EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6102
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Boundary Blowing Up Solutions for an Elliptic Neumann Problem with Nearly Critical Exponent

Rakan Almushahhin[1], Mohamed Ben Ayed[1,*]

[1] *Department of Mathematics, College of Science, Qassim University, Buraydah 51542, Saudi Arabia*

Abstract. In this paper, we investigate the nonlinear problem $(P_\varepsilon) : -\Delta u + V(x)u = fu^{\frac{n+2}{n-2}-\varepsilon}$, $u > 0$ in Ω and $\partial u/\partial\nu = 0$ on $\partial\Omega$, where Ω is a bounded regular domain in $\mathbb{R}^n$, with $n \geq 4$, ε is a small positive parameter, V and f are smooth positive functions on $\overline{\Omega}$. Under certain conditions involving the function f and the mean curvature of the boundary, we construct boundary blowing up solutions of the problem (P_ε) which converge weakly to 0 and blow up at some critical points of $f_b := f_{|\partial\Omega}$. This existence of solutions leads to a multiplicity result for (P_ε). The proof of these results involves expanding the gradient of the associated functional and testing the equation with suitable vector fields. This process imposes constraints on the concentration parameters, and a careful analysis of these constraints leads to the conclusions presented.

2020 Mathematics Subject Classifications: 35A15, 35J20, 35J25
Key Words and Phrases: Partial Differential Equations, Neumann elliptic problems, Critical Sobolev exponent

1. Introduction

In the last decades, there has been a great deal of interest in studying the following problem

$$(P_{\lambda,q}) \qquad \begin{cases} -\Delta u + \lambda u = u^q, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a smooth and bounded open set of $\mathbb{R}^n$ with $n \geq 3$, $q > 1$ and λ is a positive real number.

Problem $(P_{\lambda,q})$ is a well-known example encountered in various applied sciences. For instance, it can be interpreted as the stationary problem arising in a Keller-Segel chemotaxis model [1, 2] , originally developed to describe cell migration in response to chemical cues. Over time, this model has found broad application in engineering, supporting advances in targeted drug delivery, tissue engineering, microfluidic systems, and the design of bio-inspired robots guided by chemotactic behavior [3–5].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6102

Email addresses: `441112378@qu.edu.sa` (R. Almushahhin), `M.benayed@qu.edu.sa` (M. Ben Ayed)

From a mathematical standpoint, problem $(P_{\lambda,q})$ is an interesting model due to its solutions often exhibiting the bubbling phenomenon, with the center of the bubble located on the boundary. This boundary bubbling highlights a strong interaction between the geometry of the boundary and the solutions of $(P_{\lambda,q})$.

A large body of research has explored problem $(P_{\lambda,q})$ when the exponent q is fixed and λ is treated as the parameter. A noteworthy aspect of the problem $(P_{\lambda,q})$ is the existence of solution families, denoted $u_{\lambda,q}$, that exhibit blow up phenomena as λ changes. More precisely, these solutions blow up around certain points within Ω or on its boundary $\partial\Omega$, while remaining negligibly small elsewhere.

For the subcritical case, where $1 < q < (n+2)/(n-2)$, the only solution to $(P_{\lambda,q})$ for small λ is the constant one. However, as λ grows larger, non-constant solutions arise, which blow up at one or more points as $\lambda \to \infty$ [6]. The least energy solution experiences, for large λ, a blow-up at a boundary point where the mean curvature of the boundary is maximized [6–9]. Numerous works, such as [6, 10–13], have analyzed higher-energy solutions of $(P_{\lambda,q})$ that exhibit this asymptotic profile, whether blow up at boundary or interior points, as $\lambda \to \infty$. In particular, solutions with any desired number of blow up points, both interior and boundary, have been shown to exist.

The case when $q = (n+2)/(n-2)$, the critical exponent, differs significantly. For $n \in \{4, 5, 6\}$ and small λ, $(P_{\lambda,q})$ admits non-constant solutions [14–16]. On the other hand, the limiting equation of problem $(P_{\lambda,q})$, which arises when studying the asymptotic behavior of the least-energy solution as $\lambda \to \infty$, does not have any solutions. Nevertheless, least-energy solutions $u_{\lambda,q}$ still exist for large λ, and concentration appears in the form [17, 18]

$$\omega_{a_\lambda,\mu_\lambda}(x),$$

where $a_\lambda \in \partial\Omega$ behaves as in the subcritical case, converging to the point that maximizes the mean curvature of the boundary. In this context, for any $a \in \mathbb{R}^n$ and $\mu \in (0, \infty)$, the function $\omega_{a,\mu}$ represents the standard bubble defined by

$$\omega_{a,\mu}(x) := \beta_0 \frac{\mu^{(n-2)/2}}{(1 + \mu^2|x-a|^2)^{(n-2)/2}}, \quad \text{where} \quad \beta_0 := [n(n-2)]^{(n-2)/4} \tag{1}$$

which are the only solutions [19] of

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n.$$

Higher-energy solutions of $(P_{\lambda,q})$ with concentration on the boundary as $\lambda \to \infty$ have been constructed in several studies, such as [17, 18, 20–27] and the references therein. Unlike the subcritical scenario, at least one blow up point must lie on the boundary [28].

Another interesting avenue of research for problem $(P_{\lambda,q})$ involves studying blow up phenomena by fixing λ while letting the exponent q approaches the critical exponent, i.e., $q = \frac{n+2}{n-2} \pm \varepsilon$, where ε is a small positive parameter. This was first explored by Rey and Wei. For $n \geq 4$ and $q = \frac{n+2}{n-2} + \varepsilon$, they demonstrated the existence of a solution that blows up at a boundary point where the mean curvature is maximized [29]. They also showed the existence of a solution that blows up at a boundary point where the mean curvature is minimized when $q = \frac{n+2}{n-2} - \varepsilon$ and Ω is not convex [29]. In dimension 3, they found a solution with single interior blow-up point [30]. More recently, it was shown that for $n \geq 4$ and $q = \frac{n+2}{n-2} + \varepsilon$, there are no solutions that exhibit blow-up exclusively at interior points when ε is a small positive number [31]. Furthermore, in [32], the authors extended the problem by replacing the constant λ with a function V and studied the problem

$$(P_{V,\varepsilon}) \qquad \begin{cases} -\Delta u + Vu = u^{\frac{n+2}{n-2}-\varepsilon}, \quad u > 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a smooth, bounded subset of $\mathbb{R}^n$, $n \geq 6$, V is a positive C^2-function on $\overline{\Omega}$, and ε is a small positive parameter.

They constructed interior bubbling solutions, where the interior blow-up points of these solutions converge, as $\varepsilon \to 0$, to the critical points of the function V. More recently, in [33], the authors studied the case where a function f is introduced in front of the nonlinear term. More precisely, they considered the following problem

$$(P_\varepsilon) \quad \begin{cases} -\Delta u + Vu = fu^{p-\varepsilon}, \quad u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, $n \geqslant 4$, V and f are positive C^2-functions on $\overline{\Omega}$, ε is a small positive parameter and $p+1 = (2n)/(n-2)$ is the critical Sobolev exponent for the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$.

They constructed solutions of (P_ε) with multi-blow up points located in the interior. A natural question arises: is it possible to construct solutions with boundary blow-up points? As mentioned earlier, this question was partially addressed by Rey and Wei [29], who constructed a solution with a single boundary blow-up point when Ω is non-convex and the function f is equal to 1. In this paper, our objective is to construct solutions to (P_ε) with multiple boundary blow-up points and to present a multiplicity result for this problem. More precisely, the main results of our work are stated as follow:

Theorem 1. *Let $n \geq 4$ and $b_1, \cdots, b_N$ be N non-degenerate critical points of $f_1 := f_{|\partial\Omega}$. We assume that*

$$\frac{c_5}{f(b_k)} \frac{\partial f}{\partial \nu}(b_k) - \Big(\frac{c_1}{2} - c_4\Big)\mathcal{H}(b_k) > 0 \qquad \forall\, k \in \{1, \cdots, N\}, \tag{2}$$

where $\mathcal{H}$ is the mean curvature of the boundary $\partial\Omega$ and c_1, c_4 and c_5 are defined in Lemmas 4, 6 and 8 respectively. Then, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ and for any subset $\{b_{i_1}, \cdots, b_{i_\ell}\} \subset \{b_1, \cdots, b_N\}$, Problem (P_ε) has a solution u_ε which converges weakly to zero and blows up at the points b_{i_j}'s with the following properties

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0} \int_{\Omega\cap B(b_{i_j},\rho)} f u_\varepsilon^{2n/(n-2)} = f(b_{i_j}) S_n \qquad \text{for each } \ j \in \{1, \cdots, \ell\},$$

where S_n is an universal constant defined in (22).
More precisely, for any $\ell \leq N$, there exist $\mu_{1,\varepsilon}$,..., $\mu_{\ell,\varepsilon}$ having the same order as $\varepsilon^{-1/2}$ for $n \geq 5$, and as $\varepsilon^{-1/2}|\ln \varepsilon|^{1/2}$ for $n = 4$ and ℓ points $a_{j,\varepsilon} \to b_{i_j}$ for all j such that

$$\Big|\Big|u_\varepsilon - \sum_{j=1}^{\ell} \omega_{a_{j,\varepsilon},\mu_{j,\varepsilon}}\Big|\Big|_{H^1(\Omega)} \to 0, \qquad as \quad \varepsilon \to 0$$

where the function $\omega_{a,\mu}$ is defined in (1).

Theorem 1 allows us to obtain the following multiplicity result for problem (P_ε) in relation to the number of non-degenerate critical points of the restriction of the function f on the boundary of Ω.

Theorem 2. *Let $n \geq 4$ and assume that the restriction of f to the boundary has N non-degenerate critical points $b_1, \cdots, b_N$ satisfying assumption* (2)*. Then, for small positive ε, the number of solutions to (P_ε) that blow up on the boundary is at least $2^N - 1$.*

Remark 1. *As examples of functions satisfying the assumption* (2)*, let $\Omega := B(0, 1)$ and g be a positive C^2-function on $\overline{\Omega}$ such that $g_{|\partial\Omega}$ has only non-degenerate critical points. Let*

$$f(x) := g(x) + \gamma|x|^2.$$

It easy to see that

$$\frac{\partial f}{\partial \nu}(y) = \frac{\partial g}{\partial \nu}(y) + 2\gamma \quad \forall y \in \partial\Omega \qquad and \qquad f_{|\partial\Omega} = g_{|\partial\Omega} + \gamma.$$

Since the function $\mathcal{H}$ is constant on $\partial\Omega$, it follows that, for γ large, the function f satisfies the assumption (2).

The proof of our results relies on certain balancing conditions satisfied by the concentration parameters, which are relationships that ensure equilibrium between the various factors influencing the blow-up behavior of the solutions. These conditions are derived by performing an asymptotic expansion of the gradient of the Euler-Lagrange functional associated with the problem and testing the equation with appropriate vector fields. This process leads to constraints on both the concentration points and the corresponding blow-up rates of the solution. Through a careful analysis of these conditions, we derive our results.

Note that traditional blow-up analysis methods depend on precise point-wise C^0-estimates and the use of Pohozaev identities. In contrast, our method, used in this paper, deviates from these techniques. Bypassing the need for point-wise estimates and Pohozaev identities, our method holds significant promise for handling non-compact variational problems that involve more intricate blow-up behaviors, as the existence of non-simple blow-up points. Moreover, the method developed in this paper is specifically tailored to the variational problem and does not directly extend to non-variational settings.

The paper is structured as follows: In Section 2, we introduce the necessary preliminaries for studying the problem (P_ε). Section 3 focuses on the analysis of the infinite-dimensional part of the solutions. In Section 4, we carry out an asymptotic expansion of the gradient of the Euler-Lagrange functional associated with (P_ε). Section 5 presents the proof of our main results and Section 6 explores possible avenues for future research. Finally, the proofs rely on some technical facts, which are provided in the appendix in Section 7 for the reader's convenience.

2. Preliminaries

In this section, we proceed with the parametrization of the variational problem under consideration. Indeed, problem (P_ε) is a variational one and its solutions are the positive critical points of the functional

$$J_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega V u^2 - \frac{n-2}{2n-\varepsilon(n-2)}\int_\Omega f|u|^{\frac{2n}{n-2}-\varepsilon}, \quad u \in H^1(\Omega). \tag{3}$$

The space $H^1(\Omega)$ is equipped with the scalar product and its corresponding norm defined by:

$$\langle u_1, u_2\rangle := \int_\Omega \nabla u_1 \nabla u_2 + \int_\Omega V u_1 u_2; \quad \|u\|^2 := \int_\Omega |\nabla u|^2 + \int_\Omega V u^2.$$

Since V is a bounded positive continuous function on $\overline{\Omega}$, it follows that this norm is equivalent to the standard norm of $H^1(\Omega)$.

