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Solving Partial Differential Equations via the Conformable Double ARA-Sawi Transform

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Abstract. This paper presents a new method the conformable double ARA-Sawi transform to solve fractional partial differential equations that arise in physics and engineering. These equations often involve derivatives based on conformable calculus, which generalizes classical derivatives to fractional orders. We explore the core properties of the transform, such as its linearity and interaction with fractional derivatives, and establish conditions for its applicability. To demonstrate its practical use, we apply the method to solve two key equations: the conformable Klein-Gordon equation, which models wave propagation in quantum fields, and the conformable telegraph equation, governing signal transmission in dissipative systems. The results highlight the transform's ability to simplify complex fractional equations into manageable algebraic forms, offering a systematic tool for researchers. This work bridges theoretical advancements with real-world applications, paving the way for future studies in areas like nonlinear dynamics and material science.

2020 Mathematics Subject Classifications: 44A05

Key Words and Phrases: ARA transform, Sawi transform, the double ARA-Sawi transform, the conformable double ARA-Sawi transform.

1. Introduction

Fractional partial differential equations play a crucial role in modeling complex systems in physics, electrical circuits, fluid dynamics, optics, and mathematical biology. One important development in this field is the conformable fractional derivative, introduced in [1], which retains many key properties of standard derivatives.

To solve conformable fractional partial differential equations, researchers have explored several techniques. Among these are the conformable double Laplace transform [2, 3], the Conformable Double Laplace-Sawi Transform [4] and the conformable double Sumudu transform [5].

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The double ARA-Sawi transform [6] was proposed, offering a new perspective in handling boundary value problems for integer-order partial differential equations. For further insights into integral transforms, see [7–13].

However, when dealing with fractional derivatives, especially of conformable type, there remained a need for a structured and efficient method. The conformable double ARA-Sawi transform (CA-SW) was introduced to fill this gap. Inspired by the success of the double Sawi transform in solving boundary value problems [6], the CA-SW transform adapts and extends the method to the conformable framework, allowing researchers to manage more complex fractional models.

The motivation behind this study stems from the increasing demand for robust analytical tools that can systematically address fractional differential equations without relying on cumbersome or purely numerical techniques. Traditional methods often struggle when fractional derivatives appear with variable coefficients or non-standard boundary conditions. By developing the CA-SW, we aim to offer a reliable and systematic analytical tool that simplifies these equations into algebraic forms that are easier to solve.

This paper focuses on establishing the theoretical foundation of the CA-SW transform. We define its existence conditions, explore its linearity, and derive its interaction with conformable partial derivatives. Several basic examples demonstrate how to compute the transform explicitly for elementary functions.

Applications are a key component of this work. We apply the CA-SW transform to solve two important conformable partial differential equations: the conformable Klein-Gordon equation and the conformable telegraph equation. These applications showcase how the CA-SW transform can simplify complex equations, reduce the number of steps needed for their solution, and provide explicit forms of the solutions.

By extending the classical ARA-Sawi transform to the conformable setting and demonstrating its practical utility, this study bridges theoretical advancements with real-world applications, paving the way for further developments in fractional dynamics, nonlinear systems, signal transmission, and material science.

The CA-SW systematically reduces complex conformable fractional equations into solvable algebraic forms while preserving key properties like linearity and scaling. It extends classical transforms to the conformable setting, offering broader applicability to equations with variable coefficients and non-standard conditions. This makes it a powerful tool for solving advanced problems in physics and engineering.

2. Fundamental Definitions and Theorems

This section covers essential definitions and theorems related to conformable fractional derivatives. It defines the conformable fractional derivative, explores its key properties, and presents fundamental theorems needed for applying the CA-SW.

Definition 1. *[1] Let $0 < \beta \leq 1$ and $s : (0,\infty) \to \mathbb{R}$. The conformable fractional derivative of order β is defined as:*

$$\frac{d^{\beta}}{d\xi^{\beta}} s(\xi) = \lim_{n\to 0} \frac{s(\xi + n\xi^{1-\beta}) - s(\xi)}{n},$$

where $\xi > 0$, and $\frac{\partial^{\beta}}{\partial\xi^{\beta}}$ is referred to as the fractional derivative of order β.

Definition 2. *[14] Let $0 < \beta_1, \beta_2 \leq 1$ and $s(\xi,\rho) : (0,\infty)\times(0,\infty) \to \mathbb{R}$. The conformable partial derivatives of orders β_1 and β_2 of the function $s(\xi,\rho)$ are defined as:*

$$\frac{\partial^{\beta_1}}{\partial\xi^{\beta_1}} s(\xi,\rho) = \lim_{n\to 0} \frac{s(\xi + n\xi^{1-\beta_1},\rho) - s(\xi,\rho)}{n},$$

$$\frac{\partial^{\beta_2}}{\partial\rho^{\beta_2}} s(\xi,\rho) = \lim_{n\to 0} \frac{s(\xi, \rho + n\rho^{1-\beta_2}) - s(\xi,\rho)}{n},$$

where $\xi, \rho > 0$, $\frac{\partial^{\beta_1}}{\partial\xi^{\beta_1}}$ and $\frac{\partial^{\beta_2}}{\partial\rho^{\beta_2}}$ are referred to as fractional derivatives of orders β_1 and β_2, respectively.

