contributions of this paper lie in its comprehensive analysis of the stability of three-dimensional cubic functional equations across distinct mathematical frameworks. The study introduces (n, β)-normed spaces and non-Archimedean (n, β)-normed spaces as novel settings for conducting stability investigations, thereby extending classical approaches into new theoretical domains. Furthermore, the inclusion of random normed spaces adds a stochastic perspective, enhancing the scope and relevance of the analysis in probabilistic contexts. A key feature of this work is the comparative evaluation of stability behavior across these frameworks, which uncovers both similarities and critical differences. In addition, the theoretical results are supported by experimental validation, demonstrating the practical applicability of the proposed stability conditions. Collectively, these contributions offer a broader and deeper understanding of functional equation stability, bridging deterministic and stochastic approaches in a unified framework. The motivation for this study stems from the need to explore how the nature of the underlying space influences the stability of functional equations, particularly in higher-dimensional and non-traditional settings. The novelty lies not only in the introduction of the (n, β)-normed and non-Archimedean variants but also in the integration of a stochastic viewpoint through random normed spaces, which has not been extensively considered in previous stability analyses. This unified and comparative approach opens new avenues for theoretical advancements and practical applications in the study of functional equations.

This article is organized into seven sections. Section 1 presents the introduction and background of the study. In Sections 2 and 3, we investigate the stability of the cubic functional equation (1) within the frameworks of (n, β)-normed spaces and non-Archimedean (n, β)-normed spaces, respectively. Section 4 is devoted to the analysis of stability in random normed spaces. Sections 5 and 6 provide the experimental results and a comparative evaluation of the findings across the different spaces. Finally, the conclusions of the study are presented in Section 7.

Throughout this article, U is a linear sapce, V is a complete (n, β)-NS, W is a NA-(n, β)-NS, (X, μ, min) is a complete RNS and (Z, μ', min) is a RNS. PMI stands for the principle of mathematical induction.

2. Stability in (n, β)-Normed Space

The (n, β)-normed space ((n, β)-NS) is a generalization of normed linear spaces, where instead of a single norm, a family of norms is considered. The parameters n and β provide flexibility in defining the size metric and influence the behavior of the norms. (n, β)-NS is used to explore the stability and convergence of solutions in functional equations. These spaces help analyze the characteristics of the functions involved, allowing us to determine how small perturbations affect the solutions. Using the approach from [19], we address the stability problem for the given functional equation (1) in (n, β)-NS.

Theorem 1. *Let* $\Omega : U \times U \times U \to [0, \infty)$ *be a mapping holds*

$$\Phi(u) = \sum_{j=1}^{\infty} \frac{1}{2^{3j\beta}} \Omega(2^{j-1}u, 0, 0) < \infty \tag{2}$$

and

$$lim_{m\to\infty} \frac{1}{2^{3m\beta}} \Omega(2^m u, 0, 0) = 0. \tag{3}$$

If $\psi : V^{n-1} \to [0, \infty)$ *is a function and* $f : U \to V$ *is an odd mapping holding*

$$\begin{aligned}
&\|f(2u + w + v) - 3f(v + w + u) - f(w - u + v) - 2f(u + v)\\
&-2f(w + u) + 6f(u - v) + 6f(u - w) + 3f(w + v) - 2f(2u - v)\\
&-2f(2u - w) + 6f(v) + 6f(w) + 18f(u), \delta_1, ..., \delta_{n-1}\|_\beta\\
&\leq \Omega(u, v, w)\psi(\delta_1, ..., \delta_{n-1})
\end{aligned} \tag{4}$$

for all $u, v, w \in U$ *and* $\delta_1, ..., \delta_{n-1} \in V$*, then there is a unique cubic mapping* $Q : U \to V$ *satisfying*

$$||f(u) - Q(u), \delta_1, ..., \delta_{n-1}||_\beta \leq \Phi(u)\psi(\delta_1, ..., \delta_{n-1}). \tag{5}$$

Proof. Putting $(u, v, w) = (u, 0, 0)$ in (4) and multiplying both side by $\frac{1}{3^{-\beta}}$, we have

$$\|f(2u) - 2^3 f(u), \delta_1, ..., \delta_{n-1}\|_\beta \leq 3^{-\beta}\ell(u, 0, 0)\psi(\delta_1, ..., \delta_{n-1}).$$

Let

$$\Omega(u, 0, 0)\psi(\delta_1, ..., \delta_{n-1}) = M(u, \delta_1, ..., \delta_{n-1}).$$

This implies that

$$\|f(2u) - 2^3 f(u), \delta_1, ..., \delta_{n-1}\|_\beta \leq 3^{-\beta} M(u, \delta_1, ..., \delta_{n-1}). \tag{6}$$

Replace u by $2u$ in (6)

$$||f(2^2u) - 2^3 f(2u), \delta_1, ..., \delta_{n-1}||_\beta \leq 3^{-\beta} M(2u, \delta_1, ..., \delta_{n-1}). \tag{7}$$

Now, using (6) and (7)

$$\begin{aligned}
&\|f(2^2u) - (2^3)^2 f(u), \delta_1, ..., \delta_{n-1}\|_\beta\\
&= ||f(2^2u) - 2^3 f(2u) + 2^3 f(2u) - (2^3)^2 f(u), \delta_1, ..., \delta_{n-1}||_\beta\\
&\leq ||f(2^2u) - 2^3 f(2u), \delta_1, ..., \delta_{n-1}||_\beta + 2^{3\beta}||f(2u) - (2^3)f(u), \delta_1, ..., \delta_{n-1}||_\beta\\
&\leq 3^{-\beta} M(2u, \delta_1, ..., \delta_{n-1}) + 3^{-\beta}2^{3\beta} M(u, \delta_1, ..., \delta_{n-1}).
\end{aligned}$$

By using PMI on m, we assume that

$$||f(2^m u) - (2^3)^m f(u), \delta_1, ..., \delta_{n-1}||_\beta \leq 3^{-\beta} \sum_{j=1}^{m} 2^{3(j-1)\beta} M(2^{m-j}u, \delta_1, ..., \delta_{n-1}) \tag{8}$$

for all $u \in U, \delta_1, ..., \delta_{n-1} \in V$ and $m \in N$. From (6), we can say that inequality (8) holds for $m = 1$.
Suppose (8) is true for certain $m > 1$. Changing u by $2u$ in (8), we get

$$||f(2^{m+1}u) - (2^3)^m f(2u), \delta_1, ..., \delta_{n-1}||_\beta \leq 3^{-\beta} \sum_{j=1}^{m} 2^{3(j-1)\beta} M(2^{m+1-j}u, \delta_1, ..., \delta_{n-1}).$$

Hence, it follows from (8)

$$\begin{aligned}
&||f(2^{m+1}u) - (2^3)^{m+1} f(u), \delta_1, ..., \delta_{n-1}||_\beta \\
&\leq ||f(2^{m+1}u) - (2^3)^m f(2u), \delta_1, ..., \delta_{n-1}||_\beta + 2^{3m\beta}||f(2u) - 2^3 f(u), \delta_1, ..., \delta_{n-1}||_\beta \\
&\leq 3^{-\beta} \sum_{j=1}^{m} 2^{3(j-1)\beta} M(2^{m+1-j}u, \delta_1, ..., \delta_{n-1}) + 3^{-\beta} 2^{3m\beta} M(u, \delta_1, ..., \delta_{n-1}) \\
&= 3^{-\beta} \sum_{j=1}^{m+1} 2^{3(j-1)\beta} M(2^{m+1-j}u, \delta_1, ..., \delta_{n-1}).
\end{aligned}$$