Observe that, if u_ε is a solution of (P_ε), satisfying $u_\varepsilon \rightharpoonup 0$ (converges weakly to zero), by the concentration compactness principle [34], it follows that u_ε has to be close to some bubbles as $\varepsilon \to 0$, that is, there exist $q \in \mathbb{N}$, $\mu_1, \cdots, \mu_q \longrightarrow \infty$ (as $\varepsilon \to 0$) and $a_1, \ldots, a_q \in \overline{\Omega}$ such that, as $\varepsilon \to 0$,

$$\left\| u_\varepsilon - \sum_{i=1}^{q} f(a_i)^{(2-n)/4}\omega_{a_i,\mu_i} \right\| \to 0, \quad \text{and} \quad \frac{\mu_i}{\mu_j} + \frac{\mu_j}{\mu_i} + \mu_i\mu_j\, |a_i - a_j|^2 \longrightarrow \infty.$$

In this paper, we want to construct some solutions blowing up at some boundary points. To this aim, we introduce the following set:

Let $n \geqslant 4, \eta_0$ be a small positive real, γ_0 be a fixed small positive constant and $q \in \mathbb{N}$, we define

$$\vartheta\,(q, \gamma_0, \eta_0) := \Big\{ (a, \mu, \alpha) \in (\partial\Omega)^q \times \left(\eta_0^{-1}, \infty\right)^q \times (0, \infty)^q : |a_i - a_j| \geqslant \gamma_0 \;\; \forall i \neq j,$$
$$\varepsilon \ln \mu_i < \eta_0 \text{ and } \;|1 - \alpha_i f(a_i)^{(n-2)/4}| \leqslant \eta_0 \;\forall i \Big\}.$$

Furthermore, for $a \in \partial\Omega$ and $\mu > \eta_0^{-1}$, we define

$$F_{a,\mu} := \left\{ v \in H^1(\Omega) : \int_\Omega \nabla v \nabla \omega_{a,\mu} = \int_\Omega \nabla v \nabla \frac{\partial \omega_{a,\mu}}{\partial \mu} = \int_\Omega \nabla v \nabla \frac{\partial \omega_{a,\mu}}{\partial \tau_j} = 0;\, 1 \le j \le n-1 \right\} \tag{4}$$

where the $\tau_j's$, for $j = 1, \ldots, n-1$, build an orthonormal system of coordinates of the tangent space to $\partial\Omega$ at the point a. In addition, for $(a, \mu, \alpha) \in \vartheta(q, \gamma_0, \eta_0)$, we introduce

$$\mathcal{F}_{a,\mu} := \bigcap_{1 \le i \le q} F_{a_i,\mu_i}. \tag{5}$$

Our aim is to construct solutions u having the form $u = \sum_{i=1}^{q} \alpha_i \omega_{a_i,\mu_i} + v$, with $(a, \mu, \alpha) \in \vartheta(q, \gamma_0, \eta_0)$ and $v \in \mathcal{F}_{a,\mu}$.

3. Study of the infinite-dimensional part of the solutions

In this section, we take $(a, \mu, \alpha) \in \vartheta(q, \gamma_0, \eta_0)$ and we are going to study the v-part of the solution u. In the sequel, we denote by

$$\widetilde{u} := \sum_{i=1}^{q} \alpha_i \omega_{a_i,\mu_i}, \quad \text{for} \quad (a, \mu, \alpha) \in \vartheta(q, \gamma_0, \eta_0). \tag{6}$$

Furthermore, for $(a, \mu, \alpha) \in \vartheta(q, \gamma_0, \eta_0)$, we denote by $B_i := B(a_i, \gamma_0/2)$. It follows that

$$\omega_{a_i,\mu_i} \le \frac{c}{\mu_i^{(n-2)/2}} \quad \text{in } \Omega \setminus B_i \qquad \text{and} \qquad \widetilde{u} = \alpha_i \omega_{a_i,\mu_i} + O\Big(\sum_{j \ne i} \frac{c}{\mu_j^{(n-2)/2}}\Big) \quad \text{in } B_i. \tag{7}$$

In the following, we will use the estimate given below, the proof of which is derived by applying Taylor's expansion. For $t_1, t_2 \in \mathbb{R}$ and $\gamma \ge 2$, we have

$$|t_1 + t_2|^\gamma = |t_1|^\gamma + \gamma |t_1|^{\gamma-2} t_1 t_2 + \frac{1}{2}\gamma(\gamma-1) |t_1|^{\gamma-2} t_2^2 + \begin{cases} O\left(|t_1|^{\gamma-3} |t_2|^3 + |t_2|^\gamma\right) & \text{if } \gamma > 3, \\ O(|t_2|^\gamma) & \text{if } \gamma \leqslant 3. \end{cases} \tag{8}$$

Thus, for $u = \widetilde{u} + v$ with $v \in \mathcal{F}_{a,\mu}$, using Eq. (8), , the expansion of J_ε, defined by (3), is as follows

$$J_\varepsilon(u) = J_\varepsilon(\widetilde{u}) - L_\varepsilon(v) + \frac{1}{2} Q_\varepsilon(v) + \mathcal{R}_\varepsilon(v), \qquad \text{with} \tag{9}$$

$$Q_\varepsilon(v) := \|v\|^2 - (p - \varepsilon) \int_\Omega f \widetilde{u}^{p-\varepsilon-1} v^2, \tag{10}$$

$$L_\varepsilon(v) := \int_\Omega f \widetilde{u}^{p-\varepsilon} v, \tag{11}$$

$$\mathcal{R}_\varepsilon(v) = o\left(\|v\|^2\right), \quad \mathcal{R}'_\varepsilon(v) = o(\|v\|) \quad \text{and} \quad \mathcal{R}''_\varepsilon(v) = o(1).$$

Next, we will prove the coercivity of the form Q_ε. More precisely, we have:

Lemma 1. *Let $(a, \mu, \alpha) \in \vartheta(q, \gamma_0, \eta_0)$. Then the following fact holds*

$$Q_\varepsilon(v) = Q(v) + o\left(\|v^2\|\right) \quad \text{where} \quad Q(v) := \|v\|^2 - \frac{n+2}{n-2} \sum_{i=1}^{q} \int_\Omega \omega_{a_i,\mu_i}^{\frac{4}{n-2}} v^2.$$

Proof. By using the following formula derived from Taylor's expansion

$$\left|\sum t_i\right|^\gamma = \sum |t_i|^\gamma + O\left(\sum_{j \ne i} |t_i t_j|^{\gamma/2}\right) \quad \forall t_i \in \mathbb{R}, \quad \forall \gamma \in (0, 2],$$

we derive that

$$\int_\Omega f\tilde{u}^{p-\varepsilon-1}v^2=\sum_{i=1}^{q}\alpha_i^{p-\varepsilon-1}\int_\Omega f\omega_{a_i,\mu_i}^{\frac{4}{n-2}-\varepsilon}v^2+\sum_{j\neq i}O\left(\int_\Omega\left(\omega_{a_i,\mu_i}\omega_{a_j,\mu_j}\right)^{\frac{2}{n-2}}v^2\right).$$

Expanding f around a_i, we obtain

$$\begin{aligned}\int_\Omega f\omega_{a_i,\mu_i}^{\frac{4}{n-2}-\varepsilon}v^2&=f(a_i)\int_\Omega\omega_{a_i,\mu_i}^{\frac{4}{n-2}-\varepsilon}v^2+O\left(\int_\Omega|x-a_i|\,\omega_{a_i,\mu_i}^{\frac{4}{n-2}-\varepsilon}v^2\right)\\&=f(a_i)\int_\Omega\omega_{a_i,\mu_i}^{\frac{4}{n-2}-\varepsilon}v^2+o\left(\|v\|^2\right).\end{aligned}$$

Furthermore, since $\varepsilon\ln\mu_i$ is small, we have

$$\begin{aligned}\omega_{a_i,\mu_i}^{-\varepsilon}&=\beta_0^{-\varepsilon}\mu_i^{-\varepsilon\frac{n-2}{2}}\left(1+\frac{n-2}{2}\varepsilon\ln\left(1+\mu_i^2|x-a_i|^2\right)\right)+O\left(\varepsilon^2\ln^2\left(1+\mu_i^2|x-a_i|^2\right)\right) &(12)\\&=1+o(1). &(13)\end{aligned}$$

Thus, since $\left|1-\alpha_i f(a_i)^{(n-2)/4}\right|$ is small, we get

$$\alpha_i^{\frac{4}{n-2}-\varepsilon}\int_\Omega f\omega_{a_i,\mu_i}^{\frac{4}{n-2}-\varepsilon}v^2=\alpha_i^{\frac{4}{n-2}}f(a_i)\int_\Omega\omega_{a_i,\mu_i}^{\frac{4}{n-2}}v^2+o\left(\|v\|^2\right)=\int_\Omega\omega_{a_i,\mu_i}^{\frac{4}{n-2}}v^2+o\left(\|v\|^2\right).$$

This completes the proof of Lemma 1.

At this point, we require the following important result regarding the uniform coercivity of the quadratic form Q.

Proposition 1. *Let $(a,\mu,\alpha)\in\vartheta(q,\gamma_0,\eta_0)$. There exists $\beta_3>0$ (independent of α,μ and a) such that*

$$Q(v)\geqslant\beta_3\|v\|^2\quad\forall v\in\mathcal{F}_{a,\mu}.$$

The proof of this proposition will be presented in Subsection 7.2.

Combining Lemma 1 and Proposition 1, we deduce that the quadratic form Q_ε is coercive, that is

$$Q_\varepsilon(v)\geqslant\frac{1}{2}\beta_3\|v\|^2\quad\forall v\in\mathcal{F}_{a,\mu}. \tag{14}$$

Now, we need to estimate the norm of the linear form L_ε. More precisely, we have:

Lemma 2. *Let $a\in\partial\Omega$, μ be a large real satisfying $\varepsilon\ln\mu$ is small. Then, for ψ and v satisfying*

$$\psi\in\left\{\omega_{a,\mu},\ \mu\frac{\partial\omega_{a,\mu}}{\partial\mu},\ \frac{1}{\mu}\frac{\partial\omega_{a,\mu}}{\partial a}\right\}\quad,\quad v\in H^1(\Omega)\ \textit{with}\ \int_\Omega\nabla v\cdot\nabla\psi=0, \tag{15}$$

we have

$$\left|\int_\Omega f\omega_{a,\mu}^{\frac{4}{n-2}-\varepsilon}\psi v\right|\leqslant c\|v\|\left(\varepsilon+\frac{1}{\mu}\right).$$

Proof. Observe that, since $\varepsilon \ln \mu$ is small, then Equation (13) holds true. Therefore, using the fact that $|\psi| \leqslant c\omega_{a,\mu}$, we get

$$\begin{aligned}\int_\Omega f\omega_{a,\mu}^{\frac{4}{n-2}-\varepsilon}\psi v &= f(a)\int_\Omega \omega_{a,\mu}^{\frac{4}{n-2}-\varepsilon}\psi v + O\left(\int_\Omega |x-a|\omega_{a,\mu}^{\frac{n+2}{n-2}}|v|\right)\\ &= \beta_0^{-\varepsilon}\mu^{-\varepsilon\frac{n-2}{2}}f(a)\int_\Omega \omega^{\frac{4}{n-2}}\psi v + O\left(\varepsilon\int_\Omega \omega_{a,\mu}^{\frac{n+2}{n-2}}\ln\left(1+\mu^2|x-a|^2\right)|v| + \frac{\|v\|}{\mu}\right). \qquad (16)\end{aligned}$$

Easy computation leads to

$$\int_\Omega \omega_{a,\mu}^{\frac{2n}{n-2}}\left(\ln\left(1+\mu^2|x-a|^2\right)\right)^\gamma \le c \qquad \forall\, \gamma > 0 \qquad (17)$$

which implies that

$$\varepsilon\int_\Omega \omega_{a,\mu}^{\frac{n+2}{n-2}}\ln\left(1+\mu^2|x-a|^2\right)|v| \le c\varepsilon\|v\|. \qquad (18)$$

To estimate the first integral in the right hand side of (16), we distinguish three cases:

- Case 1. If $\psi = \omega_{a,\mu}$, using (15), Holder's inequality, Lemma 7 and the continuity of the embedding $H^1(\Omega) \hookrightarrow L^{\frac{2n-2}{n-2}}(\partial\Omega)$, we get

$$\begin{aligned}\int_\Omega \omega_{a,\mu}^{\frac{4}{n-2}}\psi v &= \int_\Omega \omega_{a,\mu}^{\frac{n+2}{n-2}} v = \int_\Omega -\Delta\omega_{a,\mu} v = \int_\Omega \nabla\omega_{a,\mu}\nabla v - \int_{\partial\Omega}\frac{\partial\omega_{a,\mu}}{\partial\nu}v\\ &= O\left(\left[\int_{\partial\Omega}|v|^{\frac{2n-2}{n-2}}\right]^{\frac{n-2}{2n-2}}\left[\int_{\partial\Omega}\left|\frac{\partial\omega_{a,\mu}}{\partial\nu}\right|^{\frac{2n-2}{n}}\right]^{\frac{n}{2n-2}}\right) = O\left(\frac{\|v\|}{\mu}\right). \qquad (19)\end{aligned}$$

  Combining Eqs. (16), (18) and (19), the proof of the lemma is completed in the case where $\psi = \omega_{a,\mu}$.