Theorem 1. *[15]Suppose that $s(\xi,\rho)$ be differentiable at a point $\xi, \rho > 0$, $0 < \beta_1, \beta_2 \leq 1$, then:*

$$\frac{\partial^{\beta_1} s}{\partial\xi^{\beta_1}} = \xi^{1-\beta_1}\frac{\partial s}{\partial\xi},$$

$$\frac{\partial^{\beta_2} s}{\partial\rho^{\beta_2}} = \rho^{1-\beta_2}\frac{\partial s}{\partial\rho}.$$

3. Conformable Double ARA-Sawi transform

This section serves to introduce the CA-SW. We commence by delineating its fundamental properties, encompassing aspects like linearity. Subsequently, we reveal a novel result associated with partial derivatives. Ultimately, we illustrate how these insights enable us to compute the CA-SW for various essential functions.

Definition 3. *Let $s(\xi,\rho)$ be a continuous function on $(0,\infty)\times(0,\infty)$. Then*
1- The Conformable ARA transformation (CA) of $s(\xi,\rho)$, denoted by $A_{\xi}^{\beta}[s(\xi,\rho)]$, is defined as:

$$J(\psi) = A_{\xi}^{\beta}(s(\xi,\rho)) = \psi \int_{0}^{\infty} e^{-\psi\frac{\xi^{\beta}}{\beta}} s(\xi,\rho)\xi^{\beta-1} d\xi, \ \psi \in \mathbb{C}.$$

2- The Conformable Sawi transformation (CSW) of $s(\xi,\rho)$, denoted by $A_{\rho}^{\beta}[s(\xi,\rho)]$, is defined as:

$$L(\varkappa) = W_{\rho}^{\beta}(s(\xi,\rho)) = \frac{1}{\varkappa^2}\int_{0}^{\infty} e^{-\frac{\rho^{\beta}}{\varkappa\beta}} s(\xi,\rho)\rho^{\beta-1} d\rho, \ \varkappa \in \mathbb{C}.$$

3- CA-SW of $s(\xi,\rho)$, denoted by $A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}[s(\xi,\rho)]$, is defined as:

$$S(\psi,\varkappa)=A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}[s(\xi,\rho)]=\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty}e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)}s(\xi,\rho)\xi^{\beta_1-1}\rho^{\beta_2-1}d\xi d\rho.$$

Theorem 2. *Assume that $s:(0,\infty)\times(0,\infty)\to\mathbb{R}$ such that $S(\psi,\varkappa)=A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}[s(\frac{\xi^{\beta_1}}{\beta_1},\frac{\rho^{\beta_2}}{\beta_2})]$ exist, then*

$$A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}[s(\frac{\xi^{\beta_1}}{\beta_1},\frac{\rho^{\beta_2}}{\beta_2})]=A_{\xi}W_{\rho}[s(\xi,\rho)],$$

where

$$A_{\xi}W_{\rho}[s(\xi,\rho)]=\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty}e^{-\left(\psi\xi+\frac{\rho}{\varkappa}\right)}s(\xi,\rho)\,d\xi\,d\rho.$$

Lemma 1. *$A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}(s(\xi,\rho))$ is a linear transformation.*

Proof. for nonzero constants λ and ν, we have

$$\begin{aligned}
&A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}(\lambda s_1(\xi,\rho)+\nu s_2(\xi,\rho))\\
&=\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty}e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)}(\lambda s_1(\xi,\rho)+\nu s_2(\xi,\rho))\xi^{\beta_1-1}\rho^{\beta_2-1}d\xi d\rho,\\
&=\lambda\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty}e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)}s_1(\xi,\rho)\xi^{\beta_1-1}\rho^{\beta_2-1}d\xi d\rho+\nu\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty}e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)}s_2(\xi,\rho)\xi^{\beta_1-1}\rho^{\beta_2-1}d\xi d\rho\\
&=\lambda A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}(s_1(\xi,\rho))+\nu A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}(s_2(\xi,\rho)).
\end{aligned}$$

If $s(\xi,\rho)$ can be written as $s(\xi,\rho)=p(\xi)q(\rho)$ for some continuous functions p and q, then $A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}(s(\xi,\rho))=A_{\xi}^{\beta_1}(p(\xi))W_{\rho}^{\beta_2}(q(\rho))$. In fact

$$\begin{aligned}
A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}(s(\xi,\rho)) &= A_{\xi}^{\beta_1}W_{\rho}^{\beta_2}(p(\xi)q(\rho))\\
&=\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty}e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)}p(\xi)q(\rho)\xi^{\beta_1-1}\rho^{\beta_2-1}d\xi d\rho\\
&=\left(\psi\int_0^{\infty}e^{-\psi\frac{\xi^{\beta_1}}{\beta_1}}p(\xi)\xi^{\beta_1-1}d\xi\right)\left(\frac{1}{\varkappa^2}\int_0^{\infty}e^{-\frac{\rho^{\beta_2}}{\varkappa\beta_2}}q(\rho)\rho^{\beta_2-1}d\rho\right)\\
&=A_{\xi}^{\beta_1}(p(\xi))W_{\rho}^{\beta_2}(q(\rho)).
\end{aligned}$$

3.1. CA-SW for some basic functions

(i)

$$A_{\xi}^{\beta_1} W_{\rho}^{\beta_2}[c] = A_{\xi} W_{\rho}[c] = \frac{c}{\varkappa}, \quad c \in \mathbb{R},$$

(ii)

$$A_{\xi}^{\beta_1} W_{\rho}^{\beta_2}\left[\left(\frac{\xi^{\beta_1}}{\beta_1}\right)^{\lambda}\left(\frac{\rho^{\beta_2}}{\beta_2}\right)^{\nu}\right]$$

$$\begin{aligned} &= A_{\xi} W_{\rho}[\xi^{\lambda}\rho^{\nu}] \\ &= \frac{\varkappa^{\nu-1}}{\psi^{\lambda}}\Gamma(\lambda+1)\Gamma(\nu+1), \ \operatorname{Re}(\psi) > 0 \text{ and } \operatorname{Re}(\lambda) > -1, \end{aligned}$$