By using (8), we get

$$||2^{-3m} f(2^m u) - f(u), \delta_1, ..., \delta_{n-1}||_\beta \leq 3^{-\beta} \sum_{j=1}^{m} 2^{3(j-m-1)\beta} M(2^{m-j}u, \delta_1, ..., \delta_{n-1}). \tag{9}$$

If $k, m \in N$ with $k > m$ then authors observe from (6) that

$$\begin{aligned}
&||2^{-3k} f(2^k u) - 2^{-3m} f(2^m u), \delta_1, ..., \delta_{n-1}||_\beta \\
&\leq 3^{-\beta} \sum_{j=m}^{k-1} ||2^{-3j} f(2^j u) - 2^{-3(j+1)} f(2^{j+1}u), \delta_1, ..., \delta_{n-1}||_\beta \\
&= 3^{-\beta} \sum_{j=m}^{k-1} 2^{-3(j+1)\beta} ||2^3 f(2^j u) - f(2^{j+1}u), \delta_1, ..., \delta_{n-1}||_\beta \\
&\leq 3^{-\beta} \sum_{j=m}^{k-1} 2^{-3(j+1)\beta} M(2^j u, \delta_1, ..., \delta_{n-1}) \\
&= 3^{-\beta} \sum_{j=m}^{k-1} 2^{-3(j+1)\beta} \Omega(u, 0, 0) \psi(\delta_1, ..., \delta_{n-1}).
\end{aligned}$$

By using Lemma(1) and applying limit as $m, k \to \infty$, we get

$$lim_{m,k\to\infty}||2^{-3k}f(2^k u) - 2^{-3m}f(2^m u), \delta_1, ..., \delta_{n-1}||_\beta = 0.$$

According to definition(2), we say that $\{2^{-3m}f(2^m u)\}$ is a Cauchy sequence in V (complete (n, β)-NS), therefore this sequence converges to the point $Q(u) \in V$. Now, we consider a mapping $Q : U \to V$ such that

$$Q(u) = lim_{m\to\infty}2^{-3m}f(2^m u).$$

Replacing (u, v, w) by $(2^m u, 2^m v, 2^m w)$ in (4) and multiplying both side by $\frac{1}{2^{3m\beta}}$, we have

$$\begin{aligned}
&2^{-3m\beta}||f(2^m(2u + w + v)) - 3f(2^m(v + w + u)) - f(2^m(w - u + v))\\
&-2f(2^m(u + v)) - 2f(2^m(w + u)) + 6f(2^m(u - v)) + 6f(2^m(u - w))\\
&+3f(2^m(w + v)) - 2f(2^m(2u - v)) - 2f(2^m(2u - w)) + 18f(2^m u) + 6f(2^m v)\\
&+6f(2^m w), \delta_1, ..., \delta_{n-1}||_\beta \le 3^{-\beta}2^{-3m\beta}\Omega(2^m u, 2^m v, 2^m w)\psi(\delta_1, ..., \delta_{n-1}).
\end{aligned}$$

Thus by using equation (3) and Lemma (1), we get

$$\begin{aligned}
&||Q(2u + w + v) - Q(w - u + v) - 3Q(w + u + v) - 2Q(u + v)\\
&+6Q(u - v) - 2Q(w + u) + 6Q(u - w) + 3Q(w + v) - 2Q(2u - v)\\
&-2Q(2u - w) + 18Q(u) + 6Q(v) + 6Q(w), v_1, ..., v_{n-1}||_\beta\\
&= lim_{m\to\infty}||2^{-3m}f(2u + w + v)) - 3.2^{-3m}f(2^m(w + u + v))\\
&-2^{-3m}f(2^m(w - u + v)) - 2.2^{-3m}f(2^m(u + v))\\
&-2.2^{-3m}f(2^m(u + w)) + 6.2^{-3m}f(2^m(u - v)) + 6.2^{-3m}f(2^m(u - w))\\
&-2.2^{-3m}f(2^m(2u - v)) + 3.2^{-3m}f(2^m(w + v)) - 2f(2^{-3m}(2u - w))\\
&+18.2^{-3m}f(2^m u) + 6.2^{-3m}f(2^m v) + 6.2^{-3m}f(2^m w), \delta_1, ..., \delta_{n-1}||_\beta\\
&\le lim_{m\to\infty}3^{-\beta}.2^{-3m\beta}\Omega(2^m u, 2^m v, 2^m w)\psi(\delta_1, ..., \delta_{n-1}) = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&||Q(2u + w + v) - 3Q(v + w + u) - Q(w - u + v) - 2Q(v + w)\\
&+6Q(u - v) - 2Q(w + u) + 6Q(u - w) + 3Q(w + v) - 2Q(2u - v)\\
&-2Q(2u - w) + 6Q(v) + 18Q(u) + 6Q(w), \delta_1, ..., \delta_{n-1}||_\beta = 0.
\end{aligned}$$

In (9), putting the limit as $m \to \infty$, we get

$$\begin{aligned}
&||Q(u) - f(u), \delta_1, ..., \delta_{n-1}||_\beta\\
&\le lim_{m\to\infty}\sum_{j=1}^{m}2^{j-1-m}\Omega(2^{m-j}u, 0, 0)\psi(\delta_1, ..., \delta_{n-1})\\
&= \Phi(u)\psi(\delta_1, ..., \delta_{n-1})
\end{aligned}$$

which shows (5). To establish uniqueness of the mapping Q, suppose another cubic mapping $Q' : U \to V$ satisfies (5). We get

$$\begin{aligned}&||Q(u) - Q'(u), v_1, ..., v_{n-1}||_\beta \leq 3^{-\beta}.2^{-m\beta}||Q(2^m u) - f(2^m u), \delta_1, ..., \delta_{n-1}||_\beta\\&+3^{-\beta}.2^{-m\beta}||f(2^m u) - Q'(2^m u), \delta_1, ..., \delta_{n-1}||_\beta\\&\leq 3^{-\beta}.2^{-m\beta+1}\Phi(2^m u)\psi(\delta_1, ..., \delta_{n-1}) \to 0 \quad as \quad m \to \infty,\end{aligned}$$

now, with the help of Lemma(1), it is proved that $Q(u) = Q'(u)$.

Corollary 1. *If $f : U \to V$ is a function satisfying*

$$\begin{aligned}&\|f(2u + v + w) - 3f(u + v + w) - f(-u + v + w) - 2f(u + v) - 2f(u + w)\\&+6f(u - v) + 6f(u - w) + 3f(v + w) - 2f(2u - v) - 2f(2u - w) + 18f(u)\\&+6f(v) + 6f(w), \delta_1, ..., \delta_{n-1}\|_\beta \leq \epsilon(||u||_\beta + ||v||_\beta + ||w||_\beta),\end{aligned}$$

then there is a unique cubic function $Q : U \to V$ holding

$$||f(u) - Q(u), \delta_1, ..., \delta_{n-1}||_\beta \leq \epsilon\frac{1}{6^\beta(2^{2\beta} - 1)}||u||_\beta.$$

Proof. Put $\Omega(u, 0, 0) = (||u||_\beta + ||v||_\beta + ||w||_\beta)$ and $\psi(\delta_1, ..., \delta_{n-1}) = \epsilon$ in above Theorem, We get the intended outcome.