- Case 2. If $\psi = \mu\left(\partial\omega_{a,\mu}/\partial\mu\right)$, it holds:

$$\begin{aligned}\int_\Omega \omega_{a,\mu}^{\frac{4}{n-2}}\psi v &= \frac{n-2}{n+2}\int_\Omega -\Delta\left(\mu\frac{\partial\omega_{a,\mu}}{\partial\mu}\right)v\\ &= \frac{n-2}{n+2}\left(\int_\Omega \nabla\left(\mu\frac{\partial\omega_{a,\mu}}{\partial\mu}\right)\nabla v - \int_{\partial\Omega}\frac{\partial}{\partial\nu}\left(\mu\frac{\partial\omega_{a,\mu}}{\partial\mu}\right)v\right)\\ &= O\left(\left[\int_{\partial\Omega}|v|^{\frac{2n-2}{n-2}}\right]^{\frac{n-2}{2n-2}}\left[\int_{\partial\Omega}\left|\mu\frac{\partial^2\omega_{a,\mu}}{\partial\nu\partial\mu}\right|^{\frac{2n-2}{n}}\right]^{\frac{n}{2n-2}}\right) = O\left(\frac{\|v\|}{\mu}\right),\end{aligned}$$

  by using Claims (i) and (ii) of Lemma 7.
  This completes the proof of of the lemma in the case where $\psi = \mu\left(\partial\omega_{a,\mu}/\partial\mu\right)$.

- Case 3. If $\psi = \mu^{-1}\left(\partial\omega_{a,\mu}/\partial a_j\right)$, the proof can be done in the same way. Hence, we omit it.

The proof of Lemma 2 is thereby completed.

Now, we are ready to present the estimate of the linear form L_ε defined in (11).

Proposition 2. *Let $(a,\mu,\alpha) \in \vartheta\left(q,\gamma_0,\eta_0\right)$. Then, we have*

$$|L_\varepsilon(v)| := \left|\int_\Omega f\widetilde{u}^{p-\varepsilon}v\right| \leqslant c\|v\|\left(\varepsilon + \sum\frac{1}{\mu_i}\right) \quad \forall v \in \mathcal{F}_{a,\mu}.$$

Proof. We will use the following formula, the proof of which follows from Taylor expansion. For $t_j > 0$,

$$\left(\sum t_j\right)^\gamma = \sum t_j^\gamma + \sum_{i\neq j} \begin{cases} O\left((t_i t_j)^{\gamma/2}\right) & \text{if } \gamma \leqslant 2, \\ O\left(t_i^{\gamma-1} t_j\right) & \text{if } \gamma > 2. \end{cases} \tag{20}$$

Observe that, if $n \geqslant 6$, it follows that $p := \frac{(n+2)}{(n-2)} \leqslant 2$, and therefore, using (20) and (13), we deduce that

$$L_\varepsilon(v) = \sum_{i=1}^{q} \alpha_i^{p-\varepsilon} \int_\Omega f \omega_{a_i,\mu_i}^{p-\varepsilon} v + \sum_{i\neq j} O\left(\int_\Omega \left(\omega_{a_i,\mu_i}\omega_{a_j,\mu_j}\right)^{p/2} |v|\right).$$

Thus, using Lemma 2 and Holder's inequality, we get

$$|L_\varepsilon(v)| \leqslant c\|v\| \left(\varepsilon + \sum \frac{1}{\mu_i}\right) + c\|v\| \left(\int_\Omega \omega_{a_i,\mu_i}^{\frac{n}{n-2}} \omega_{a_j,\mu_j}^{\frac{n}{n-2}}\right)^{\frac{n+2}{2n}}.$$

But, since $|a_i - a_j| \geqslant \gamma_0$, let $B_k = B\left(a_k, \gamma_0/2\right)$, using (7), it follows that

$$\int_\Omega \omega_{a_i,\mu_i}^{\frac{n}{n-2}} \omega_{a_j,\mu_j}^{\frac{n}{n-2}} \leqslant \frac{c}{\mu_j^{\frac{n}{2}}} \int_{B_i} \omega_{a_i,\mu_i}^{\frac{n}{n-2}} + \frac{c}{\mu_i^{\frac{n}{2}}} \int_{B_j} \omega_{a_j,\mu_j}^{\frac{n}{n-2}} + \frac{c}{(\mu_i\mu_j)^{\frac{n}{2}}} \int_{\Omega\setminus(B_i\cup B_j)} dx \leqslant c\frac{\ln(\mu_i\mu_j)}{(\mu_i\mu_j)^{\frac{n}{2}}}.$$

This completes the proof for $n \geqslant 6$. However, for $n \leqslant 5$, we need to estimate

$$\int_\Omega \omega_{a_i,\mu_i}^{\frac{4}{n-2}} \omega_{a_j,\mu_j} |v| \leqslant c\|v\| \left(\int_\Omega \omega_{a_i,\mu_i}^{\frac{8n}{(n-2)(n+2)}} \omega_{a_j,\mu_j}^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}}.$$

But, since $|a_i - a_j| \geqslant \gamma_0$, using (7), we get

$$\begin{aligned} \int_\Omega \omega_{a_i,\mu_i}^{\frac{8n}{(n-2)(n+2)}} \omega_{a_j,\mu_j}^{\frac{2n}{n+2}} &\leqslant \frac{c}{\mu_j^{\frac{n(n-2)}{n+2}}} \int_{B_i} \omega_{a_i,\mu_i}^{\frac{8n}{(n-2)(n+2)}} + \frac{c}{\mu_i^{\frac{4n}{n+2}}} \int_{B_j} \omega_{a_j,\mu_j}^{\frac{2n}{n+2}} + \frac{c}{\mu_i^{\frac{4n}{n+2}} \mu_j^{\frac{n(n-2)}{(n+2)}}} \\ &\leqslant \frac{c}{(\mu_i\mu_j)^{\frac{n(n-2)}{(n+2)}}}. \end{aligned}$$

This completes the proof of the proposition.

Proposition 3. *Let $(a, \mu, \alpha) \in \vartheta\left(q, \gamma_0, \eta_0\right)$. Then, for ε small, there exists a unique $\overline{v} \in \mathcal{F}_{a,\mu}$ verifying*

$$\langle \nabla J_\varepsilon(\widetilde{u} + \overline{v}), v\rangle = 0 \quad \forall v \in \mathcal{F}_{a,\mu}. \qquad \text{and} \qquad \|\overline{v}\| \leqslant c\left(\varepsilon + \sum \frac{1}{\mu_i}\right).$$

Proof. The proof follows from (9), (14) and Proposition 2 by using the implicit function theorem.

4. Asymptotic expansion of the gradient in the potential sets

This section is devoted to the asymptotic expansion of the gradient of the functional J_ε defined in (3). To this aim, by easy computation, we see that

$$\langle \nabla J_\varepsilon(u), g\rangle = \langle u, g\rangle - \int_\Omega f|u|^{p-\varepsilon-1} u g \quad \forall u, g \in H^1(\Omega). \tag{21}$$

We start by the expansion with respect to the variable α.

Proposition 4. *Let $(a,\mu,\alpha)\in\vartheta(q,\gamma_0,\eta_0)$, $\tilde u$ be defined in (6) and $v\in\mathcal{F}_{a,\mu}$ where $\mathcal{F}_{a,\mu}$ is defined in (5). Then, for ε small and $i\in\{1,\dots,q\}$, it holds*

$$\langle\nabla J_\varepsilon(\tilde u+v),\omega_{a_i,\mu_i}\rangle=\alpha_i S_n\left(1-\mu_i^{-\varepsilon\frac{n-2}{2}}\alpha_i^{p-\varepsilon-1}f(a_i)\right)+O\left(\|v\|^2+\frac{1}{\mu_i}+\varepsilon\right),$$

$$\text{with}\qquad S_n:=\frac{1}{2}[n(n-2)]^{n/2}\int_{\mathbb{R}^n}\frac{1}{(1+|x|^2)^n}dx. \tag{22}$$

Proof. Since $v\in\mathcal{F}_{a,\mu}$, using Lemmas 4 and 5, it follows that

$$\langle\tilde u+v,\omega_{a_i,\mu_i}\rangle=\sum_{j=1}^{q}\alpha_j\langle\omega_{a_j,\mu_j}\omega_{a_i,\mu_i}\rangle=\alpha_i\left(S_n+O\left(\frac{1}{\mu_i}\right)\right)+\sum_{j\neq i}^{q}O\left(\frac{1}{(\mu_i\mu_j)^{(n-2)/2}}\right). \tag{23}$$

Now, observe that

$$|s+t|^\gamma(s+t)z=|s|^\gamma sz+(\gamma+1)|s|^\gamma tz+O(|s|^\gamma t^2+|t|^{\gamma+2})\quad\forall\, s,t\in\mathbb{R},|z|\le|s|\text{ and }\gamma>0.$$

Thus, for each ψ_i satisfying $|\psi_i|\leqslant c\omega_{a_i,\mu_i}$, in $B_i:=B\left(a_i,\gamma_{0/2}\right)\cap\Omega$, using (7) and (13), it holds

$$\begin{aligned}|\widetilde u+v|^{p-\varepsilon-1}(\widetilde u+v)\psi_i=&\ \alpha_i^{p-\varepsilon}\omega_{a_i,\mu_i}^{p-\varepsilon}\psi_i+(p-\varepsilon)\left(\alpha_i\omega_{a_i,\mu_i}\right)^{p-\varepsilon-1}\left(\sum_{j\neq i}\alpha_j\omega_{a_j,\mu_j}+v\right)\psi_i\\&+O\left(\omega_{a_i,\mu_i}^{p-1}\left[\sum\frac{1}{\mu_j^{n-2}}+|v|^2\right]+\sum\frac{1}{\mu_j^n}+|v|^{p+1-\varepsilon}\right)\qquad\text{in }B_i.\end{aligned}\tag{24}$$

But, in $\Omega\backslash B_i$, we have

$$|\widetilde u+v|^{p-\varepsilon}|\psi_i|\leqslant c\left(|v|^{p-\varepsilon}+\sum\omega_{a_j,\mu_j}^p\right)|\psi_i|\quad\text{in}\quad\Omega\backslash B_i. \tag{25}$$

Thus, the integral in Eq. (21) becomes

$$\begin{aligned}&\int_\Omega f|\widetilde u+v|^{p-\varepsilon-1}(\widetilde u+v)\omega_{a_i,\mu_i}\\&=\alpha_i^{p-\varepsilon}\int_{B_i}f\omega_{a_i,\mu_i}^{p+1-\varepsilon}+(p-\varepsilon)\alpha_i^{p-\varepsilon-1}\Big[\sum_{j\neq i}\alpha_j\int_{B_i}f\omega_{a_i,\mu_i}^{p-\varepsilon}\omega_{a_j,\mu_j}+\int_{B_i}f\omega_{a_i,\mu_i}^{p-\varepsilon}v\Big]\\&+O\left(\|v\|^2+\sum\frac{1}{\mu_j^n}+\sum\frac{1}{\mu_j^{n-2}}\int_{B_i}\omega_{a_i,\mu_i}^{p-1}+\frac{\|v\|^{p-\varepsilon}}{\mu_i^{\frac{n-2}{2}}}+\frac{1}{\mu_i^{\frac{n-2}{2}}}\sum\int_{\Omega\backslash B_i}\omega_{a_j,\mu_j}^p\right).\end{aligned}\tag{26}$$

Using (13) and Lemma 5, we get

$$\int_{B_i}\omega_{a_i,\mu_i}^{p-\varepsilon}\omega_{a_j,\mu_j}\leqslant c\int_{B_i}\omega_{a_i,\mu_i}^{p}\omega_{a_j,\mu_j}\leqslant\frac{c}{(\mu_i\mu_j)^{(n-2)/2}}. \tag{27}$$

In addition, using (13), (17) and Lemma 6, we get

$$\begin{aligned}\int_{B_i}f\omega_{a_i,\mu_i}^{p+1-\varepsilon}=&\ \beta_0^{-\varepsilon}\mu_i^{-\varepsilon(n-2)/2}f(a_i)\int_\Omega\omega_{a_i,\mu_i}^{p+1}\\&+O\Big(\int_{\Omega\backslash B_i}\omega_{a_i,\mu_i}^{p+1}+\varepsilon\int_\Omega\omega_{a_i,\mu_i}^{p+1}\ln\left(1+\mu_i^2|x-a_i|^2\right)+\int_{B_i}|x-a_i|\omega_{a_i,\mu_i}^{p+1}\Big)\end{aligned}$$

$$= \beta_0^{-\varepsilon}\mu_i^{-\varepsilon(n-2)/2} f(a_i) S_n + O\left(\frac{1}{\mu_i} + \varepsilon\right). \tag{28}$$

Furthermore, taking $\psi = \omega_{a,\mu}$ in Lemma 2 and using (7), we deduce that

$$\begin{aligned}\int_{B_i} f\omega_{a_i,\mu_i}^{p-\varepsilon} v &= \int_\Omega f\omega_{a_i,\mu_i}^{p-\varepsilon} v + O\Big(\int_{\Omega\setminus B_i} \omega_{a_i,\mu_i}^{p}|v|\Big)\\ &= O\Big(\|v\|\Big[\varepsilon + \frac{1}{\mu_i}\Big]\Big) + O\Big(\frac{\|v\|}{\mu_i^{(n+2)/2}}\Big) = O\Big(\|v\|\Big[\varepsilon + \frac{1}{\mu_i}\Big]\Big).\end{aligned} \tag{29}$$

Combining (29), (28) and (27), the equation (26) becomes

$$\int_{B_i} f|\tilde{u}+v|^{p-\varepsilon-1}(\tilde{u}+v)\omega_{a_i,\mu_i} = \mu_i^{-\varepsilon(n-2)/2}\alpha_i^{p-\varepsilon} f(a_i) S_n + O\left(\frac{1}{\mu_i} + \varepsilon + \|v\|^2\right). \tag{30}$$

Combining (23) and (30), the proof of Proposition 4 follows.