(iii)

$$A_{\xi}^{\beta_1} W_{\rho}^{\beta_2}\left[e^{\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}}\right]$$

$$= A_{\xi} W_{\rho}[e^{\lambda\xi+\nu\rho}] = \frac{\psi}{\varkappa\left(\psi-\lambda\right)\left(1-\nu\varkappa\right)}, \operatorname{Re}(\psi) > \operatorname{Re}(\lambda).$$

3.2. Existence condition for CA-SW

Definition 4. *Let $0 < \beta_1, \beta_2 \leq 1$. Then a function $s(\xi,\rho)$ is said to be of conformable exponential orders λ and ν on $0 < \xi < \infty$ and $0 < \rho < \infty$. If there exist $K, X, Y > 0$ such that $|s(\xi,\rho)| \leq K e^{\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}}$, for all $\frac{\xi^{\beta_1}}{\beta_1} > X$, $\frac{\rho^{\beta_2}}{\beta_2} > Y$.*

Theorem 3. *Let $0 < \beta_1, \beta_2 \leq 1$ and $s(\xi,\rho)$ be a continuous function on the region $(0,\infty)\times(0,\infty)$ of conformable exponential orders λ and ν. Then $S(\psi,\varkappa) = A_{\xi}^{\beta_1} W_{\rho}^{\beta_2}[s(\xi,\rho)]$ exists for $\psi, \varkappa$ whenever $\operatorname{Re}(\psi) > \lambda$ and $\operatorname{Re}\left(\frac{1}{\varkappa}\right) > \nu$.*

Proof. We have

$$\begin{aligned} |S(\psi,\varkappa)| &= \left|\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty} e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)} s(\xi,\rho)\xi^{\beta_1-1}\rho^{\beta_2-1}\, d\xi d\rho\right| \\ &\leq \frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty} e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)} |s(\xi,\rho)|\, \xi^{\beta_1-1}\rho^{\beta_2-1} d\xi d\rho \\ &\leq K\frac{\psi}{\varkappa^2}\int_0^{\infty}\int_0^{\infty} e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)} e^{\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}} \xi^{\beta_1-1}\rho^{\beta_2-1} d\xi d\rho \end{aligned}$$

$$
\begin{aligned}
&= K\left(\psi\int_0^\infty e^{-(\psi-\lambda)\frac{\xi^{\beta_1}}{\beta_1}}\xi^{\beta_1-1}d\xi\right)\left(\frac{1}{\varkappa^2}\int_0^\infty e^{-(\frac{1}{\varkappa}-\nu)\frac{\rho^{\beta_2}}{\beta_2}}\rho^{\beta_2-1}d\rho\right)\\
&= \frac{K\psi}{\varkappa(\psi-\lambda)(1-\nu\varkappa)},
\end{aligned}
$$

where $\mathrm{Re}(\psi)>\lambda$ and $\mathrm{Re}\left(\frac{1}{\varkappa}\right)>\nu$.

3.3. Derivatives properties

Now, we present some basic properties of the CA-SW

Let $S(\psi,\varkappa) = A_\xi^{\beta_1}W_\rho^{\beta_2}(s(\xi,\rho))$ where $s(\xi,\rho)$ is a continuous function on $(0,\infty)\times(0,\infty)$. Then

(i)

$$A_\xi^{\beta_1}W_\rho^{\beta_2}\left(\frac{\partial^{\beta_1}s(\xi,\rho)}{\partial\xi^{\beta_1}}\right) = \psi S(\psi,\varkappa) - \psi W_\rho^{\beta_2}(s(0,\rho)), \tag{1}$$

(ii)

$$A_\xi^{\beta_1}W_\rho^{\beta_2}\left(\frac{\partial^{2\beta_1}s(\xi,\rho)}{\partial\xi^{2\beta_1}}\right) = \psi^2 S(\psi,\varkappa) - \psi^2 W_\rho^{\beta_2}(s(0,\rho)) - \psi W_\rho^{\beta_2}(\frac{\partial^{\beta_1}s(0,\rho)}{\partial\xi^{\beta_1}}), \tag{2}$$

(iii)

$$A_\xi^{\beta_1}W_\rho^{\beta_2}\left(\frac{\partial^{\beta_2}s(\xi,\rho)}{\partial\rho^{\beta_2}}\right) = \frac{1}{\varkappa}S(\psi,\varkappa) - \frac{1}{\varkappa^2}A_\xi^{\beta_1}(s(\xi,0)), \tag{3}$$

(iv)

$$A_\xi^{\beta_1}W_\rho^{\beta_2}\left(\frac{\partial^{2\beta_2}s(\xi,\rho)}{\partial\rho^{2\beta_2}}\right) = \frac{1}{\varkappa^2}S(\psi,\varkappa) - \frac{1}{\varkappa^3}A_\xi^{\beta_1}(s(\xi,0)) - \frac{1}{\varkappa^2}A_\xi^{\beta_1}(\frac{\partial^{\beta_2}s(\xi,0)}{\partial\rho^{\beta_2}}). \tag{4}$$