Example 1. *Let $\phi : V^{n-1} \to [0, \infty)$ be a constant mapping such that*

$$\phi(\delta_1, ..., \delta_{n-1}) = 1$$

for all $\delta_1, ...\delta_{n-1} \in V$ and $f : U \to V$ be a mapping defined as $f(u) = u^3 + ||u||_\beta u_0$, where u_0 is the unit vector in U. An easy calculation demonstrates that

$$\begin{aligned}&||f(2u + w + v) - f(w - u + v) - 3f(w + u + v) - 2f(u + v) - 2f(w + u)\\&+6f(u - v) + 6f(u - w) - 2f(2u - v) + 3f(w + v) - 2f(2u - w) + 18f(u)\\&+6f(v) + 6f(w), \delta_1, ..., \delta_{n-1}\|_\beta \leq (48||u||_\beta + 24||v||_\beta + 24||w||_\beta)u_0,\end{aligned}$$

and

$$\phi(u) = \sum_{j=1}^{\infty} \frac{1}{2^{3j\beta}}(48||2^{j-1}u||_\beta u_0) = \frac{24^\beta}{4^\beta - 1}||u||_\beta u_0, \quad \lim_{m\to\infty} \frac{1}{2^{3m\beta}}(||2^m u||_\beta u_0) = 0.$$

Consequently, all the requirements of Theorem 2.1 are satisfied, implying the existence of a unique cubic mapping $Q : U \to V$ such that

$$||f(u) - Q(u), \delta_1, ...\delta_{n-1}||_\beta \leq \frac{8^\beta}{4^\beta - 1}||u||_\beta u_0.$$

3. Stability in Non-Archimedean-(n, β)-Normed Spaces

In NA-(n, β)-NS, the triangular property is modified such that the distance between two points can be dominated by the maximum magnitudes of the vectors rather than their sum of two distances. This space provides a framework for analyzing the stability of solutions to FEs over NA fields such as the p-adic numbers. Also, the convergence behavior in this space helps characterize the behavior of solutions and their limiting properties. By motivating the approach used in [20], we find stability problems for FE (1) in NA-(n, β)-NS.

In the following theorem 2 the , $\theta \geq 0$: A constant controlling the "degree" of approximation. $p, r, q > 0$: Exponents that influence how the right-hand side scales with respect to the norms of u, v, w. And $\phi : V^{n-1} \to [0, \infty)$ is a control function depending on parameters $\delta_1, ..., \delta_{n-1}$ used to describe perturbation or control of deviation.

Theorem 2. *Let $\theta \geq 0$, $p, r, q > 0$ with $(p + r + q)\beta_1 < 3\beta$, $0 < \beta, \beta_1 \leq 1$. and $\phi : V^{n-1} \to [0, \infty)$ be a function. If $f : V \to W$ is an odd function that fulfills inequality*

$$\begin{aligned} &||f(2u + w + v) - 3f(v + w + u) - f(w - u + v) - 2f(u + v) - 2f(u + w) \\ &+6f(u - v) + 6f(u - w) - 2f(2u - v) + 3f(v + w) - 2f(2u - w) + 18f(u) \\ &+6f(v) + 6f(w), \delta_1, ..., \delta_{n-1}||_\beta \leq \theta||u||^p_{\beta_1}||v||^q_{\beta_1}||w||^r_{\beta_1}\phi(\delta_1, ..., \delta_{n-1}), \end{aligned} \tag{10}$$

then there is a unique cubic mapping $Q : W \to V$ such that

$$||f(u) - Q(u), \delta_1, ..., \delta_{n-1}||_\beta \leq \theta 2^{-3\beta}||u||^p_{\beta_1}\phi(\delta_1, ..., \delta_{n-1}) \tag{11}$$

for all $u, v, w \in W$ and $\delta_1, ..., \delta_{n-1} \in V$.

Proof. Putting $(u, v, w) = (u, 0, 0)$ in (10) and multiplying both side by $\frac{1}{3^\beta.2^{3\beta}}$, we get

$$||\frac{f(2u)}{2^3} - f(u), \delta_1, ..., \delta_{n-1}||_\beta \leq \theta 3^{-\beta}2^{-3\beta}||u||^p_{\beta_1}\phi(\delta_1, ..., \delta_{n-1}). \tag{12}$$

By putting $u = 2^m u$ in (12) and multiplying both side by $\frac{1}{2^{3m\beta}}$, we get

$$\begin{aligned} &||\frac{f(2^{m+1}u)}{2^{3m+3}} - \frac{f(2^m u)}{2^{3m}}, \delta_1, ..., \delta_{n-1}||_\beta \leq \theta 3^{-\beta}2^{-3\beta}2^{-3m\beta}||2^m u||^p_{\beta_1}\phi(\delta_1, ..., \delta_{n-1}) \\ &= \theta 3^{-\beta}2^{-3\beta}2^{-3m\beta}2^{mp\beta}||u||^p_{\beta_1}\phi(\delta_1, ..., \delta_{n-1}) \\ &= \theta 3^{-\beta}2^{-3\beta}2^{-3m\beta}(2^{(p\beta_1 - 3\beta)})^m||u||^p_{\beta_1}\phi(\delta_1, ..., \delta_{n-1}). \end{aligned}$$

Since $p\beta_1 < 3\beta$, taking limit as $m \to \infty$, we have

$$||\frac{f(2^{m+1}u)}{2^{3m+3}} - \frac{f(2^m u)}{2^{3m}}, \delta_1, ..., \delta_{n-1}||_\beta = 0.$$

Hence, in complete space V, the sequence $< 2^{-3m}f(2^m u) >$ is a Cauchy sequence. Therefore, this sequence converges to $Q(u) \in V$. Now, we consider $Q : W \to V$ a mapping holds

$$Q(u) = lim_{m\to\infty}2^{-3m}f(2^m u). \tag{13}$$

Now, we have to prove that Q is a cubic. By using (10), (13) and Lemma (1), we get

$$\begin{aligned}
&||Q(2u+v+w)-3Q(v+w+u)-Q(w-u+v)-2Q(w+u)-2Q(u+v)\\
&+6Q(u-w)+6Q(u-v)+3Q(w+v)-2Q(2u-v)-2Q(2u-w)+18Q(u)\\
&+6Q(v)+6Q(w),\delta_1,...,\delta_{n-1}||_\beta = lim_{m\to\infty}|2^{-3m\beta}|||f(2^m(2u+w+v))\\
&-3f(2^m(w+u+v))-f(2^m(w-u+v))-2f(2^m(u+v))-2f(2^m(u+w))\\
&+6f(2^m(u-w))+6f(2^m(u-v))+3f(2^m(v+w))-2f(2^m(2u-v))\\
&-2f(2^m(2u-w))+18f(2^m u)+6f(2^m v)+6f(2^m w),\delta_1,...,\delta_{n-1}||_\beta\\
&\le lim_{m\to\infty}\theta(2^{-3m\beta})||2^m u||^p_{\beta_1}||2^m v||^q_{\beta_1}||2w||^r_{\beta_1}\phi(\delta_1,...,\delta_{n-1})\\
&= lim_{m\to\infty}\theta(2^{((p+q+r)\beta_1-3\beta)})^m||u||^p_{\beta_1}||v||^q_{\beta_1}||w||^r_{\beta_1}\phi(\delta_1,...,\delta_{n-1}).
\end{aligned}$$