Next, we deal with the expansion with respect to μ.

Proposition 5. *Let $(a,\mu,\alpha) \in \vartheta(q,\gamma_0,\eta_0)$ and $v \in \mathcal{F}_{a,\mu}$. For ε small and $i \leqslant q$, we have*

$$\begin{aligned}\Big\langle \nabla J_\varepsilon(\tilde{u}+v), \mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i}\Big\rangle &= \frac{n-2}{4}c_6\mu_i^{-\varepsilon\frac{n-2}{2}}\alpha_i^{p-\varepsilon} f(a_i)\varepsilon + \alpha_i\frac{\mathcal{H}(a_i)}{\mu_i}\left(\frac{c_1}{2} - \mu_i^{-\varepsilon\frac{n-2}{2}}\alpha_i^{p-\varepsilon-1} f(a_i) c_4\right)\\ &\quad - \frac{c_5}{\mu_i}\frac{\partial f}{\partial\nu}(a_i)\mu_i^{-\varepsilon\frac{n-2}{2}}\alpha_i^{p-\varepsilon} + \underset{(n=4)}{O}\left(\frac{\ln\mu_i}{\mu_i^2}\right) + O\left(\|v\|^2 + \varepsilon^2 + \frac{1}{\mu_i^2} + \sum\frac{1}{\mu_j^{n-2}}\right),\end{aligned}$$

where $\underset{(n=4)}{O}$ appears only if $n=4$ and the constants c_1, c_4, c_5 and c_6 are defined in (55), (56), (57), (58) *respectively.*

Proof. Observe that, using Lemmas 4, 5, 9 and the fact that $v \in \mathcal{F}_{a,\mu}$, we obtain

$$\begin{aligned}\Big\langle \tilde{u}+v, \mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i}\Big\rangle &= \sum_{j=1}^q \alpha_j\Big\langle \omega_{a_j,\mu_j}, \mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i}\Big\rangle\\ &= \frac{c_1}{2}\alpha_i\frac{\mathcal{H}(a_i)}{\mu_i} + O\left(\frac{1}{\mu_i^2} + \sum\frac{1}{\mu_k^{n-2}}\right) + \underset{(n=4)}{O}\left(\frac{\ln\mu_i}{\mu_i^2}\right).\end{aligned} \tag{31}$$

In addition, using (24) and (25), we get

$$\begin{aligned}\int_\Omega f|\tilde{u}+v|^{p-\varepsilon-1}(\tilde{u}+v)\mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} &= \alpha_i^{p-\varepsilon}\int_{B_i} f\omega_{a_i,\mu_i}^{p-\varepsilon}\mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i}\\ &\quad + (p-\varepsilon)\alpha_i^{p-\varepsilon-1}\int_{B_i} f\omega_{a_i,\mu_i}^{p-\varepsilon-1} v\mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} + O\Big(\sum_{j\neq i}\frac{1}{\mu_j^{(n-2)/2}}\int_{B_i}\omega_{a_i,\mu_i}^p\\ &\quad + \sum_{j\neq i}\frac{1}{\mu_j^{n-2}}\int_{B_i}\omega_{a_i,\mu_i}^{p-1} + \sum\frac{1}{\mu_j^n} + \|v\|^2 + \frac{1}{\mu_i^{(n-2)/2}}\Big(\|v\|^{p-\varepsilon} + \sum\int_\Omega \omega_{a_j,\mu_j}^p\Big)\Big).\end{aligned} \tag{32}$$

Notice that, the remainder term can be estimated as

$$O\Big(\sum_{j=1}^q \frac{1}{\mu_j^{n-2}} + \|v\|^2\Big).$$

Furthermore, taking $\psi = \mu\partial\omega_{a,\mu}/\partial\mu$ in Lemma 2 and using (7), we get

$$\int_{B_i} f\omega_{a_i,\mu_i}^{p-\varepsilon-1} v\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} = \int_{\Omega} f\omega_{a_i,\mu_i}^{p-\varepsilon-1} v\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} + O\Big(\int_{\Omega\setminus B_i} \omega_{a_i,\mu_i}^{p}|v|\Big)$$
$$= O\Big(\|v\|\Big[\varepsilon + \frac{1}{\mu_i}\Big]\Big) + O\Big(\frac{\|v\|}{\mu_i^{(n+2)/2}}\Big) = O\Big(\|v\|\Big[\varepsilon + \frac{1}{\mu_i}\Big]\Big). \quad (33)$$

To complete the estimate of (32), using (12), we write

$$\int_{B_i} f\omega_{a_i,\mu_i}^{p-\varepsilon}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} = \beta_0^{-\varepsilon}\mu_i^{-\varepsilon\frac{n-2}{2}} \Bigg[\int_{B_i} f\omega_{a_i,\mu_i}^{p}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} + \frac{n-2}{2}\varepsilon \times$$
$$\times \int_{B_i} f\omega_{a_i,\mu_i}^{p}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} \ln\left(1+\mu_i^2\left|x-a_i\right|^2\right)\Bigg] + O\left(\varepsilon^2 \int_{B_i} \omega_{a_i,\mu_i}^{p+1} \ln^2\left(1+\mu_i^2\left|x-a_i\right|^2\right)\right). \quad (34)$$

Expanding f around a_i and using Lemmas 6 and 8, we obtain

$$\int_{B_i} f\omega_{a_i,\mu_i}^{p}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i}$$
$$= f(a_i)\int_{B_i} \omega_{a_i,\mu_i}^{p}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} + \nabla f(a_i)\int_{B_i}(x-a_i)\,\omega_{a_i,\mu_i}^{p}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} + O\left(\int_{B_i}\left|x-a_i\right|^2\omega_{a_i,\mu_i}^{p}\right)$$
$$= f(a_i)\left(c_4\frac{\mathcal{H}(a_i)}{\mu_i} + O\left(\frac{1}{\mu_i^2}\right)\right) + \frac{\partial f}{\partial\nu}(a_i)\left(\frac{c_5}{\mu_i} + O\left(\frac{1}{\mu_i^2}\right)\right) + O\left(\frac{1}{\mu_i^2}\right).$$

For the second integral in the right hand side of (34), expanding f around a_i and using Lemma 8, we get

$$\int_{B_i} f\omega_{a_i,\mu_i}^{p}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i}\ln\left(1+\mu_i^2\left|x-a_i\right|^2\right) = f(a_i)\left(-\frac{c_6}{2} + O\left(\frac{1}{\mu_i}\right)\right) + O\left(\frac{1}{\mu_i}\right).$$

Thus, using (17), (34) becomes

$$\int_{B_i} f\omega_{a_i,\mu_i}^{p-\varepsilon}\mu_i \frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} = \beta_0^{-\varepsilon}\mu_i^{-\varepsilon\frac{n-2}{2}}\left[c_4 f(a_i)\frac{\mathcal{H}(a_i)}{\mu_i} + \frac{c_5}{\mu_i}\frac{\partial f}{\partial\nu}(a_i) - \frac{n-2}{4}c_6 f(a_i)\,\varepsilon\right]$$
$$+ O\left(\varepsilon^2 + \frac{1}{\mu_i^2}\right). \quad (35)$$

This completes the proof of (32) and we get (by combining (35) and (33))

$$\int_{\Omega} f|\tilde{u}+v|^{p-\varepsilon-1}(\tilde{u}+v)\mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i} = \beta_0^{-\varepsilon}\mu_i^{-\varepsilon\frac{n-2}{2}}\left[c_4 f(a_i)\frac{\mathcal{H}(a_i)}{\mu_i} + \frac{c_5}{\mu_i}\frac{\partial f}{\partial\nu}(a_i)\right.$$
$$\left. - \frac{n-2}{4}c_6 f(a_i)\,\varepsilon\right] + O\left(\|v\|^2 + \varepsilon^2 + \frac{1}{{\mu_i}^2} + \sum\frac{1}{\mu_j^{n-2}}\right). \quad (36)$$

Thus, Combining (36) and (31), the proof of Proposition 5 follows.

We end this section by expanding the gradient of J_ε with respect to the concentration point a.

Proposition 6. *Let $(a,\mu,\alpha)\in\vartheta(q,\gamma_0,\eta_0)$ and $v\in\mathcal{F}_{a,\mu}$. Let $i\in\{1,\dots,q\}$, we denote by $\left(\sigma_i^k\right)$, where $1\leqslant k\leqslant n-1$, an orthonormal system of coordinates on the tangent space to the boundary $\partial\Omega$ at a_i. It holds*

$$\left\langle \nabla J_\varepsilon(\widetilde{u}+v),\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial\sigma_i^k}\right\rangle=-c_7\mu_i^{-\varepsilon\frac{n-2}{2}}\alpha_i^{p-\varepsilon}\frac{1}{\mu_i}\frac{\partial f}{\partial\sigma_i^k}(a_i)+O\left(\varepsilon^2+\frac{1}{\mu_i^2}+\sum\frac{1}{\mu_j^{n-2}}+\|v\|^2\right)$$

where c_7 is defined in (59).

Proof. To simply the presentation, without loss of generality, we will assume that $a_i=0$ and the normal exterior vector $\nu_{a_i}=-e_n$. By this choose, we deduce that the tangent space to the boundary $\partial\Omega$ at the point $a_i=0$ is $\mathbb{R}^{n-1}\times\{0\}$. Let $k\in\{1,\cdots,n-1\}$, using Lemmas 4, 5, the proof is similar to the proof of Proposition 5. Here, we will give a sketch and precise the argument of the estimate of some integrals. Using Lemma 9 and the fact that $v\in\mathcal{F}_{a,\mu}$, we deduce that

$$\left\langle \widetilde{u}+v,\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}\right\rangle=\sum_{j=1}^{q}\alpha_j\left\langle \omega_{a_j,\mu_j},\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}\right\rangle=O\left(\frac{1}{\mu_i^2}+\sum\frac{1}{\mu_j^{n-1}}\right).$$

For the other part of the gradient, using (24) and (25), we deduce that (32), (33) and (34) hold true by taking $\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}$ instead of $\mu_i\frac{\partial\omega_{a_i,\mu_i}}{\partial\mu_i}$. Now, using Lemmas 6 and 8, it holds

$$\begin{aligned}&\int_{B_i}f\omega_{a_i,\mu_i}^p\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}\\&=f(a_i)\int_{B_i}\omega_{a_i,\mu_i}^p\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}+\nabla f(a_i)\int_{B_i}(x-a_i)\,\omega_{a_i,\mu_i}^p\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}+O\left(\int_{B_i}|x-a_i|^2\omega_{a_i,\mu_i}^{p+1}\right)\\&=\frac{c_7}{\mu_i}\frac{\partial f}{\partial x_i}(a_i)+O\left(\frac{1}{\mu_i^2}\right),\\&\int_{B_i}f\omega_{a_i,\mu_i}^p\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}\ln\left(1+\mu_i^2|x-a_i|^2\right)=O\left(\frac{1}{\mu_i}\right),\end{aligned}$$

by expanding f around a_i. Hence, we obtain

$$\int_{B_i}f\omega_{a_i,\mu_i}^{p-\varepsilon}\frac{1}{\mu_i}\frac{\partial\omega_{a_i,\mu_i}}{\partial a_{i,k}}=\beta_0^{-\varepsilon}\mu_i^{-\varepsilon(n-2)/2}\frac{c_7}{\mu_i}\frac{\partial f}{\partial x_k}(a_i)+O\left(\frac{1}{\mu_i^2}\right)+O\left(\varepsilon^2\right).$$

This completes the proof of Proposition 6.