Proof. Proof of Equation 1 $A_\xi^{\beta_1}W_\rho^{\beta_2}\left(\frac{\partial^{\beta_1}s(\xi,\rho)}{\partial\xi^{\beta_1}}\right) = \frac{\psi}{\varkappa^2}\int_0^\infty\int_0^\infty e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)}\frac{\partial^{\beta_1}s(\xi,\rho)}{\partial\xi^{\beta_1}}\xi^{\beta_1-1}\rho^{\beta_2-1}d\xi d\rho.$
By Theorem 1, we have $\frac{\partial^{\beta_1}s(\xi,\rho)}{\partial\xi^{\beta_1}} = \xi^{1-\beta_1}\frac{\partial s(\xi,\rho)}{\partial\xi}$. So,
$A_\xi^{\beta_1}W_\rho^{\beta_2}\left(\frac{\partial^{\beta_1}s(\xi,\rho)}{\partial\xi^{\beta_1}}\right) = \frac{\psi}{\varkappa^2}\int_0^\infty e^{-\frac{\rho^{\beta_2}}{\varkappa\beta_2}}\rho^{\beta_2-1}\int_0^\infty e^{-\psi\frac{\xi^{\beta_1}}{\beta_1}}\frac{\partial s(\xi,\rho)}{\partial\xi}d\xi d\rho.$
By integrating by parts, we get

$A_\xi^{\beta_1}W_\rho^{\beta_2}\left(\frac{\partial^{\beta_1}s(\xi,\rho)}{\partial\xi^{\beta_1}}\right) = \frac{\psi}{\varkappa^2}\int_0^\infty e^{-\frac{\rho^{\beta_2}}{\varkappa\beta_2}}\rho^{\beta_2-1}\left(-s(0,\rho)+\psi\int_0^\infty e^{-\psi\frac{\xi^{\beta_1}}{\beta_1}}s(\xi,\rho)\xi^{\beta_1-1}\,d\xi\right)d\rho$

$= -\frac{\psi}{\varkappa^2}\int_0^\infty e^{-\frac{\rho^{\beta_2}}{\varkappa\beta_2}} s(0,\rho)\rho^{\beta_2-1}d\rho + \frac{\psi^2}{\varkappa^2}\int_0^\infty\int_0^\infty e^{-\left(\psi\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\varkappa\beta_2}\right)} s(\xi,\rho)\xi^{\beta_1-1}\rho^{\beta_2-1}d\xi d\rho$

$= \psi S(\psi,\varkappa) - \psi W_\rho^{\beta_2}(s(0,\rho))$.

The proof of Equations 2, 3 and 4 can be obtained in the same manner.

In Table 1, we have the CA-SW of some basic functions.

Table 1: Table of CA-SW

$s(\xi,\rho)$	$A_\xi^{\beta_1} W_\rho^{\beta_2}\left(s(\xi,\rho)\right)$
c	$\frac{c}{\varkappa}$, $\mathrm{Re}(\psi) > 0$
$\left(\frac{\xi^{\beta_1}}{\beta_1}\right)^\lambda \left(\frac{\rho^{\beta_2}}{\beta_2}\right)^\nu$	$\frac{\varkappa^{\nu-1}}{\psi^\lambda}\Gamma(\lambda+1)\Gamma(\nu+1)$, $\mathrm{Re}(\psi) > 0$ and $\mathrm{Re}(\lambda) > -1$
$e^{\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}}$	$\frac{\psi}{\varkappa(\psi-\lambda)(1-\nu\varkappa)}$, $\mathrm{Re}(\psi) > \mathrm{Re}(\lambda)$
$e^{i\left(\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}\right)}$	$\frac{i\psi}{\varkappa(\psi-i\lambda)(i+\nu\varkappa)}$, $\mathrm{Im}(\lambda)+\mathrm{Re}(\psi) > 0$
$\sin\left(\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}\right)$	$\frac{\psi(\lambda+\psi\varkappa\nu)}{\varkappa(\psi^2+\lambda^2)(1+\nu^2\varkappa^2)}$, $\lvert\mathrm{Im}(\lambda)\rvert < \mathrm{Re}(\psi)$
$\cos\left(\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}\right)$	$\frac{\psi(\psi-\varkappa\lambda\nu)}{\varkappa(\psi^2+\lambda^2)(1+\nu^2\varkappa^2)}$, $\lvert\mathrm{Im}(\lambda)\rvert < \mathrm{Re}(\psi)$
$\sinh\left(\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}\right)$	$\frac{\psi(\lambda+\psi\varkappa\nu)}{\varkappa(\psi^2-\lambda^2)(1-\nu^2\varkappa^2)}$, $\mathrm{Re}(\psi) > \mathrm{Re}(\lambda)$ and $\mathrm{Re}(\psi)+\mathrm{Re}(\lambda) > 0$
$\cosh\left(\lambda\frac{\xi^{\beta_1}}{\beta_1}+\nu\frac{\rho^{\beta_2}}{\beta_2}\right)$	$\frac{\psi(\psi+\varkappa\lambda\nu)}{\varkappa(\psi^2-\lambda^2)(1-\nu^2\varkappa^2)}$, $\mathrm{Re}(\psi) > \mathrm{Re}(\lambda)$ and $\mathrm{Re}(\psi)+\mathrm{Re}(\lambda) > 0$
$p(\xi)q(\rho)$	$A_\xi^{\beta_1}(p(\xi))W_\rho^{\beta_2}(q(\rho))$

4. Applications

This section applies CA-SW to solving conformable partial differential equations.