Since $(p+r+q)\beta_1 < 3\beta$, we get

$$\begin{aligned}
&||Q(2u+v+w)-Q(-u+v+w)-3Q(u+v+w)-2Q(u+w)-2Q(u+v)\\
&+6Q(u-v)+6Q(u-w)+3Q(v+w)-2Q(2u-w)-2Q(2u-v)+18Q(u)\\
&+6Q(v)+6Q(w),\delta_1,...,\delta_{n-1}||_\beta = 0.
\end{aligned}$$

Hence, Q is a cubic mapping. By substituting u with $2u$ in (12) and multiplying both side by $\frac{1}{2^{3\beta}}$, we obtain

$$||\frac{f(2^2u)}{2^6}-\frac{f(2u)}{2^3},\delta_1,...,\delta_{n-1}||_\beta \le \theta 3^{-\beta}2^{-3\beta}||2u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1}). \tag{14}$$

Thus, by (12) and (14), we get

$$\begin{aligned}
&||f(u)-\frac{f(2^2u)}{2^6},\delta_1,...,\delta_{n-1}||_\beta\\
&\le max\{||\frac{f(2u)}{2^3}-f(u),\delta_1,...,\delta_{n-1}||_\beta, ||\frac{f(2^2u)}{2^6}-\frac{f(2u)}{2^3},\delta_1,...,\delta_{n-1}||_\beta\}\\
&\le max\{\theta 3^{-\beta}2^{-3\beta}||u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1}), \theta 3^{-\beta}2^{-3\beta}||2u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1})\}.
\end{aligned}$$

Since $(p+r+q)\beta_1 < 3\beta$, we get

$$||f(u)-\frac{f(2^2u)}{2^6},\delta_1,...,\delta_{n-1}||_\beta \le \theta 3^{-\beta}2^{-3\beta}||u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1}).$$

By applying PMI on m, we that

$$||f(u)-\frac{f(2^m u)}{2^{3m}},\delta_1,...,\delta_{n-1}||_\beta \le \theta 3^{-\beta}2^{-3\beta}||u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1}). \tag{15}$$

Taking $u = 2u$ in (15) and multiplying both side by $\frac{1}{2^{3\beta}}$, we have

$$||\frac{f(2u)}{2^3}-\frac{f(2^{m+1}u)}{2^{3m+3}},\delta_1,...,\delta_{n-1}||_\beta \le 3^{-\beta}2^{-6\beta}||u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1}). \tag{16}$$

By using (12) and (16), we have

$$||f(u)-\frac{f(2^{m+1}u)}{2^{3(m+1)}},\delta_1,...,\delta_{n-1}||_\beta \leq \theta 3^{-\beta}2^{-3\beta}||u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1}).$$

Hence, the result (15) is true for all m. Now, taking limit as $m\to\infty$ in (15), we obtained (11). Now, to prove the Q is unique, assume Q' is an another cubic mapping which follows (11),

$$\begin{aligned}&||Q(u)-Q'(u),\delta_1,...,\delta_{n-1}||_\beta = 2^{-3m\beta}||Q(2^m u)-Q'(2^m u),\delta_1,...,\delta_{n-1}||_\beta\\ &\leq 2^{-3m\beta}max\{||Q(2^m u)-f(2^m u),\delta_1,...,\delta_{n-1}||_\beta, ||f(2^m u)-Q'(2^m u),\delta_1,...,\delta_{n-1}||_\beta\}\\ &\leq 3^{-\beta}2^{-3m\beta}2^{-3\beta}\theta||2^m u||^p_{\beta_1}\phi(\delta_1,...,\delta_{n-1}).\end{aligned}$$

Applying limit as $m\to\infty$, we find

$$||Q(u)-Q'(u),\delta_1,...,\delta_{n-1}||_\beta = 0.$$

By using Lemma (1), we find the uniqueness of mapping Q.

Theorem 3. *Let $\phi : U\times U\times U \to [0,\infty)$ be a mapping holds*

$$lim_{m\to\infty}|\frac{1}{2^{3m\beta}}|\phi(2^m u, 2^m v, 2^m w) = 0 \tag{17}$$

and $\psi : V^{n-1}\to[0,\infty)$ be a mapping. The limit

$$lim_{m\to\infty}max\{2^{-3j\beta}\psi(2^{j-1}u,0,0) : m\geq j\geq 1\} \tag{18}$$

exists and it is denoted by $\tilde{\phi}(u)$. Let $f : U\to V$ be an odd function fulfilling

$$\begin{aligned}&||f(2u+w+v)-f(v-u+w)-3f(v+w+u)-2f(w+u)-2f(w+v)\\ &+6f(u-v)+3f(w+v)+6f(u-w)-2f(2u-w)-2f(2u-v)+6f(v)\\ &+18f(u)+6f(w),\delta_1,...,\delta_{n-1}\|_\beta \leq \phi(u,v,w)\psi(\delta_1,...,\delta_{n-1}).\end{aligned} \tag{19}$$

Then, there is exactly one cubic mapping $Q : U\to V$ holds

$$||f(u)-Q(u),\delta_1,...,\delta_{n-1}||_\beta \leq 3^{-\beta}\tilde{\phi}(u)\psi(\delta_1,...,\delta_{n-1}) \tag{20}$$

and

$$lim_{k\to\infty}lim_{m\to\infty}max\{2^{-3\beta}\phi(u,0,0) : 1+k;\leq j\leq k+m\} = 0 \tag{21}$$

for all $u\in U$ and $\delta_1,...,\delta_{n-1}\in V$.