5. Proof of Theorems 1 and 2

Since Theorem 2 is a direct consequence of Theorem 1, it is sufficient to prove the latter. Adopting the proof strategy from [35], let $N\in\mathbb{N}$, $b_1,\dots,b_N$ be as defined in Theorem 1 and $\varepsilon>0$ be small. We consider the set

$$\begin{aligned}\mathcal{D}_{\varepsilon,N}:=\Big\{&(a,\mu,\alpha,v)\in(\partial\Omega)^N\times(0,\infty)^N\times(0,\infty)^N\times H^1(\Omega):\ |a_i-b_i|<\sqrt{\varepsilon}\ ;\\&M_1^{-1}\leqslant\mu_i\varepsilon\leqslant M_1;\quad\left|1-\alpha_i f(a_i)^{(n-2)/4}\right|<\varepsilon\ln^2\varepsilon, v\in\mathcal{F}_{a,\mu}\ \text{ and }\ \|v\|<\sqrt{\varepsilon}\Big\}\end{aligned}$$

where M_1 is a fixed large constant. Let $\tilde{J}_\varepsilon$ be the function defined by

$$\tilde{J}_\varepsilon:\ \mathcal{D}_{\varepsilon,N}\longrightarrow\mathbb{R}\quad;\quad\Lambda:=(a,\mu,\alpha,v)\longmapsto\tilde{J}_\varepsilon(\Lambda):=J_\varepsilon\Big(\sum_{i=1}^{N}\alpha_i w_{a_i,\mu_i}+v\Big).$$

There exists a biunivoque relation between the critical points of $\tilde{J}_\varepsilon$ and the ones of J_ε.

Proposition 7. *Let $\Lambda := (a, \mu, \alpha, v) \in \mathcal{D}_{\varepsilon,N}$. $u = \sum_{i=1}^N \alpha_i w_{a_i,\mu_i} + v$ is a critical point of J_ε if and only if Λ is a critical point of $\tilde{J}_\varepsilon$, i.,e., there exists $(\gamma, \eta, \sigma) \in \left(\mathbb{R}^{n-1}\right)^N \times \mathbb{R}^N \times \mathbb{R}^N$ such that the following system is satisfied:*

$$\begin{aligned}
&(A_k)\ \frac{\partial \tilde{J}_\varepsilon}{\partial \alpha_k}(\Lambda) = 0 \quad \forall\ k,\\
&(V)\ \frac{\partial \tilde{J}_\varepsilon}{\partial v}(\Lambda) = \sum_{k=1}^N \Big(\eta_k \frac{\partial g_{1,k}}{\partial v}(\Lambda) + \sigma_k \frac{\partial g_{2,k}}{\partial v}(\Lambda) + \sum_{j=1}^{n-1} \gamma_{k,j} \frac{\partial g_{3,k,j}}{\partial v}(\Lambda)\Big),\\
&(M_k)\ \frac{\partial \tilde{J}_\varepsilon}{\partial \mu_k}(\Lambda) = \sigma_k \int_\Omega \nabla v \cdot \nabla\Big(\mu_k \frac{\partial^2 w_{a_k,\mu_k}}{\partial \mu_k^2}\Big) + \sum_{j=1}^{n-1} \gamma_{k,j} \int_\Omega \nabla v \cdot \nabla\Big(\frac{1}{\mu_k} \frac{\partial^2 w_{a_k,\mu_k}}{\partial \mu_k \partial \tau_{k,j}}\Big), \forall\ k,\\
&(T_k)\ \frac{\partial \tilde{J}_\varepsilon}{\partial \tau_{k,l}}(\Lambda) = \sigma_k \int_\Omega \nabla v \cdot \nabla\Big(\mu_k \frac{\partial^2 w_{a_k,\mu_k}}{\partial \mu_k \partial \tau_{k,l}}\Big) + \sum_{j=1}^{n-1} \gamma_{k,j} \int_\Omega \nabla v \cdot \nabla\Big(\frac{1}{\mu_k} \frac{\partial^2 w_{a_k,\mu_k}}{\partial \tau_{k,j} \partial \tau_{k,l}}\Big),\ \forall k, \forall\ l.
\end{aligned} \tag{37}$$

Proof. Observe that $\mathcal{D}_{\varepsilon,N}$ is not an open set in $(\partial\Omega)^N \times (0,\infty)^N \times (0,\infty)^N \times H^1(\Omega)$ since the elements Λ of $\mathcal{D}_{\varepsilon,N}$ have to satisfy the following orthogonality constraints:

$$\begin{aligned}
g_{1,k}(\Lambda) &:= \int_\Omega \nabla v \cdot \nabla w_{a_k,\mu_k} = 0, \qquad k \in \{1, \cdots, N\},\\
g_{2,k}(\Lambda) &:= \int_\Omega \nabla v \cdot \nabla\Big(\mu_k \frac{\partial w_{a_k,\mu_k}}{\partial \mu_k}\Big) = 0, \quad k \in \{1, \dots, N\},\\
g_{3,k,j}(\Lambda) &:= \int_\Omega \nabla v \cdot \nabla\Big(\frac{1}{\mu_k} \frac{\partial w_{a_k,\mu_k}}{\partial \tau_{k,j}}\Big) = 0, \quad k \in \{1, \dots, N\},\ j \in \{1, \dots, n-1\}.
\end{aligned}$$

Therefore, it follows from the multiplier Lagrange theorem that Λ is a critical point of $\tilde{J}_\varepsilon$ in $\mathcal{D}_{\varepsilon,N}$ if and only if there exist some constants $\gamma \in \left(\mathbb{R}^{n-1}\right)^N, \eta \in \mathbb{R}^N$ and $\sigma \in \mathbb{R}^N$ such that

$$\nabla \tilde{J}_\varepsilon(\Lambda) = \sum_{k=1}^N \Big(\eta_k \nabla g_{1,k}(\Lambda) + \sigma_k \nabla g_{2,k}(\Lambda) + \sum_{1 \le j \le n-1} \gamma_{k,j} \nabla g_{3,k,j}(\Lambda)\Big). \tag{38}$$

Notice that

$$\nabla \tilde{J}_\varepsilon(\Lambda) = \Big(\Big(\frac{\partial \tilde{J}_\varepsilon(\Lambda)}{\partial \tau_{k,1}}\Big)_{k \le N}, \cdots, \Big(\frac{\partial \tilde{J}_\varepsilon(\Lambda)}{\partial \tau_{k,n-1}}\Big)_{k \le N}, \Big(\frac{\partial \tilde{J}_\varepsilon(\Lambda)}{\partial \mu_k}\Big)_{k \le N}, \Big(\frac{\partial \tilde{J}_\varepsilon(\Lambda)}{\partial \alpha_k}\Big)_{k \le N}, \frac{\partial \tilde{J}_\varepsilon(\Lambda)}{\partial v}\Big). \tag{39}$$

Combining (38), (39) and the fact that the functions $g_{1,k}$, $g_{2,k}$ and $g_{3,k}$ are independent of the variable α, we easily derive the result.

To prove Theorem 1, we observe that Proposition 7 implies that it is sufficient to study the system (37) and demonstrate that (37) has a solution. First, for $\Lambda \in \mathcal{D}_{\varepsilon,N}$, let $u = \sum_{i=1}^N \alpha_i w_{a_i \mu_i} + v$, the definition of $\tilde{J}_\varepsilon$ implies that, for each $k \in \{1, \dots, N\}$,

$$\begin{aligned}
&\frac{\partial \tilde{J}_\varepsilon}{\partial v}(\Lambda) = \nabla J_\varepsilon(u), && \frac{\partial \tilde{J}_\varepsilon}{\partial \mu_k}(\Lambda) = \Big\langle \nabla J_\varepsilon(u), \alpha_k \frac{\partial w_{a_k,\mu_k}}{\partial \mu_k}\Big\rangle,\\
&\frac{\partial \tilde{J}_\varepsilon}{\partial \alpha_k}(\Lambda) = \langle \nabla J_\varepsilon(u), w_{a_k,\mu_k}\rangle, && \frac{\partial \tilde{J}}{\partial \tau_{k,j}}(\Lambda) = \Big\langle \nabla J_\varepsilon(u), \alpha_k \frac{\partial w_{a_k,\mu_k}}{\partial \tau_{k,j}}\Big\rangle, 1 \le j \le n-1.
\end{aligned} \tag{40}$$

Second, using Proposition 3, for each $(a, \mu, \alpha, 0) \in \mathcal{D}_{\varepsilon,N}$, there exists $\overline{v} := \bar{v}_{\varepsilon,a,\mu,\alpha} \in \mathcal{F}_{a,\mu}$ such that

$$\Big\langle \nabla J_\varepsilon\Big(\sum_{i=1}^N \alpha_i w_{a_i \mu_i} + \overline{v}\Big), h\Big\rangle = 0 \quad \forall h \in \mathcal{F}_{a,\mu} \quad \text{and} \quad \|\overline{v}\| \leqslant c\left(\varepsilon + \sum \frac{1}{\mu_i}\right). \tag{41}$$

Therefore, Eqs. (40) and (41) imply the existence of $\eta \in \mathbb{R}^N, \sigma \in \mathbb{R}^N$ and $\gamma \in \left(\mathbb{R}^{n-1}\right)^N$ such that

$$\nabla J_\varepsilon(\bar{u}) = \nabla \tilde{J}_\varepsilon(\bar{\Lambda}) = \sum_{k=1}^{N} \left(\eta_k w_{a_k,\mu_k} + \sigma_k \mu_k \frac{\partial w_{a_k,\mu_k}}{\partial \mu_k} + \sum_{j=1}^{n-1} \gamma_{k,j} \frac{1}{\mu_k} \frac{\partial w_{a_k,\mu_k}}{\partial \tau_{k,j}} \right) \tag{42}$$

where $\bar{\Lambda} = (a, \mu, \alpha, \bar{v})$ and $\bar{u} = \sum_{i=1}^N \alpha_i w_{a_i,\mu_i} + \bar{v}$. Thus, (42) implies that the second equation of (37) is satisfied for each $(a, \mu, \alpha, \bar{v}) \in \mathcal{D}_{\varepsilon,N}$.

Hence, it remains to solve the three other equations. To do so, we start by giving the estimate of the multiplier Lagrange coefficients (η, σ, γ).

Lemma 3. *The multiplier Lagrange coefficients* (η, σ, γ) *found in* (42) *satisfy:*

$$|\eta_k| \leqslant c\varepsilon \ln^2 \varepsilon; \quad |\sigma_k| \leqslant c\varepsilon; \quad |\gamma_{k,j}| \leqslant c\varepsilon^{3/2},$$

for each $k \in \{1, \ldots, N\}$ *and each* $j \in \{1, \ldots, n-1\}$.

Proof. Using $\bar{\Lambda} := (a, \mu, \alpha, \bar{v}) \in \mathcal{D}_{\varepsilon,N}$ and Propositions 4, 5 and 6, it follows that

$$\begin{aligned}
&\langle \nabla J_\varepsilon(\bar{u}), w_{a_k,\mu_k} \rangle = O\left(\varepsilon + \left|1 - \alpha_i^{4/(n-2)} f(a_i)\right| + \varepsilon |\ln \varepsilon|\right) = O\left(\varepsilon |\ln \varepsilon|^2\right), \\
&\left\langle \nabla J_\varepsilon(\bar{u}), \mu_k \frac{\partial w_{a_k,\mu_k}}{\partial \mu_k} \right\rangle = O(\varepsilon), \\
&\left\langle \nabla J_\varepsilon(\bar{u}), \frac{1}{\mu_k} \frac{\partial w_{a_k,\mu_k}}{\partial \tau_{k,j}} \right\rangle = O\left(\varepsilon |a_k - b_k| + \varepsilon^2\right) = O\left(\varepsilon^{3/2}\right).
\end{aligned}$$

Furthermore, for

$$\psi_k, \psi_l \in \bigcup_{i=1}^{N} \left\{ w_{a_i,\mu_i}, \mu_i \frac{\partial w_{a_i,\mu_i}}{\partial \mu_i}, \frac{1}{\mu_i} \frac{\partial w_{a_i,\mu_i}}{\partial \tau_{i,j}}, \quad j \in \{1, \ldots, n-1\} \right\},$$

using Lemmas 4, 5 and 9, we deduce that

$$\langle \psi_k, \psi_l \rangle = \begin{cases} c + O(\varepsilon) & \text{if } k = l, \\ O(\varepsilon) & \text{if } k \neq l, \end{cases}$$

for some positive constant c. Thus, the scalar products of (42) with w_{a_i,μ_i}, $\mu_i \frac{\partial w_{a_i,\mu_i}}{\partial \mu_i}$ and $\frac{1}{\mu_i} \frac{\partial w_{a_i,\mu_i}}{\partial \tau_{i,j}}$, respectively, give the following quasi-diagonal system:

$$\begin{aligned}
c\eta_i + \sum_k O\left(\varepsilon\left(|\gamma_k| + |\sigma_k| + |\eta_k|\right)\right) &= O\left(\varepsilon \ln^2 \varepsilon\right), \\
c\sigma_i + \sum_k O\left(\varepsilon\left(|\gamma_k| + |\sigma_k| + |\eta_k|\right)\right) &= O(\varepsilon), \\
c\gamma_{ij} + O\left(\varepsilon\left(|\gamma_k| + |\sigma_k| + |\eta_k|\right)\right) &= O\left(\varepsilon^{3/2}\right),
\end{aligned}$$

which implies the result.