Example 1. *Consider the conformable Klein-Gordon equation*

$$3\frac{\partial^{2\beta_2}s(\xi,\rho)}{\partial\rho^{2\beta_2}} + \frac{\partial^{2\beta_1}s(\xi,\rho)}{\partial\xi^{2\beta_1}} + s(\xi,\rho) = 0, \text{ where } \xi,\rho > 0, \tag{5}$$

With ICs

$s(\xi,0) = \sin\left(\frac{2\xi^{\beta_1}}{\beta_1}\right)$, $\frac{\partial^{\beta_2}s(\xi,0)}{\partial\rho^{\beta_2}} = -\sin\left(\frac{2\xi^{\beta_1}}{\beta_1}\right)$,

and BCs

$s(0,\rho) = 0$, $\frac{\partial^{\beta_1}s(0,\rho)}{\partial\xi^{\beta_1}} = 2e^{-\frac{\rho^{\beta_2}}{\beta_2}}$.

Solution 1. *By applying the CA to the ICs and the CSW to the BCs, we get*
$A_\xi^{\beta_1}\left(\sin\left(\frac{2\xi^{\beta_1}}{\beta_1}\right)\right) = \frac{2\psi}{\psi^2+4}$, $A_\xi^{\beta_1}\left(-\sin\left(\frac{2\xi^{\beta_1}}{\beta_1}\right)\right) = \frac{-2\psi}{\psi^2+4}$, $W_\rho^{\beta_2}(0) = 0$, $W_\rho^{\beta_2}(2e^{-\rho}) = \frac{2}{\varkappa(1+\varkappa)}$.
Apply the CA-SW to Equation 5, we get

$$\frac{3}{\varkappa^2}S - \frac{6\psi}{\varkappa^3(\psi^2+4)} + \frac{6\psi}{\varkappa^2(\psi^2+4)} + \psi^2 S - \frac{2\psi}{\varkappa(1+\varkappa)} + S = 0.$$

So,

$$S(\psi,\varkappa)=\frac{\frac{6\psi}{\varkappa^3(\psi^2+4)}-\frac{6\psi}{\varkappa^2(\psi^2+4)}+\frac{2\psi}{\varkappa(1+\varkappa)}}{\frac{3}{\varkappa^2}+\psi^2+1}.$$

By simplify,

$$S(\psi,\varkappa)=\frac{2\psi}{\varkappa\left(1+\varkappa\right)\left(\psi^2+4\right)}.$$

So,

$$s(\xi,\rho)=\left(A_{\xi}^{\beta_1}\right)^{-1}\left(W_{\rho}^{\beta_2}\right)^{-1}\left(\frac{2\psi}{\varkappa\left(1+\varkappa\right)\left(\psi^2+4\right)}\right)=e^{-\frac{\rho^{\beta_2}}{\beta_2}}\sin\left(\frac{2\xi^{\beta_1}}{\beta_1}\right).$$

The following figures show the 3D representation of the solution at $\beta_1=\beta_2=0.5,1$.

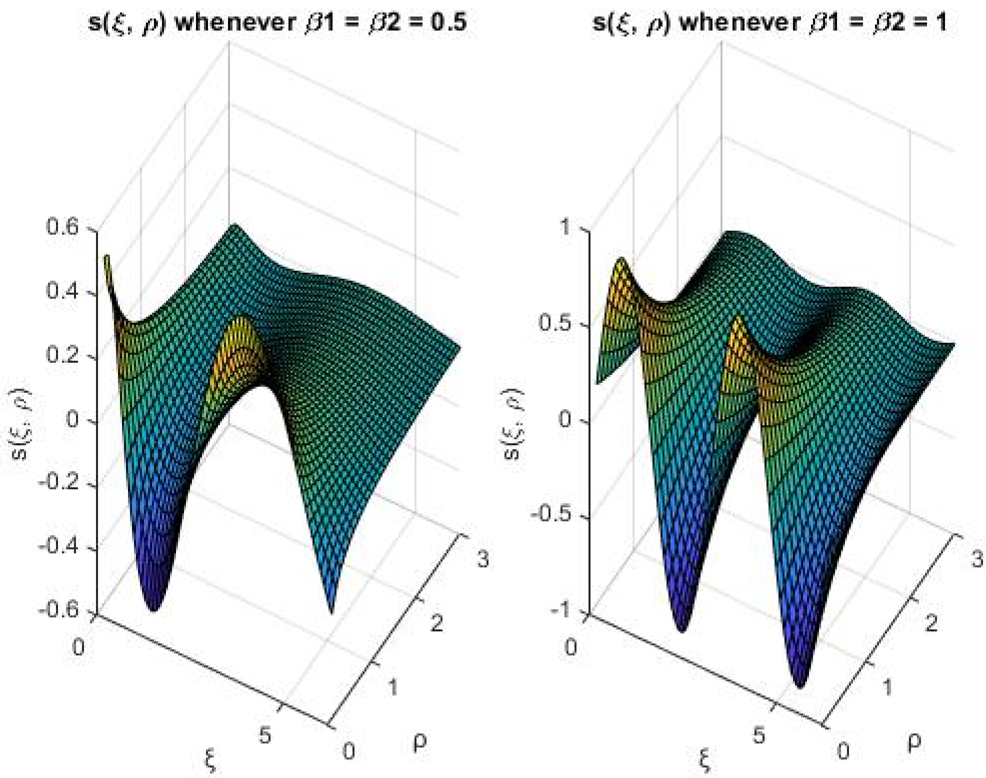

Figure 1: The 3D representation of the solution at $\beta_1=\beta_2=0.5,1$.