Proof. Putting $(u, v, w) = (u, 0, 0)$ in (19) and multiplying both side by $3^{-\beta}2^{-3\beta}$, we have

$$||\frac{f(2u)}{2^3} - f(u), \delta_1, ..., \delta_{n-1}||_\beta \leq 3^{-\beta}2^{-3\beta}\phi(u, 0, 0)\psi(\delta_1, ..., \delta_{n-1}). \tag{22}$$

Changing u by $2^m u$ in (22) and multiplying both side by $2^{-3m\beta}$, we get

$$||\frac{f(2^{m+1}u)}{2^{3m+3}} - \frac{f(2^m u)}{2^{3m}}, \delta_1, ..., \delta_{n-1}||_\beta \leq 3^{-\beta}2^{-3\beta}2^{-3m\beta}\phi(2^m u, 0, 0)\psi(\delta_1, ..., \delta_{n-1}).$$

Using (17) and take limit $m \to \infty$, we get

$$||\frac{f(2^{m+1}u)}{2^{3m+3}} - \frac{f(2^m u)}{2^{3m}}, \delta_1, ..., \delta_{n-1}||_\beta = 0.$$

Using definition (2), we obtain that $< 2^{-3m}f(2^m u) >$ is a Cauchy sequence in complete space V, therefore, this sequence has convergence point $Q(u) \in V$. Now, we consider a function $Q : U \to V$ holds

$$Q(u) = lim_{m\to\infty}2^{-3m}f(2^m u).$$

Now, we have to prove that Q is a cubic. Using (17), we get

$$\begin{aligned}
&||Q(2u + w + v) - 3Q(v + w + u) - Q(w - u + v) - 2Q(u + v) - 2Q(w + u) \\
&+6Q(u - v) + 6Q(u - w) + 3Q(v + w) - 2Q(2u - v) - 2Q(2u - w) + 18Q(u) \\
&+6Q(v) + 6Q(w), \delta_1, ..., \delta_{n-1}||_\beta = \lim_{m\to\infty} 2^{-3m\beta}||f(2^m(2u + w + v)) \\
&-3f(2^m(v + w + u)) - f(2^m(w - u + v)) - 2f(2^m(u + v)) - 2f(2^m(w + u)) \\
&+6f(2^m(u - v)) + 6f(2^m(u - w)) + 3f(2^m(w + v)) - 2f(2^m(2u - v)) \\
&-2f(2^m(2u - w)) + 18f(2^m u) + 6f(2^m v) + 6f(2^m w), \delta_1, ..., \delta_{n-1}||_\beta \\
&\leq \lim_{m\to\infty} 2^{-3m\beta}\phi(2^m u, 2^m v, 2^m w)\psi(\delta_1, ..., \delta_{n-1}) = 0,
\end{aligned}$$

hence

$$\begin{aligned}
&||Q(2u + v + w) - 3Q(v + w + u) - Q(w - u + v) - 2Q(u + v) - 2Q(w + u) \\
&+6Q(u - v) + 6Q(u - w) + 3Q(w + v) - 2Q(2u - v) - 2Q(2u - w) + 18Q(u) \\
&+6Q(v) + 6Q(w), \delta_1, ..., \delta_{n-1}||_\beta = 0.
\end{aligned}$$

Using Lemma (1), we obtain Q is cubic mapping. Next, substituting u by $2u$ in (22) and multiplying both side by $2^{-3\beta}$, the authors get

$$||\frac{f(2^2u)}{2^6} - \frac{f(2u)}{2^3}, \delta_1, ..., \delta_{n-1}||_\beta \leq |3|^{-\beta}|2|^{-6\beta}\phi(2u, 0, 0)\psi(\delta_1, ..., \delta_{n-1}).$$

By using (22), we get

$$||f(u) - \frac{f(2^2u)}{2^6}, v_1, ..., v_{n-1}||_\beta$$

$$\leq max\{3^{-\beta}2^{-3\beta}\phi(u,0,0)\psi(\delta_1,...,\delta_{n-1}),3^{-\beta}2^{-6\beta}\phi(2u,0,0)\psi(\delta_1,...,\delta_{n-1})\}.$$

Apply PMI on m, we have

$$\begin{aligned}
&||f(u)-\frac{f(2^m u)}{2^{3m}},\delta_1,...,\delta_{n-1}||_\beta\\
&\leq max\left\{3^{-\beta}\frac{\phi(2^{k-1}u,0,0)}{2^{3k\beta}}:1\leq k\leq m\right\}\psi(\delta_1,...,\delta_{n-1}). \qquad (23)
\end{aligned}$$

Changing u by $2u$ in (23) and multiplying both side by $2^{-3\beta}$, we have

$$\begin{aligned}
&||f(u)-\frac{f(2^{m+1}u)}{2^{3(m+1)}},\delta_1,...,\delta_{n-1}||_\beta\\
&\leq max\left\{3^{-\beta}\frac{\phi(u,0,0)}{2^{3\beta}},3^{-\beta}\frac{\phi(2^k u,0,0)}{2^{3(k+1)\beta}}:m\geq k\geq 1\right\}\psi(\delta_1,...,\delta_{n-1})\\
&=max\left\{3^{-\beta}\frac{\phi(2^k u,0,0)}{2^{3(k+1)\beta}}:m\geq k\geq 0\right\}\psi(\delta_1,...,\delta_{n-1})\\
&=max\left\{3^{-\beta}\frac{\phi(2^{k-1}u,0,0)}{2^{3k\beta}}:m+1\geq k\geq 1\right\}\psi(\delta_1,...,\delta_{n-1}).
\end{aligned}$$

Hence, the result (23) is true for all m. Taking limit as $m\to\infty$ in (23), we get (20). To prove Q is unique mapping. Assume Q' is another cubic mapping which follows (20). Now

$$\begin{aligned}
&lim_{k\to\infty}3^{-\beta}2^{-3k\beta}\phi(2^k u)\\
&=lim_{k\to\infty}3^{-\beta}\frac{1}{2^{3k\beta}}lim_{m\to\infty}max\{2^{-3j\beta}\psi(2^{j+k-1}u,0,0):1\leq k\leq m\}\\
&=lim_{k\to\infty}lim_{m\to\infty}max\{3^{-\beta}2^{-3j\beta}\psi(2^{j-1}u,0,0):1+k\leq j\leq m+k\},
\end{aligned}$$

by using (21), we have

$$\begin{aligned}
&||Q(u)-Q'(u),\delta_1,...,\delta_{n-1}||_\beta\\
&=lim_{k\to\infty}2^{-3k\beta}||Q(2^k u)-Q'(2^k u),\delta_1,...,\delta_{n-1}||_\beta\\
&\leq lim_{k\to\infty}2^{-3k\beta}max\{||Q(2^k u)-f(2^k u),\delta_1,...,\delta_{n-1}||_\beta,||f(2^k u)-Q'(2^k u),\delta_1,...,\delta_{n-1}||_\beta\}\\
&\leq lim_{k\to\infty}2^{-3k\beta}\phi(2^k u)\psi(\delta_1,...,\delta_{n-1})=0
\end{aligned}$$

By using Lemma (1), we find the uniqueness of mapping Q.

These are generalized Hyers-Ulam-Rassias stability of a cubic functional equation in a non-Archimedean $(n,\beta)-$ normed space. The original inequality shows an approximate cubic behavior. The conclusion guarantees the existence and uniqueness of a true cubic mapping nearby. The bound tells how close f is to this true cubic mapping.

Corollary 2. *If $f:U\to V$ is a function satisfying*

$$\|f(2u+w+v)-3f(u+v+w)-f(w-u+v)-2f(u+v)$$

$$-2f(w+u)+6f(u-v)+6f(u-w)+3f(w+v)-2f(2u-v)$$
$$-2f(2u-w)+6f(v)+18f(u)+6f(w),\delta_1,...,\delta_{n-1}||_\beta \leq (||u||+||v||+||w||)\epsilon,$$

then there is a unique cubic function $Q : U \to V$ *such that*

$$||f(u)-Q(u),\delta_1,...,\delta_{n-1}||_\beta \leq 24^{-\beta}||u||\epsilon.$$

Proof. Put $\phi(u,0,0)=||u||+||v||+||w||$ and $\psi(\delta_1,...,\delta_{n-1})=\epsilon$ in above Theorem, we obtained desired outcome.