Now, we are ready to solve the equations (A_k, M_k, T_k) for $k \in \{1, \ldots, N\}$ defined in Proposition 7.

First, using Lemma 3 and Eq. (41), for $\bar{\Lambda}=(a,\mu,\alpha,\bar{v})$, the equations (A_k,M_k,T_k) in the system (37) are equivalent to:

$$\begin{aligned} &\frac{\partial \tilde{J}_\varepsilon}{\partial \alpha_k}(\bar{\Lambda})=0, \quad \forall k\in\{1,\dots,N\},\\ &\mu_k\frac{\partial \tilde{J}_\varepsilon}{\partial \mu_k}(\bar{\Lambda})=O\left(\left(|\sigma_k|+\sum|\gamma_{k,j}|\right)\|\bar{v}\|\right)=O\left(\varepsilon^2\right),\forall k\in\{1,\dots;N\},\\ &\frac{1}{\mu_k}\frac{\partial \tilde{J}_\varepsilon}{\partial \tau_{k,j}}(\bar{\Lambda})=O\left(\left(|\sigma_k|+\sum|\gamma_{k,j}|\right)\|\bar{v}\|\right)=O\left(\varepsilon^2\right),\ \forall k\in\{1,\dots,N\},\forall j\in\{1,\dots,n-1\}. \end{aligned} \tag{43}$$

Second, using (41) and Propositions 4, 5 and 6, we derive that the system (43) is equivalent to

$$\begin{aligned} &1-\mu_k^{-\varepsilon(n-2)/2}\alpha_k^{p-1-\varepsilon}f\left(a_k\right)=O(\varepsilon), \quad \forall k,\\ &\frac{n-2}{4}c_6\varepsilon+\left(\frac{c_1}{2}-c_4\right)\frac{\mathcal{H}\left(a_k\right)}{\mu_k}-\frac{c_5}{\mu_k}\frac{1}{f\left(a_k\right)}\frac{\partial f}{\partial \nu}\left(a_k\right)=\begin{cases}O\left(\varepsilon^2\right) & \text{if} \quad n\geqslant 5,\\ O\left(\varepsilon^2|\ln\varepsilon|\right) & \text{if} \quad n=4,\end{cases} \quad \forall k,\\ &\frac{1}{\mu_k}\nabla f_1\left(a_k\right)=O\left(\varepsilon^2\right), \quad \forall k. \end{aligned} \tag{44}$$

At this step, to solve the system (44), it is better to take a change of variables to obtain an easier system to solve. Notice that $\alpha_k\in(0,\infty)$ and $\mu_k\in(0,\infty)$, however $a_k\in\partial\Omega$ and therefore, we need to be move careful in the change of variables for a_k. To be more precise, let $y\in\partial\Omega$ and $\left(e_1',\dots,e_{n-1}',-\nu_y\right)$ be an orthonormal basis of $\mathbb{R}^n$. In this basis, the tangent space to $\partial\Omega$ at y is $\mathbb{R}^{n-1}\times\{0\}$. Written $y=(y',y_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$, since Ω is a C^2-domain, there exist $\rho>0$ (small) and a C^2-function $g:B_{n-1}(0,\rho)\subset\mathbb{R}^{n-1}\to\mathbb{R}$ such that:

- $g(0)=0,\ \nabla g(0)=0$ and therefore $|g\left(z'\right)|\leqslant c\left|z'\right|^2 \ \forall z'$,
- $\Omega\cap B_n\left(y,\rho\right)=\left\{\left(y'+z',y_n+z_n\right)\in\mathbb{R}^{n-1}\times\mathbb{R}:|(z',z_n)|<\rho \text{ and } z_n>g\left(z'\right)\right\}$,
- $\partial\Omega\cap B_n\left(y,\rho\right)=\left\{\left(y'+z',y_n+z_n\right)\in\mathbb{R}^{n-1}\times\mathbb{R}:|(z',z_n)|<\rho \text{ and } z_n=g\left(z'\right)\right\}$.

Furthermore, assume that y is a critical point of $f_1:=f_{|\partial\Omega}$ (the restriction of f on the boundary), for $a\in\partial\Omega\cap B(y,\rho)$, written a as

$$a:=\left(a',a_n\right)=\left(y'+z',y_n+g\left(z'\right)\right) \quad \text{with} \quad z'\in B_{n-1}(0,\rho), \tag{45}$$

then it holds that

$$\nabla_T f(a)=\nabla f_1(a)=D^2f_1(y)\left(\left(z',0\right),\cdot\right)+O\left(|z'|^2\right). \tag{46}$$

Recall that $(a,\mu,\alpha,0)\in\mathcal{D}_{\varepsilon,N}$ which implies that a_k is close to b_k (which is a critical point of f_1) and $\alpha_k f\left(a_k\right)^{n-2/4}$ is close to 1 for each k. Hence, let us consider the following change of variables:

$$\rho_k:=1-\alpha_k f\left(b_k\right)^{n-2/4}, \quad k\in\{1,\dots,N\}, \tag{47}$$

$$\frac{1}{\mu_k}:=\left[\frac{c_5}{f\left(b_k\right)}\frac{\partial f}{\partial\nu}\left(b_k\right)-\left(\frac{c_1}{2}-c_4\right)\mathcal{H}\left(b_k\right)\right]^{-1}\frac{n-2}{4}c_6\varepsilon\left(1+\lambda_k\right), \tag{48}$$

$$a_k:=\left(b_k'+z_k',\left(b_k\right)_n+g\left(z_k'\right)\right), \quad k\in\{1,\dots,N\}, \tag{49}$$

by using the notation of (45). Using this change of variables, we get:

$$f\left(a_k\right)=f_1\left(a_k\right)=f_1\left(b_k\right)+O\left(\left|a_k-b_k\right|^2\right)=f_1\left(b_k\right)+O\left(\left|z_k'\right|^2\right),$$

$$\mathcal{H}\left(a_k\right)=\mathcal{H}\left(b_k\right)+O\left(\left|z_k'\right|\right),$$
$$\frac{1}{f(a_k)}\frac{\partial f}{\partial \nu}\left(a_k\right)=\frac{1}{f\left(b_k\right)}\frac{\partial f}{\partial \nu}\left(b_k\right)+O\left(\left|z_k'\right|\right),$$

$$\begin{aligned}1-\mu_k^{-\varepsilon(n-2)/2}\alpha_k^{\frac{4}{n-2}-\varepsilon}f\left(a_k\right)&=1-\alpha_k^{\frac{4}{n-2}}f_1\left(b_k\right)+O\left(\left|z'\right|^2+\varepsilon\left|\ln\varepsilon\right|\right)\\&=\frac{4}{n-2}\rho_k+O\left(\rho_k^2+\left|z'\right|^2+\varepsilon\left|\ln\varepsilon\right|\right),\end{aligned}\qquad(50)$$

$$\begin{aligned}&\frac{n-2}{4}c_6\varepsilon+\frac{1}{\mu_k}\left[\left(\frac{c_1}{2}-c_4\right)\mathcal{H}\left(a_k\right)-\frac{c_5}{f\left(a_k\right)}\frac{\partial f}{\partial \nu}\left(a_k\right)\right]\\&=\frac{n-2}{4}c_6\varepsilon-\frac{1}{\mu_k}\left[\frac{c_5}{f\left(b_k\right)}\frac{\partial f}{\partial \nu}\left(b_k\right)-\left(\frac{c_1}{2}-c_4\right)\mathcal{H}\left(b_k\right)\right]+O\left(\frac{1}{\mu_k}\left|z'\right|\right)\\&=-\frac{n-2}{4}c_6\varepsilon\lambda_k+O\left(\varepsilon\left|z'\right|\right).\end{aligned}\qquad(51)$$

Thus, using (46), (50) and (51), the system (44) becomes equivalent to:

$$\begin{aligned}&\rho_k=O\left(\rho_k^2+\left|z'\right|^2+\varepsilon\left|\ln\varepsilon\right|\right),\quad k\in\{1,\ldots,N\},\\&\lambda_k=O\Big(\left|z'\right|+(\text{if }\ n\geqslant 5)\,\varepsilon+(\text{if }\ n=4)\,\varepsilon\left|\ln\varepsilon\right|\Big)\\&D^2f_1\left(b_k\right)\left(\left(z',0\right),\cdot\right)=O\left(\varepsilon+\left|z'\right|^2\right),\quad k\in\{1,\ldots,N\}.\end{aligned}\qquad(52)$$

Since $D^2f_1\left(b_k\right)$ is assumed to be non-degenerate, the last equation in (52) implies that $|z'|\leqslant c\left(\varepsilon+\left|z'\right|^2\right)$, and therefore, the system (52) can be rewritten as

$$\begin{aligned}\rho_k&=O\left(\rho_k^2+\left|z'\right|^2+\varepsilon\left|\ln\varepsilon\right|\right),\quad k\in\{1,\ldots,N\},\\\lambda_k&=O\left(\left|z'\right|^2+(\text{if }\ n\geqslant 5)\,\varepsilon+(\text{if }\ n=4)\left|\varepsilon\ln\varepsilon\right|\right),\\D^2f_1\left(b_k\right)&=O\left(\varepsilon+\left|z'\right|^2\right),\quad k\in\{1,\ldots,N\}.\end{aligned}\qquad(53)$$

Since, $D^2f_1\left(b_k\right)$ is non-degenerate, using Brouwer's fixed point theorem, we deduce that (53) has a solution $\left(\rho^\varepsilon,\lambda^\varepsilon,(z')^\varepsilon\right)$. Furthermore, it holds, for each $k\in\{1,\ldots,N\}$,

$$\rho_k^\varepsilon=O\left(\varepsilon\left|\ln\varepsilon\right|\right)\ ;\quad\lambda_k^\varepsilon=O\Big((\text{if }\ n\geqslant 5)\,\varepsilon+(\text{if }\ n=4)\left|\varepsilon\ln\varepsilon\right|\Big)\ ;\quad\left(z_k'\right)^\varepsilon=O(\varepsilon).$$

Taking $\alpha_k^\varepsilon,\mu_k^\varepsilon$ and a_k^ε by using the equations (47), (48) and (49) and taking $u_\varepsilon=\sum_{k=1}^N\alpha_k^\varepsilon w_{a_k^\varepsilon,\mu_k^\varepsilon}+\bar{v}_\varepsilon$, we deduce that u_ε is a critical point of I_ε and therefore it satisfies

$$\begin{cases}(-\Delta+V)u_\varepsilon=f|u_\varepsilon|^{\frac{4}{n-2}-\varepsilon}u_\varepsilon & \text{in }\Omega,\\\partial u_\varepsilon/\partial\nu=0 & \text{on }\partial\Omega.\end{cases}$$

Finally, we have to prove that $u_\varepsilon>0$. To this aim, let $u_\varepsilon^-:=\max(0,-u_\varepsilon)$, it follows that $0\leq u_\varepsilon^-\leq|\overline{v}_\varepsilon|$. Furthermore, multiplying the previous equation by u_ε^- and integrating over Ω, we obtain

$$\|u_\varepsilon^-\|^2=\int_\Omega\nabla u_\varepsilon\nabla u_\varepsilon^-+\int_\Omega Vu_\varepsilon u_\varepsilon^-$$

$$= \int_\Omega (-\Delta + V) u_\varepsilon u_\varepsilon^- = \int_\Omega f|u_\varepsilon|^{p-\varepsilon-1} u_\varepsilon u_\varepsilon^- = \int_\Omega f(u_\varepsilon^-)^{p-\varepsilon+1} \tag{54}$$

which implies that $\|u_\varepsilon^-\|^2 = o(1)$ (since $0 \le u_\varepsilon^- \le |\overline{v}_\varepsilon|$). Now, using the Holder's inequality, we obtain

$$\|u_\varepsilon^-\|^2 \le c\|u_\varepsilon^-\|^{p+1-\varepsilon}.$$

Thus, u_ε^- has to be zero and therefore, by the maximum principle, we derive that $u_\varepsilon > 0$ in Ω. Hence u_ε is a solution of Problem (P_ε). This completes the proof of Theorem 1.

6. Conclusion

By expanding the gradient of the associated functional and testing the equation with appropriate vector fields, we were able to construct boundary blow-up solutions for the problem (P_ε), which exhibit isolated bubbles. This construction exploits the structure of the problem, using asymptotic analysis to capture the intricate behavior of the concentration points and the corresponding blow-up rates of the solution as the perturbation parameter ε approaches zero. By carefully analyzing the interaction between the nonlinearities of the equation and the boundary conditions, we establish a connection between the number of isolated bubbles and the topology of the problem. This approach ultimately leads to a multiplicity result, demonstrating that the number of boundary blow-up solutions is closely related to the number of non-degenerate critical points of the restriction of the function f on the boundary of the domain Ω. This result provides a deeper understanding of the solution structure, offering insights into bifurcation behavior and the stability of solutions as the boundary conditions are varied. Nevertheless, several promising avenues for further research and open questions remain:

(i) Do boundary clustered bubble solutions exist for the problem?