Figure 1 presents the three-dimensional plot of the solution to the conformable Klein-Gordon equation at different fractional orders ($\beta_1=\beta_2=0.5$ *and* $\beta_1=\beta_2=1$*). The plot clearly shows that at* $\beta_1=\beta_2=1$*, the solution behaves according to the classical structure with sharp transitions. When the fractional orders are reduced to* $\beta_1=\beta_2=0.5$, *the surface becomes smoother and the amplitude variations are more gradual, reflecting the memory and nonlocal properties introduced by the fractional operators.*

The following two figures illustrate the 2D graph of the solution with respect to ξ *and* ρ *at* $\beta_1 = \beta_2 = 0.6, 0.8, 1$.

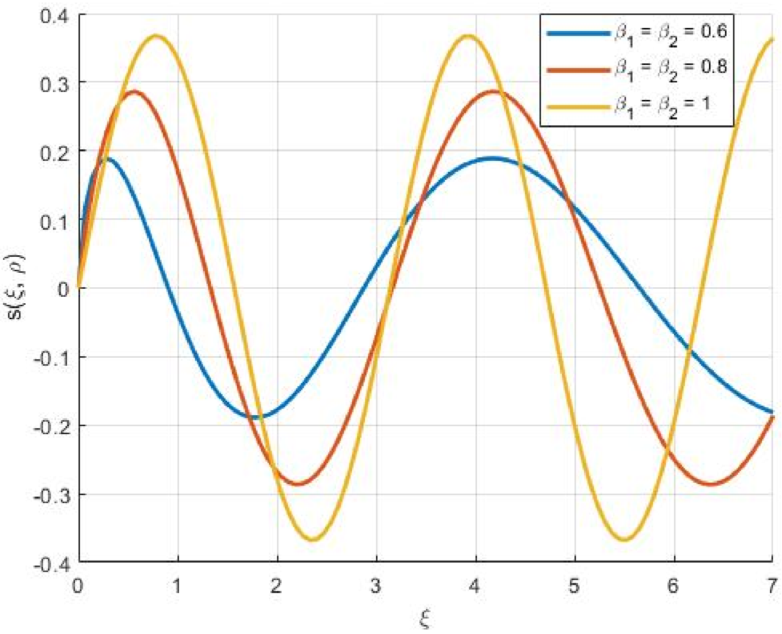

Figure 2: The 2D graph of the solution with respect to ξ at $\beta_1 = \beta_2 = 0.6, 0.8, 1$.

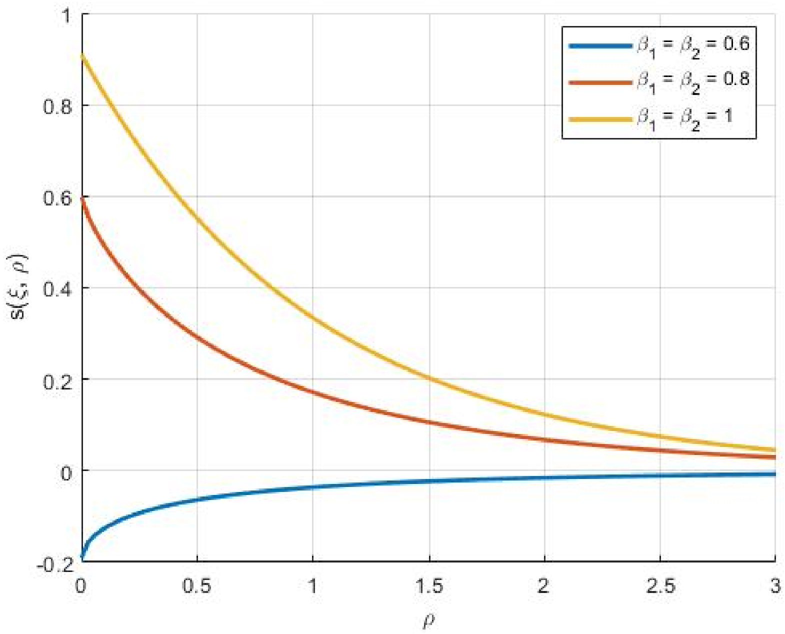

Figure 3: The 2D graph of the solution with respect to ρ at $\beta_1 = \beta_2 = 0.6, 0.8, 1$.

Figures 2 and 3 display the two-dimensional behavior of the solution with respect to ξ *and* ρ*, respectively, for different fractional orders (*$\beta_1 = \beta_2 = 0.6, 0.8, 1$*). Figure 2 shows the solution as a function of* ξ*, where lower fractional orders lead to broader and less steep transitions. Similarly, Figure 3 shows the solution as a function of* ρ*, highlighting that fractional orders lower than 1 result in slower decay and wider spreading of the solution profile.*

Example 2. *Consider the conformable telegraph equation*

$$\frac{\partial^{2\beta_1} s(\xi,\rho)}{\partial \xi^{2\beta_1}} + 3\frac{\partial^{2\beta_2} s(\xi,\rho)}{\partial \rho^{2\beta_2}} + 2\frac{\partial^{\beta_2} s(\xi,\rho)}{\partial \rho^{\beta_2}} = 9s(\xi,\rho), \text{ where } \xi, \rho > 0, \tag{6}$$

With initial conditions (ICs)
$s(\xi,0) = e^{-2\frac{\xi^{\beta_1}}{\beta_1}}$, $\frac{\partial^{\beta_2} s(\xi,0)}{\partial \rho^{\beta_2}} = e^{-2\frac{\xi^{\beta_1}}{\beta_1}}$,
and boundary conditions (BCs)
$s(0,\rho) = e^{\frac{\rho^{\beta_1}}{\beta_1}}$, $\frac{\partial^{\beta_1} s(0,\rho)}{\partial \xi^{\beta_1}} = -2e^{\frac{\rho^{\beta_1}}{\beta_1}}$.