4. Stability in Random Normed Space

An RNS is a generalization of the concept of an NS, where the norm can vary randomly. The study of RNS in functional analysis allows for investigating random variations in normed structures. In the study of the stability of FEs in RNS, one examines how small changes in the FE or its solutions behave under random variations of the norm. Using concepts from [30], we solve the stability problem for FE (1) in RNS.

Theorem 4. *Suppose* $\phi : U \times U \times U \to Z$ *is a function holds*

$$\mu'_{\phi(u,0,0)}(s) \geq \mu'_{k\phi(u,0,0)}(s) \tag{24}$$

and

$$lim_{m\to\infty}\mu'_{\phi(2^n u,2^n v,2^n w)}(8^n s)=1$$

$\forall\ u,w,v \in U,\ s>0$ *and* $0<k<8$. *If* $f : U \to X$ *is an odd function having*

$$\mu_\Theta(s) \geq \mu'_{\phi(u,v,w)}(s), \tag{25}$$

where

$$\begin{aligned}\Theta &= f(2u+w+v)-3f(v+w+u)-f(v+w-u)-2f(u+w)-2f(v+u)\\ &+ 6f(u-w)+6f(u-v)+3f(w+v)-2f(2u-v)-2f(2u-w)\\ &+ 6f(v)+18f(u)+6f(w)\end{aligned}$$

then there is exactly one cubic mapping $Q : U \to V$ *holds*

$$\mu_{f(u)-Q(u)}(s) \geq \mu'_{\phi(u,0,0)}(3(8-k)s). \tag{26}$$

Proof. Putting (u,v,w) by $(u,0,0)$ in (25), we have

$$\mu_{\frac{f(2u)}{2^3}-f(u)}(s) \geq \mu'_{\phi(u,0,0)}(24s). \tag{27}$$

Replacing u by $2^n u$ in (27), we get

$$\mu_{\frac{f(2^{n+1}u)}{2^{3(n+1)}}-\frac{f(2^n u)}{2^{3n}}}(s) \geq \mu'_{\phi(2^n u,0,0)}(3.2^{3(n+1)}s). \tag{28}$$

Since

$$\frac{f(2^n u)}{2^{3n}} - f(u) = \sum_{i=0}^{n-1} \left\{ \frac{f(2^{i+1}u)}{2^{3(i+1)}} - \frac{f(2^i u)}{2^{3i}} \right\}. \tag{29}$$

By using (28) and (29), we get

$$\mu_{\frac{f(2^n u)}{2^{3n}} - f(u)} \left(\sum_{i=0}^{n-1} \frac{sk^i}{3 \times 2^{3(n+1)}} \right) \geq T_{i=0}^{n-1} \left(\mu'_{\phi(u,0,0)}(s) \right) = \mu'_{\phi(u,0,0)}(s). \tag{30}$$

Therefore,

$$\mu_{\frac{f(2^n u)}{2^{3n}} - f(u)}(s) \geq \mu'_{\phi(u,0,0)} \left(\frac{s}{\sum_{i=0}^{n-1} \frac{k^i}{3 \times 2^{3(i+1)}}} \right). \tag{31}$$

Switching u by $2^m u$ in (31), we have

$$\mu_{(\frac{f(2^{m+n}u)}{2^{(m+n)}} - \frac{f(2^m u)}{2^m})}(s) \geq \mu'_{\phi(u,0,0)} \left(\frac{s}{\sum_{i=m}^{n+m} \frac{k^i}{3 \times 2^{3(i+1)}}} \right). \tag{32}$$

As,

$$lim_{n,m \to \infty} \mu'_{\phi(u,0,0)} \left(\frac{s}{\sum_{i=m}^{n+m} \frac{k^i}{3 \times 2^{3(i+1)}}} \right) = 1,$$

therefore, the sequence $\{\frac{f(2^n u)}{2^{3n}}\}$ is a Cauchy in X(complete-RNS), therefore there exists the convergence point $Q(u) \in X$ such that $lim_{m \to \infty} \frac{f(2^n u)}{2^{3n}} = Q(u)$. Fix $u \in U$ and taking $m = 0$ in (32), we have

$$\mu_{\frac{f(2^n u)}{2^{3n}} - f(u)}(s) \geq \mu'_{\phi(u,0,0)} \left(\frac{s}{\sum_{i=0}^{n-1} \frac{k^i}{2^{3(i+1)}}} \right) \tag{33}$$

and for every $\zeta > 0$, we have

$$\begin{aligned} \mu_{Q(u)-f(u)}(s+\zeta) &\geq T\left(\mu_{Q(u) - \frac{f(2^n u)}{2^{3n}}}(\zeta), \mu_{\frac{f(2^n u)}{2^{3n}} - f(u)}(s) \right) \\ &\geq T\left(\mu_{Q(u) - \frac{f(2^n u)}{2^{3n}}}(\zeta), \mu'_{\phi(u,0,0)} \left(\frac{s}{\sum_{i=0}^{n-1} \frac{k^i}{3 \times 2^{3(n+1)}}} \right) \right), \end{aligned} \tag{34}$$

considering limit as $n \to \infty$ in (34),

$$\mu_{Q(u)-f(u)}(s+\zeta) \geq \mu'_{\phi(u,0,0)}(3(8-k)s). \tag{35}$$

taking arbitrary $\zeta \to 0$ in (35), we have

$$\mu_{Q(u)-f(u)}(s) \geq \mu'_{\phi(u,0,0)}(3(8-k)s). \tag{36}$$

Replacing (u, v, w) by $(2^n u, 2^n v, 2^n w)$ in (25), we obtain

$$\mu_{\Theta_1}(s) \geq \mu'_{\phi(2^n u, 2^n v, 2^n w)}(2^n s), \tag{37}$$

where

$$\begin{aligned}
\Theta_1 &= f(2.2^n u + 2^n v + 2^n w) - 3f(2^n u + 2^n v + 2^n w) - f(-2^n u + 2^n v + 2^n w) \\
&- 2f(2^n u + 2^n v) - 2f(2^n u + 2^\beta w) + 6f(2^n u - 2^n v) + 6f(2^n u - 2^n w) \\
&+ 3f(2^n v + 2^n w) - 2f(2.2^n u - 2^n v) - 2f(2.2^n u - 2^n w) + 18f(2^n u) \\
&+ 6f(2^n v) + 6f(2^n w),
\end{aligned}$$

taking the limit as $n \to \infty$ in (37) and using the following equality

$$lim_{m\to\infty} \mu'_{\phi(2^n u, 2^n v, 2^n w)}(8^n s) = 1, \tag{38}$$

we conclude that Q satisfies (1).