(ii) Can we provide a complete description of the asymptotic profile of the boundary blowing up solutions?

(iii) What occurs if the critical points of the restriction f_1 of the function f on the boundary are degenerate? In particular, what occurs when f_1 satisfies certain flatness conditions?

(iv) Is it possible to get the same results presented in this paper when the solutions do not converge weakly to zero?

7. Appendix

In this section, we gather estimates for several integrals, which are crucial for refining the expansion of the gradient of the Euler-Lagrange functional J_ε. Additionally, we prove the coercivity of the quadratic form defined by (10).

7.1. Useful estimates of some integrals

We start by the following lemma which is extracted from [27] (see equations (D.6), (D.7) and (D.8)).

Lemma 4. *[27] Let $n \geqslant 3$, $a \in \partial\Omega$ and μ be a large real. We have*

(i) $$\int_\Omega |\nabla \omega_{a,\mu}|^2 = S_n - c_1 \frac{\mathcal{H}(a)}{\mu} + O\left(\frac{1}{\mu^2}\right),$$

(ii) $$\int_\Omega \nabla\omega_{a,\mu}\nabla\left(\mu\frac{\partial\omega_{a,\mu}}{\partial\mu}\right) = \frac{c_1}{2}\frac{\mathcal{H}(a)}{\mu} + O\left(\frac{1}{\mu^2}\right),$$

(iii) $$\int_\Omega \nabla\omega_{a,\mu}\nabla\left(\frac{1}{\mu}\frac{\partial\omega_{a,\mu}}{\partial\tau_j}\right) = O\left(\frac{1}{\mu^2}\right) \quad \forall j\in\{1,\dots,n-1\},$$

where $\tau_j's$, for $j=1,\dots,n-1$, build an orthonormal system of coordinates on the tangent space to $\partial\Omega$ at the point $a\in\partial\Omega$, the constant S_n is defined in (22) and the constant c_1 is defined by

$$c_1 := [n(n-2)]^{(n-2)/2}\,\frac{(n-2)^2}{4}\,\text{meas}\left(\mathbb{S}^{n-2}\right)\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{n-3}{2}\right)}{\Gamma(n)}. \tag{55}$$

We notice that, in this paper we use $\omega_{a,\mu}=\beta_0 U_{a,\mu}$ where $\beta_0=[n(n-2)]^{(n-2)/4}$ and $U_{a,\mu}$ is the function used in [27]. For this reason, there is some changes in the constants found in Lemma 4 and the following lemmas with the corresponding results in [27].

The second lemma deals with some integrals involving the bubbles.

Lemma 5. *Let $n\geqslant 4$, $a\in\partial\Omega$ and μ be a large real. It holds:*

(i) $$\int_\Omega \omega_{a,\mu}^2 \leqslant c\begin{cases}\mu^{-2} & \text{if } n\geqslant 5,\\ \mu^{-2}\ln\mu & \text{if } n=4,\end{cases}$$

(ii) $$\left|\int_\Omega \omega_{a,\mu}\mu\frac{\partial\omega_{a,\mu}}{\partial\mu}\right| \leqslant c\begin{cases}\mu^{-2} & \text{if } n\geqslant 5,\\ \mu^{-2}\ln\mu & \text{if } n=4,\end{cases}$$

(iii) $$\left|\int_\Omega \omega_{a,\mu}\frac{1}{\mu}\frac{\partial\omega_{a,\mu}}{\partial a}\right| \leqslant \frac{c}{\mu^3}.$$

Proof. Notice that, since Ω is bounded, there exists $R>0$ such that $\Omega\subset B(a,R)$.
Claim (i) follows by standard computations. Concerning Claim (ii), it follows from the first one and the fact that $\mu\left|\frac{\partial\omega_{a,\mu}}{\partial\mu}\right|\leqslant c\omega_{a,\mu}$. Finally, for Claim (iii), observe that

$$\frac{1}{\mu}\left|\frac{\partial\omega_{a,\mu}}{\partial a}\right| \leqslant \frac{1}{\mu|x-a|}\omega_{a,\mu}.$$

Hence, the result follows by standard computations.

The next lemma is extracted from [27] (see the equations (D.17), (D.18) and (D.19)).

Lemma 6. *[27] Let $n\geqslant 4, a\in\partial\Omega$ and μ be a large real. There hold:*

(i) $$\int_\Omega \omega_{a,\mu}^{\frac{2n}{n-2}} = S_n - \frac{2n}{n-2}c_4\frac{\mathcal{H}(a)}{\mu} + O\left(\frac{1}{\mu^2}\right),$$

(ii) $$\int_\Omega \omega_{a,\mu}^{\frac{n+2}{n-2}}\mu\frac{\partial\omega_{a,\mu}}{\partial\mu} = c_4\frac{\mathcal{H}(a)}{\mu} + O\left(\frac{1}{\mu^2}\right),$$

(iii) $$\int_\Omega \omega_{a,\mu}^{\frac{n+2}{n-2}}\frac{1}{\mu}\frac{\partial\omega_{a,\mu}}{\partial\tau_j} = O\left(\frac{1}{\mu^2}\right) \quad \forall j\in\{1,\dots,n-1\},$$

where

$$c_4 := \frac{n-2}{2n}[n(n-2)]^{n/2}\,\frac{1}{4}\,\text{meas}\left(\mathbb{S}^{n-2}\right)\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n)}. \tag{56}$$

We also have the following estimates:

Lemma 7. *Let $a \in \partial\Omega$ and μ be a large real. It holds:*

(i) $$\mu \left| \frac{\partial^2 \omega_{a,\mu}}{\partial \nu \partial \mu} \right| \leqslant c \left| \frac{\partial \omega_{a,\mu}}{\partial \nu} \right|,$$

(ii) $$\Big(\int_{\partial\Omega} \Big| \frac{\partial \omega_{a,\mu}}{\partial \nu} \Big|^{(2n-2)/n} \Big)^{n/(2n-2)} \leq \frac{c}{\mu},$$

(iii) $$\Big(\int_{\partial\Omega} \Big| \frac{\partial^2 \omega_{a,\mu}}{\partial \nu \partial a} \Big|^{(2n-2)/n} \Big)^{n/(2n-2)} \leqslant c.$$

Proof. Claims (ii) and (iii) are extracted from [27] (See the equations $(D.49)$ and $(D.50)$). Concerning Claim (i), it follows easily.

We end this subsection by the following two lemmas:

Lemma 8. *Let $a \in \partial\Omega$ and μ be a large real. It holds:*

(i) $$\int_\Omega (x-a)\cdot \tau_j \omega_{a,\mu}^{\frac{n+2}{n-2}} \mu \frac{\partial \omega_{a,\mu}}{\partial \mu} = O\left(\frac{1}{\mu^2}\right) \quad \forall j \in \{1, \dots, n-1\},$$

(ii) $$\int_\Omega (x-a)\cdot \nu_a \omega_{a,\mu}^{\frac{n+2}{n-2}} \mu \frac{\partial \omega_{a,\mu}}{\partial \mu} = \frac{c_5}{\mu} + O\left(\frac{1}{\mu^2}\right),$$

(iii) $$\int_\Omega \omega_{a,\mu}^{\frac{n+2}{n-2}} \mu \frac{\partial \omega_{a,\mu}}{\partial \mu} \ln\left(1+\mu^2 |x-a|^2\right) = -\frac{c_6}{2} + O\left(\frac{1}{\mu}\right),$$

(iv) $$\int_\Omega (x-a)\cdot \tau_k \omega_{a,\mu}^{\frac{n+2}{n-2}} \frac{1}{\mu} \frac{\partial \omega_{a,\mu}}{\partial \tau_j} = \begin{cases} O\left(\mu^{-2}\right) & \text{if} \quad k \neq j, \\ \frac{c_7}{\mu} + O\left(\frac{1}{\mu^2}\right) & \text{if} \quad k = j, \end{cases}$$

for each $j \in \{1, \dots, n-1\}$ and $k \in \{1, \dots, n\}$,

(v) $$\int_\Omega \omega_{a,\mu}^{\frac{n+2}{n-2}} \frac{1}{\mu} \frac{\partial \omega_{a,\mu}}{\partial \tau_j} \ln\left(1+\mu^2 |x-a|^2\right) = O\left(\frac{1}{\mu^2}\right) \quad \forall j \in \{1, \dots, n-1\},$$

where

$$c_5 := [n(n-2]^{n/2}\, \frac{n-2}{2} \int_{\mathbb{R}^n_+} x_n \frac{|x|^2-1}{(1+|x|^2)^{n+1}} dx > 0, \tag{57}$$

$$c_6 := \frac{n-2}{2} [n(n-2)]^{n/2} \int_{\mathbb{R}^n} \frac{|x|^2-1}{(1+|x|^2)^{n+1}} \ln\left(1+|x|^2\right) dx > 0, \tag{58}$$

$$c_7 := \frac{n-2}{2n} [n(n-2)]^{n/2} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{n+1}} dx. \tag{59}$$

Proof. Without loss of generality, we can assume that $a = 0$ and

$$\nu_a = -e_n. \tag{60}$$

Since we assumed that Ω is smooth, there exit $\rho > 0$ (we take it small) and a function $\varphi : B_{n-1}(0,\rho) \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that

$$\varphi(0) = 0,\ \varphi'(0) = 0 \ \text{ and } \ \Omega \cap B_n(0,\rho) = \{x = (x', x_n) \in B_{n-1}(0,\rho) \times \mathbb{R} :\ x_n > \varphi(x')\}.$$

Since $\varphi'(0) = 0$, it is easy to see that

$$\varphi(x') = O\left(|x'|^2\right) \quad \forall x' \in B_{n-1}(0,\rho). \tag{61}$$

Observe that

$$\int_{\Omega\setminus B(0,\rho)} |x|\omega_{0,\mu}^{\frac{2n}{n-2}} \leqslant \int_{\Omega\setminus B(0,\rho)} \frac{|x|}{\mu^n|x|^{2n}}dx \leqslant \frac{c}{\mu^n}. \tag{62}$$

To estimate the integral over $\Omega \cap B(0,\rho)$, we introduce the following sets

$$\begin{aligned} B^+(0,\rho) &:= \{x = (x', x_n) \in B(0,\rho) : \quad x_n > 0\}, \\ \Omega_1 &:= \{x = (x', x_n) \in B(0,\rho) : \quad 0 < x_n < \varphi(x')\}, \\ \Omega_2 &:= \{x = (x', x_n) \in B(0,\rho) : \quad \varphi(x') < x_n < 0\}, \end{aligned}$$

and we have

$$\int_{\Omega\cap B(0,\rho)} \ldots = \int_{B^+(0,\rho)} \ldots - \int_{\Omega_1} \ldots + \int_{\Omega_2} \ldots. \tag{63}$$

Proof of (i): Let $j \in \{1, \cdots, n-1\}$. By (60), it follows that $(x-a)\cdot\tau_j = x_j$. By oddness of the function, it is easy to get that the first integral is zero. Concerning the other ones, using (61), we derive that

$$|x_n| \leqslant \varphi(x') = O\left(|x'|^2\right) \quad \forall (x', x_n) \in \Omega_i, \; i = 1, 2. \tag{64}$$

Furthermore, it is easy to see that $1 + \mu^2|x|^2 \geqslant 1 + \mu^2|x'|^2$. Thus, we obtain, for $i \in \{1, 2\}$,

$$\left|\int_{\Omega_i} \ldots\right| \leqslant \int_{\Omega_i} |x'|\,\omega_{a,\mu}^{\frac{2n}{n-2}} \leqslant c\int_{\Omega_i} \frac{\mu^n|x'|}{(1+\mu^2|x'|^2}dx'dx_n \leqslant c\int_{B_{n-1}(0,\rho)} \frac{\mu^n|x'|^3}{\left(1+\mu^2|x'|^2\right)^n}dx' \leqslant \frac{c}{\mu^2}. \tag{65}$$

Hence, Eqs. (62), (63) and (65) end the proof of Claim (i).