Solution 2. *By applying the CA to the ICs and the CSW to the BCs, we get*
$A_{\xi}^{\beta_1}\left(e^{-2\frac{\xi^{\beta_1}}{\beta_1}}\right) = \frac{\psi}{\psi+2}$, $A_{\xi}^{\beta_1}\left(e^{-2\frac{\xi^{\beta_1}}{\beta_1}}\right) = \frac{\psi}{\psi+2}$, $W_{\rho}^{\beta_2}\left(e^{\frac{\rho^{\beta_1}}{\beta_1}}\right) = \frac{1}{\varkappa(1-\varkappa)}$, $W_{\rho}^{\beta_2}\left(-2e^{\frac{\rho^{\beta_1}}{\beta_1}}\right) = \frac{-2}{\varkappa(1-\varkappa)}$.
Apply the CA-SW to Equation 6, we get

$$\psi^2 S - \frac{\psi^2}{\varkappa(1-\varkappa)} + \frac{2\psi}{\varkappa(1-\varkappa)} + \frac{3}{\varkappa^2}S - \frac{3\psi}{\varkappa^3(\psi+2)} - \frac{3\psi}{\varkappa^2(\psi+2)} + \frac{2}{\varkappa}S - \frac{2\psi}{\varkappa^2(\psi+2)} = 9S.$$

So,

$$\begin{aligned} S(\psi,\varkappa) &= \frac{\frac{\psi^2-2\psi}{\varkappa(1-\varkappa)} + \frac{3\psi}{\varkappa^3(\psi+2)} + \frac{2\psi}{\varkappa^2(\psi+2)}}{\psi^2 + \frac{3}{\varkappa^2} + \frac{2}{\varkappa} - 9} \\ &= \frac{\frac{\varkappa^2(\psi-2)(\psi+2)+3(1-\varkappa)+5\varkappa(1-\varkappa)}{\varkappa^3(\psi+2)(1-\varkappa)}}{\frac{\psi^2\varkappa^2-9\varkappa^2+2\varkappa+3}{\varkappa^2}}. \end{aligned}$$

By simplify,

$$S(\psi,\varkappa) = \frac{\psi}{\varkappa(\psi+2)(1-\varkappa)}.$$

So,

$$s(\xi,\rho) = \left(A_\xi^{\beta_1}\right)^{-1}\left(W_\rho^{\beta_2}\right)^{-1}\left(\frac{\psi}{\varkappa(\psi+2)(1-\varkappa)}\right) = e^{-2\frac{\xi^{\beta_1}}{\beta_1}+\frac{\rho^{\beta_2}}{\beta_2}}.$$

The following figures show the 3D representation of the solution at $\beta_1 = \beta_2 = 0.4, 1$.

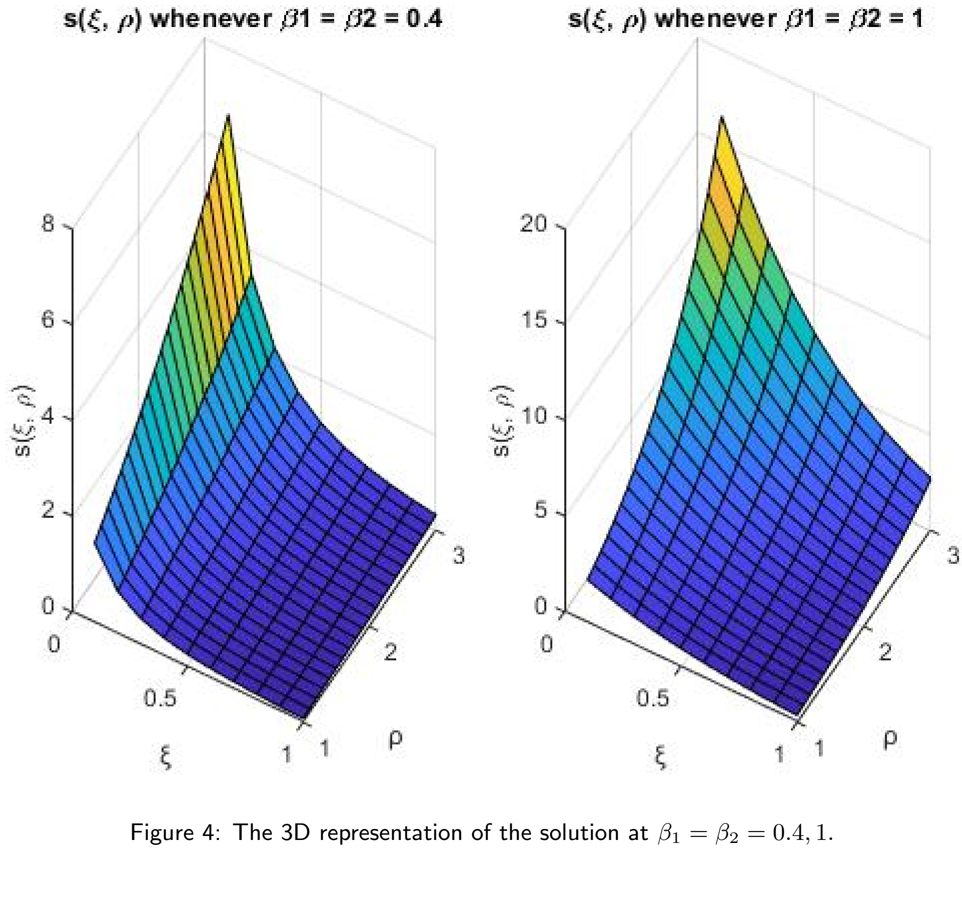

Figure 4: The 3D representation of the solution at $\beta_1 = \beta_2 = 0.4, 1$.