Uniqueness: To show the function Q is unique, we suppose that there is any other cubic mapping $Q' : U \to X$ which holds (26). Since f, Q and Q' all are cubic mapping then for every $u \in U$, we can write $Q(2^n u) = 2^{3n} Q(u)$ and $Q'(2^n u) = 2^{3n} Q(u)$. Therefore, we have

$$\mu_{Q(u)-Q'(u)}(s) = lim_{n\to\infty} \mu_{\frac{Q(2^n u)}{2^{3n}} - \frac{Q'(2^n u)}{2^{3n}}}(s). \tag{39}$$

Now,

$$\begin{aligned}
\mu_{\frac{Q(2^n u)}{2^{3n}} - \frac{Q'(2^n u)}{2^{3n}}}(s) &\geq min\left\{\mu_{\frac{Q(2^n u)}{2^{3n}} - \frac{f(2^n u)}{2^{3n}}}(\frac{s}{2}), \mu_{\frac{Q'(2^n u)}{2^{3n}} - \frac{f(2^n u)}{2^{3n}}}(\frac{s}{2})\right\} \\
&\geq \mu'_{\phi(2^n u,0,0)}\left(\frac{(3\times 8^n)(8-k)s}{2}\right) \geq \mu'_{\phi(u,0,0)}\left(\frac{(3\times 8^n)(8-k)s}{2k^n}\right).
\end{aligned}$$

Since, $\lim_{n\to\infty} \frac{2^{3n}(8-k)s}{2k^n} = \infty$, we get $lim_{n\to\infty} \mu'_{\phi(u,0,0)}(\frac{(3\times 8^n)(8-k)s}{2k^n}) = 1$.

Hence, it follows that $\mu_{\frac{Q(2^n u)}{2^{3n}} - \frac{Q'(2^n u)}{8^n}}(s) = 1$ and so $Q(u) = Q'(u)$.

Corollary 3. *Suppose* $p \in (0, 1)$ *and* $u_0 \in U$. *If* $f : U \to X$ *is a function such that*

$$\mu_\Theta(s) \geq \mu'_{||u||^p u_0}(s),$$

where

$$\begin{aligned}
\Theta &= f(2u + w + v) - 3f(w + u + v) - f(v + w - u) - 2f(u + v) - 2f(w + u) \\
&+ 6f(u - w) + 6f(u - v) + 3f(v + w) - 2f(2u - w) - 2f(2u - v) + 18f(u) \\
&+ 6f(w) + 6f(v)
\end{aligned}$$

then there is a unique cubic mapping $Q : U \to X$ *holds*

$$\mu_{Q(u)-f(u)}(s) \geq \mu'_{u_0||u||^p}(3(8 - 8^p)s).$$

Proof. Let $k = 8^p$ and $\phi : U \to X$ be defined as $\phi(u, v, w) = ||u||^p u_0$ and $\phi(u, v, w) = (||w||^p + ||v||^p + ||u||^p)u_0$ in theorem (4). We obtain the desired results.

5. Results of Experiment

In this segment, the authors discuss the graphs of the approximate and exact solutions for the given equation. It is simple to demonstrate that function $f(u) = u^3$ is an exact solution to a cubic equation. For experimental purposes, we explored an alternative function $Q(u) = u^3 + u^3(u - \lfloor u \rfloor)^3 \lfloor u \rfloor$, (Here, $\lfloor . \rfloor$ represents a floor function) which differs from a cubic function. These two functions were graphed using Matlab, and it was observed that the graphs of both functions, $f(u)$ and $Q(u)$, coincide at multiple points. This suggests that $Q(u)$ is an approximate solution to the given cubic equation. The Table 1 shows the behavior of the exact solution, the approximate solution, and the difference between their values between -1 and 1. Also, the graphs of the functions $f(u)$ and $Q(u)$ are shown in Figure 1.

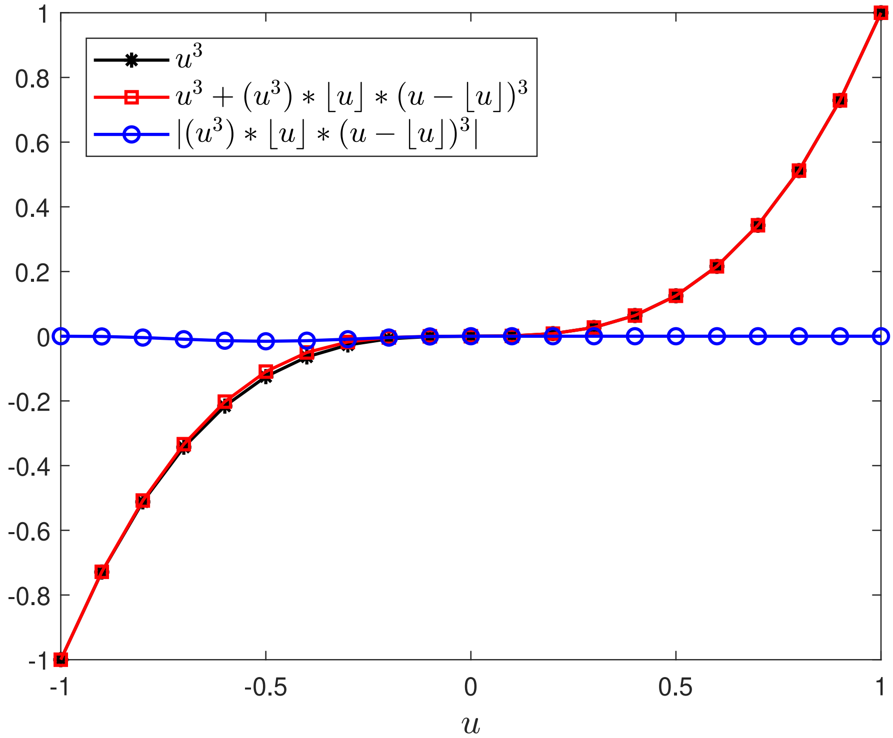

Figure 1: Graph of $f(u)$ and $Q(u)$

Table 1: Error of Approximation

Values of u	**Exact Solution** $f(u)$	**Approximate Solution** $Q(u)$	**Absolute error** $\lvert f(u)-Q(u)\rvert$
-1	-1	-1	0
-0.9	-0.729	-0.728	0.001
-0.8	-0.512	-0.508	0.004
-0.7	-0.343	-0.333	0.010
-0.6	-0.216	-0.202	0.014
-0.5	-0.125	-0.109	0.016
-0.4	-0.064	-0.050	0.014
-0.3	-0.027	-0.017	0.010
-0.2	-0.008	-0.004	0.004
-0.1	-0.001	-0.0003	0.0007
0.1	0.001	0.001	0.000
0.2	0.008	0.0008	0.000
0.3	0.027	0.027	0.000
0.4	0.064	0.0064	0.000
0.5	0.125	0.125	0.000
0.6	0.216	0.216	0.000
0.7	0.343	0.343	0.000
0.8	0.512	0.512	0.000
0.9	0.729	0.729	0.000
1	1	1	0

6. Comparative evaluation of the results

This study established the stability of given cubic FE across the spaces, including (n, β)-NS, NA-(n, β)-NS, and RNS. The main results obtained are summarized as follows:

Corollary No.	**Space setting**	**Stability result**
Corollary 1	(n, β)-NS	$\|\|f(u) - Q(u), \delta_1, ..., \delta_{n-1}\|\|_\beta \leq \frac{\epsilon}{6^\beta(2^{2\beta} - 1)}\|\|u\|\|_\beta.$
Corollary 2	NA-(n, β)-NS	$\|\|f(u) - Q(u), \delta_1, ..., \delta_{n-1}\|\|_\beta \leq 24^{-\beta}\epsilon\|\|u\|\|_\beta.$
Corollary 3	RNS	$\mu_{Q(u)-f(u)}(s) \geq \mu'_{\|\|u\|\|^p u_0}(3(8 - 8^p)s)$

Upon comparing the results presented in the table, it is evident that the approximate solution closely aligns with the exact solution within the framework of NA-(n, β)-NS since the upper bound $24^{-\beta}||u||\epsilon$ is less when compared with upper bound in (n, β)-NS and RNS. The stability results concerning Hyers-Ulam stability regarding the upper bound are obtained in Corollaries (1), (2), and (3).

This study is limited to the analysis of a specific three-dimensional cubic functional equation, and the results may not directly extend to other types. While novel spaces are introduced, other generalized normed structures remain unexplored. Additionally, the experimental validation is based on limited examples, and deeper stochastic modeling in random normed spaces is left for future work.

7. Conclusions

The study of stability for cubic functional equations has been a focal point for many mathematicians, with significant progress achieved across various mathematical spaces. In this work, we investigated the stability of three-dimensional cubic functional equations within the frameworks of (n, β)-normed spaces, non-Archimedean (n, β)-normed spaces, and random normed spaces. Through theoretical analysis and experimental validation, we confirmed the stability properties of these functional equations in each of these distinct spaces. Furthermore, a comparative analysis was performed, highlighting both the similarities and differences in stability behavior across these spaces. The introduction of (n, β)-normed and non-Archimedean (n, β)-normed spaces offers a novel perspective for understanding stability, while the incorporation of random normed spaces adds a stochastic dimension, providing a more comprehensive understanding of the stability dynamics. This comparative evaluation not only deepens the insights into stability across different normed structures but also bridges deterministic and probabilistic approaches in functional equation theory.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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References

[1] S. M. Ulam. *A Collection of the Mathematical Problems.* Interscience, New York, 1960.

[2] D. H. Hyers. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA*, 27:222–224, 1941.

[3] T. M. Rassias. On the stability of the linear mapping in banach spaces. *Proc. Am. Math. Soc.*, 72:297–300, 1978.

[4] J. Jakhar, R. Chugh, and J. Jakhar. Solution and intuitionistic fuzzy stability of 3-dimensional cubic functional equation: using two different methods. *J. Math. Comput. Sci.*, 25:103–114, 2022.

[5] J. Jakhar, R. Chugh, and J. Jakhar. Stability of various iterative type functional equation in menger-ϕ normed space. *Bull. Math. Anal. Appl.*, 13:106–120, 2021.

[6] J. Jakhar, R. Chugh, and J. Jakhar. Fuzzy stability of mixed type functional equations in modular spaces. *Math. Found. Comput.*, 2023.

[7] D. Mihet, R. Saadati, and S. M. Vaezpour. The stability of the quartic functional equation in random normed spaces. *Acta Appl. Math.*, 110:797–803, 2010.

[8] M. Vivas-Cortez, M. A. Yousif, B. A. Mahmood, P. O. Mohammed, N. Chorfi, and A. A. Lupas. High-accuracy solutions to the time-fractional kdv–burgers equation using rational non-polynomial splines. *Symmetry*, 17(1):16, 2025.

[9] M. Vivas-Cortez, M. A. Yousif, P. O. Mohammed, A. A. Lupas, I. S. Ibrahim, and N. Chorfi. Hyperbolic non-polynomial spline approach for time-fractional coupled kdv equations: A computational investigation. *Symmetry*, 16(12):1610, 2024.

[10] A. K. Mirmostafaee and M. S. Moslehian. Fuzzy approximately cubic mappings. *Inf. Sci.*, 178:3791–3798, 2008.

[11] W. G. Park. Approximate additive mappings in 2-banach spaces and related topics. *J. Math. Anal. Appl.*, 376:193–202, 2011.

[12] S. Gähler. Lineare 2-normierte räume. *Math. Nachr.*, 28:1–43, 1964.

[13] S. Gähler. Untersuchungen über verallgemeinerte m-metrische räume. i. *Math. Nachr.*, 40:165–189, 1969.

[14] H. Gunawan and M. Mashadi. On n-normed spaces. *Int. J. Math. Math. Sci.*, 27:631–639, 2001.

[15] R. Malčeski. Strong n-convex n-normed spaces. *Math. Bilt.*, 21:81–102, 1997.

[16] S. S. Kim and Y. J. Cho. Strict convexity in linear n-normed spaces. *Demonstr. Math.*, 29:739–744, 1996.

[17] A. Misiak. n-inner product spaces. *Math. Nachr.*, 140:299–319, 1989.

[18] T. Z. Xu and J. M. Rassias. On the hyers-ulam stability of a general mixed additive and cubic functional equation in n-banach spaces. *Abstr. Appl. Anal.*, 2012:Article ID 926390, 2012.

[19] X. Yang, L. Chang, G. Liu, and G. Shen. Stability of functional equations in (n, β)-normed spaces. *J. Inequal. Appl.*, 2015(1):112–130, 2015.

[20] Jyotsana, R. Chugh, S. Jaiswal, and R. Dubey. Stability of functional equations in non-archimedean (n, β)-normed spaces. *J. Anal.*, 30:1653–1669, 2022.

[21] El Fassi and S. Kabbaj. Non-archimedean random stability of σ-quadratic functional equation. *Thai J. Math.*, 14:151–165, 2016.

[22] M. Mirzavaziri and M. S. Moslehian. A fixed point approach to stability of a quadratic equation. *Bull. Braz. Math. Soc.*, 37:361–376, 2006.

[23] D. Mihet and V. Radu. On the stability of the additive cauchy functional equation in random normed spaces. *J. Math. Anal. Appl.*, 343:567–572, 2008.

[24] E. Baktash, Y. J. Cho, M. Jalili, R. Saadati, and S. M. Vaezpour. On the stability of cubic mappings and quadratic mappings in random normed spaces. *J. Inequal. Appl.*, 2008:1–11, 2008.

[25] R. Saadati, S. M. Vaezpour, and Y. J. Cho. Erratum to: A note to paper "on the stability of cubic mappings and quartic mappings in random normed spaces". *J. Inequal. Appl.*, 2009:1–6, 2009.

[26] J. M. Rassias, S. Sharma, J. Jakhar, and J. Jakhar. Direct approach to the stability of various functional equations in felbin's type non-archimedean fuzzy normed spaces. *J. Comput. Anal. Appl.*, 32:320–352, 2024.

[27] V. Govindan, C. Park, S. Pinelas, and S. Baskaran. Solution of a 3-d cubic functional equation and its stability. *AIMS Math.*, 5:1693–1705, 2020.

[28] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces.* Elsevier, North Holland, New York, 1983.

[29] A. N. Serstnev. The notion of random normed spaces. *Doki. Acad. Nauk USSR*, 149:280–283, 1963.

[30] Y. J. Cho, T. M. Rassias, and R. Saadati. *Stability of Functional Equations in Random Normed Spaces.* Springer Science & Business Media, 2013.