Figure 4 illustrates the three-dimensional plot of the solution to the conformable telegraph equation at fractional orders $\beta_1 = \beta_2 = 0.4$ *and* $\beta_1 = \beta_2 = 1$*. The figure demonstrates that for* $\beta_1 = \beta_2 = 1$*, the surface is steeper and more sharply defined, corresponding to the classical case. When* $\beta_1 = \beta_2 = 0.4$*, the surface becomes noticeably smoother and the solution exhibits a delayed and slower variation, indicating the influence of fractional derivatives. The following two figures illustrate the 2D graph of the solution with respect to* ξ *and* ρ *at* $\beta_1 = \beta_2 = 0.6, 0.8, 1$.

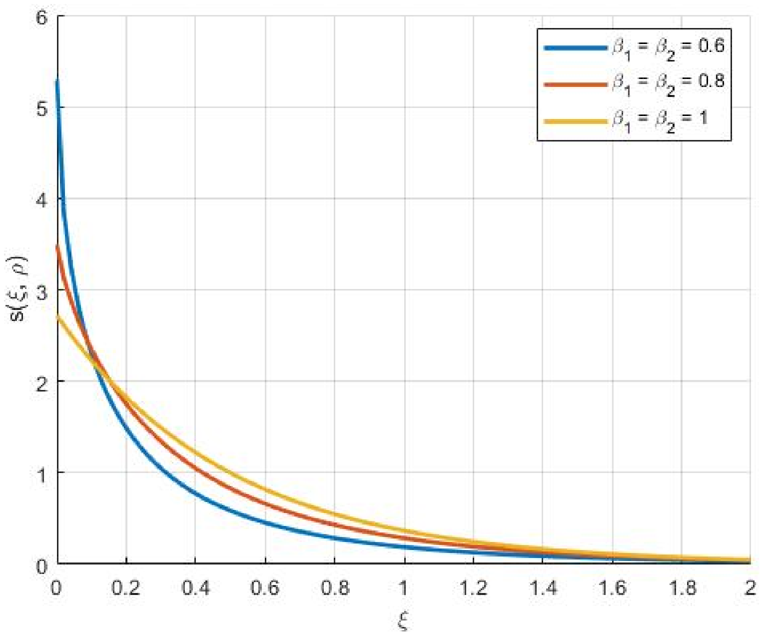

Figure 5: The 2D graph of the solution with respect to ξ at $\beta_1 = \beta_2 = 0.6, 0.8, 1$.

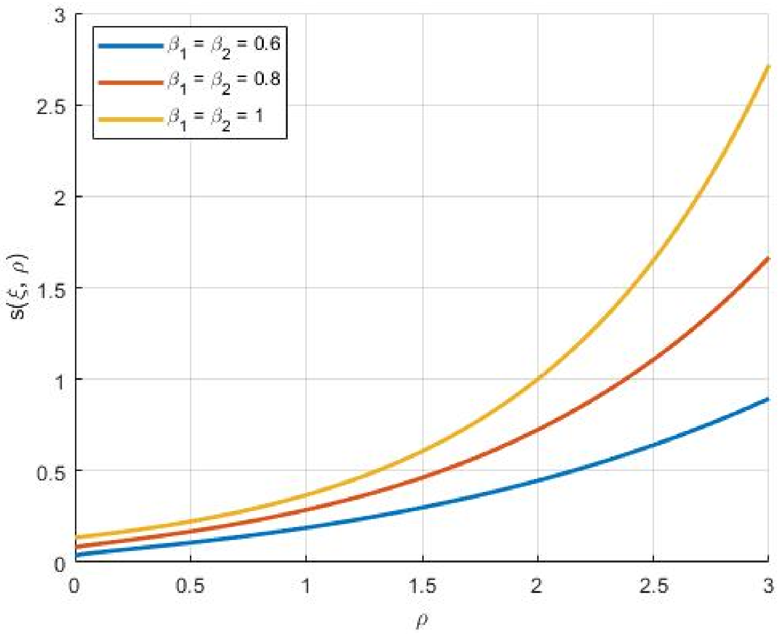

Figure 6: The 2D graph of the solution with respect to ρ at $\beta_1 = \beta_2 = 0.6, 0.8, 1$.

Figures 5 and 6 depict two-dimensional profiles of the solution of the telegraph equation with respect to ξ and ρ, respectively, for fractional orders ($\beta_1 = \beta_2 = 0.6, 0.8, 1$). Figure 5 shows that as ξ increases, the solution's behavior changes more gradually for lower fractional orders. Figure 6 reveals a similar effect along the ρ direction, where lower fractional orders result in a slower and smoother evolution of the solution, confirming the theoretical expectations based on the properties of conformable derivatives.

5. Conclusion

This study introduced the CA-SW and examined its application to conformable fractional partial differential equations. We demonstrated its effectiveness in solving these equations, highlighting its potential as a useful mathematical tool. As a newly developed approach, CA-SW presents several open problems and opportunities for further research. Its ability to handle complex fractional models makes it valuable for applications in physics, engineering, and related fields.

Author contribution statement
The listed authors have played a key role in developing and writing this article.

Data availability statement
This research did not involve the use of any data.

Conflict of interest

The authors confirm that there are no conflicts of interest.

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