Proof of (ii): From (60), we deduce that $(x-a)\cdot\nu_a = -x_n$. As in the proof of Claim (i), for $i \in \{1, 2\}$, we have (using (64))

$$\left|\int_{\Omega_i} x_n\omega_{a,\mu}^{\frac{n+2}{n-2}}\mu\frac{\partial\omega_{a,\mu}}{\partial\mu}\right| \leqslant c\int_{\Omega_i} \frac{\mu^n|x_n|}{\left(1+\mu^2|x'|^2\right)^n}dx'dx_n \leqslant c\int_{B_{n-1}(0,\rho)} \frac{\mu^n|x'|^4}{\left(1+\mu^2|x'|^2\right)^n}dx' \leqslant \frac{c}{\mu^3}. \tag{66}$$

For the integral over $B^+(0,\rho)$, it holds

$$\begin{aligned} \int_{B^+(0,\rho)} -x_n\omega_{0,\mu}^{\frac{n+2}{n-2}}\mu\frac{\partial\omega_{0,\mu}}{\partial\mu} &= \int_{B^+(0,\rho)} -x_n\left(\frac{n-2}{2}\right)\omega_{0,\mu}^{\frac{2n}{n-2}}\frac{1-\mu^2|x|^2}{1+\mu^2|x|^2}dx \\ &= -\beta_0^{\frac{2n}{n-2}}\frac{n-2}{2}\int_{B^+(0,\rho)} \mu^n x_n\frac{1-\mu^2|x|^2}{(1+\mu^2|x|^2)^{n+1}}dx \\ &= -\beta_0^{\frac{2n}{n-2}}\frac{n-2}{2}\frac{1}{\mu}\int_{\mathbb{R}^n_+} x_n\frac{1-|x|^2}{(1+|x|^2)^{n+1}}dx + O\left(\frac{1}{\mu^n}\right). \end{aligned} \tag{67}$$

Combining Eqs. (62), (66) and (67), the proof of Claim (ii) follows.

Proof of (iii): Following the proof of the previous claims, we need to estimate:

$$\left|\int_{\Omega\setminus B(0,\rho)} \ldots\right| \leqslant \int_{\Omega\setminus B(0,\rho)} \omega_{a,\mu}^{\frac{2n}{n-2}}\ln\left(1+\mu^2|x-a|^2\right) \leqslant c\frac{\ln\mu}{\mu^n}, \tag{68}$$

$$\int_{B^+(0,\rho)} \dots = \beta_0^{\frac{2n}{n-2}} \frac{n-2}{2} \int_{B^+(0,\rho)} \frac{\mu^n \left(1-\mu^2|x|^2\right)}{\left(1+\mu^2|x|^2\right)^{n+1}} \ln\left(1+\mu^2|x|^2\right) dx$$
$$= \beta_0^{\frac{2n}{n-2}} \frac{n-2}{2} \int_{(B^+(0,\lambda\rho)} \frac{1-|x|^2}{\left(1+|x|^2\right)^{n+1}} \ln\left(1+|x|^2\right) dx$$
$$= -\frac{1}{2}\beta_0^{\frac{2n}{n-2}} \frac{n-2}{2} \int_{\mathbb{R}^n} \frac{|x|^2-1}{\left(1+|x|^2\right)^{n+1}} \ln\left(1+|x|^2\right) dx + O\left(\frac{\ln\mu}{\mu^n}\right). \tag{69}$$

For the integrals over $\Omega_i, i=1,2$, note that, using Eq. (64), we have $|x_n| \leq c\,|x'|^2$, which implies that

$$1+\mu^2|x|^2 = 1+\mu^2\,|x'|^2 + \mu^2 x_n^2 \leqslant 1+\mu^2\,|x'|^2\left(1+c\,|x'|^2\right) \leqslant 2(1+\mu^2\,|x'|^2), \tag{70}$$

since $|x'| < \rho$ which is small. Thus we obtain

$$\left|\int_{\Omega_i} \dots\right| \leqslant c\int_{\Omega_i} \mu^n \frac{\ln\left(1+\mu^2|x|^2\right)}{\left(1+\mu^2|x|^2\right)^n} dx \leq c\int_{B_{n-1}(0,\rho)} \mu^n \frac{|x'|^2 \ln\left(1+\mu^2\,|x'|^2\right)}{\left(1+\mu^2\,|x'|^2\right)^n} dx'$$
$$\leqslant \frac{c}{\mu}\int_{\mathbb{R}^{n-1}} \frac{|x'|^2 \ln\left(1+|x'|^2\right)}{\left(1+|x'|^2\right)^n} dx' \leqslant \frac{c}{\mu}. \tag{71}$$

Hence, (68), (69) and (71) imply the proof of Cham (iii).

Proof of (iv): Note that, by (60), it follows that $(x-a)\cdot\tau_k = x_k$ and $\frac{\partial\omega_{a,\mu}}{\partial\tau_j} = \frac{\partial\omega_{a,\mu}}{\partial a_j}$. As before, we compute:

$$\left|\int_{\Omega\setminus B(0,\rho)} \dots\right| \leqslant \int_{\Omega\setminus B(0,\rho)} |x|\frac{1}{\mu|x|}\omega_{0,\mu}^{\frac{2n}{n-2}} \leqslant \frac{c}{\mu^{n+1}}, \tag{72}$$

where we have uses the fact that $\left|\frac{\partial\omega_{a,\mu}}{\partial a}\right| \leqslant c\frac{\omega_{a,\mu}}{|x-a|}$.

Concerning the integral over $\Omega\cap B(0,\rho)$, using (63), we need to compute:

$$\int_{B^+(0,\rho)} \dots = (n-2)\beta_0^{\frac{2n}{n-2}} \int_{B^+(0,\rho)} x_k \frac{\mu^{n+1} x_j}{\left(1+\mu^2|x|^2\right)^{n+1}} dx = 0, \quad k\neq j, \tag{73}$$

(by oddness with respect to the variable x_j). However, if $k=j$, we obtain

$$\int_{B^+(0,\rho)} \dots = (n-2)\beta_0^{\frac{2n}{n-2}} \int_{B^+(0,\rho)} \frac{\mu^{n+1} x_j^2}{\left(1+\mu^2|x|^2\right)^{n+1}} dx$$
$$= \frac{1}{2}\frac{n-2}{\mu}\beta^{\frac{2n}{n-2}} \int_{B(0,\mu\rho)} \frac{x_j^2}{\left(1+|x|^2\right)^{n+1}} dx$$
$$= \frac{1}{2\mu}\frac{n-2}{n}\beta_0^{\frac{2n}{n-2}} \int_{B(0,\mu\rho)} \frac{|x|^2}{\left(1+|x|^2\right)^{n+1}} dx = \frac{c_7}{\mu} + O\left(\frac{1}{\mu^{n+1}}\right). \tag{74}$$

It remains the integrals over Ω_i, $i=1,2$. Using (70), it holds

$$\left|\int_{\Omega_i} \dots\right| \leqslant c\int_{\Omega_i} |x_k|\,\frac{1}{\mu|x|}\omega_{a,\mu}^{\frac{2n}{n-2}} \leqslant c\int_{\Omega_i} \frac{\mu^{n-1}}{\left(1+\mu^2|x'|^2\right)^n} dx' dx_n$$

$$\leqslant c \int_{B_{n-1}(0,\rho)} \frac{\mu^{n-1} |x'|^2}{\left(1 + \mu^2 |x'|^2\right)^n} \leqslant \frac{c}{\mu^2}. \tag{75}$$

Thus, Combining (72) - (75), the proof of Claim (iv) follows.
Proof of (v): It can be done in the same way than the proof of Claims (iii) and (iv). Hence, we omit it.

Lemma 9. *Let $a_1, a_2 \in \partial\Omega$ with $|a_1 - a_2| \geqslant c > 0$ and μ_1, μ_2 be large reals. We have:*

(i) $$\int_\Omega |\nabla \omega_{a_1,\mu_1}| |\nabla \omega_{a_2,\mu_2}| \leqslant \frac{c}{(\mu_1\mu_2)^{(n-2)/2}} \leqslant \frac{c}{\mu_1^{n-2}} + \frac{c}{\mu_2^{n-2}},$$

(ii) $$\int_\Omega |\nabla \omega_{a_1,\mu_1}| \left|\nabla\left(\mu_2 \frac{\partial \omega_{a_2,\mu_2}}{\partial \mu_2}\right)\right| \leqslant \frac{c}{(\mu_1\mu_2)^{(n-2)/2}} \leqslant \frac{c}{\mu_1^{n-2}} + \frac{c}{\mu_2^{n-2}},$$

(iii) $$\int_\Omega |\nabla \omega_{a_1,\mu_1}| \left|\nabla\left(\frac{1}{\mu_2} \frac{\partial \omega_{a_2,\mu_2}}{\partial a_2}\right)\right| \leqslant \frac{c}{\mu_1^{(n-2)/2}} \frac{\ln \mu_2}{\mu_2^{n/2}},$$

(iv) $$\int_\Omega \omega_{a_1,\mu_1} \omega_{a_2,\mu_2} \leqslant \frac{c}{(\mu_1\mu_2)^{(n-2)/2}} \leqslant \frac{c}{\mu_1^{n-2}} + \frac{c}{\mu_2^{n-2}},$$

(v) $$\int_\Omega \omega_{a_1,\mu_1} \left|\mu_2 \frac{\partial \omega_{a_2,\mu_2}}{\partial \mu_2}\right| \leqslant \frac{c}{(\mu_1\mu_2)^{(n-2)/2}} \leqslant \frac{c}{\mu_1^{n-2}} + \frac{c}{\mu_2^{n-2}},$$

(vi) $$\int_\Omega \omega_{a_1,\mu_1} \left|\frac{1}{\mu_2} \frac{\partial \omega_{a_2,\mu_2}}{\partial a_2}\right| \leqslant \frac{c}{\mu_1^{(n-2)/2} \mu_2^{n/2}} \leqslant \frac{c}{\mu_1^{n-1}} + \frac{c}{\mu_2^{n-1}},$$

(vii) $$\int_\Omega \omega_{a_1,\mu_1}^{\frac{n+2}{n-2}} \omega_{a_2,\mu_2} \leqslant \frac{c}{(\mu_1\mu_2)^{(n-2)/2}} \leqslant \frac{c}{\mu_1^{n-2}} + \frac{c}{\mu_2^{n-2}},$$

(viii) $$\int_\Omega \omega_{a_1,\mu_1}^{\frac{n+2}{n-2}} \mu_2 \left|\frac{\partial \omega_{a_2,\mu_2}}{\partial \mu_2}\right| \leqslant \frac{c}{(\mu_1\mu_2)^{(n-2)/2}} \leqslant \frac{c}{\mu_1^{n-2}} + \frac{c}{\mu_2^{n-2}},$$

(ix) $$\int_\Omega \omega_{a_1,\mu_1}^{\frac{n+2}{n-2}} \frac{1}{\mu_2} \left|\frac{\partial \omega_{a_2,\mu_2}}{\partial a_2}\right| \leqslant \frac{c}{\mu_1^{(n-2)/2} \mu_2^{n/2}} \leqslant \frac{c}{\mu_1^{n-1}} + \frac{c}{\mu_2^{n-1}}.$$

Proof. We will focus on the proof of the first one and the other proofs can be done in the same way. Note that

$$\left|\nabla \omega_{a_i,\mu_i}\right| \leqslant c \frac{\mu^{\frac{n+2}{2}} |x - a_i|}{\left(1 + \mu_i^2 |x - a_i|^2\right)^{n/2}} \leqslant \frac{c}{\mu_i^{(n-2)/2} |x - a_i|^{n-1}}.$$

Thus, let $\rho := |a_1 - a_2|/2$, it holds :

$$\int_\Omega |\nabla \omega_{a_1,\mu_1}| |\nabla \omega_{a_2,\mu_2}| \leqslant \frac{1}{\mu_1^{(n-2)/2} \mu_2^{(n-2)/2}} \left(\sum_{i=1,2} \int_{B(a_i,\rho)} \frac{dx}{|x - a_i|^{n-1}} + \int_{\Omega \setminus \cup B(a_i,\rho)} 1 dx\right).$$
$$\leqslant \frac{c}{(\mu_1\mu_2)^{(n-2)/2}} \leqslant c\left(\frac{1}{\mu_1^{n-2}} + \frac{1}{\mu_2^{n-2}}\right).$$

Hence, the proof of Claim (i) is completed.

7.2. Coercivity of the quadratic form

The goal of this subsection is to prove Proposition 1. To this aim, for $\mu > 0$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote by

$$\begin{aligned} \psi_1(x) &:= \omega_{0,\mu}(x) := \beta_0 \frac{\mu^{(n-2)/2}}{(1+\mu^2|x|^2)^{(n-2)/2}}, \\ \psi_2(x) &:= \mu \frac{\partial \omega_{0,\mu}}{\partial \mu}(x) = \frac{n-2}{2}\beta_0 \frac{\mu^{(n-2)/2}\left(1-\mu^2|x|^2\right)}{(1+\mu^2|x|^2)^{n/2}}, \\ \psi_j(x) &:= (n-2)\beta_0 \frac{\mu^{n/2} x_{j-2}}{(1+\mu^2|x|^2)^{n/2}}, \quad \text{for } j \in \{3, \cdots, n+2\}. \end{aligned} \tag{76}$$

We begin by the following lemma:

Lemma 10. *Let $\rho > 0$ be a small radius and*

$$B_\rho^+ := \left\{ x := (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < \rho \quad \text{and} \quad x_n > 0 \right\}.$$

For μ large and $\bar{\gamma} > 0$, let us define

$$Q_+(v) := \int_{B_\rho^+} |\nabla v|^2 + \bar{\gamma} \int_{B_\rho^+} v^2 - \frac{n+2}{n-2} \int_{B_\rho^+} \omega_{0,\mu}^{\frac{4}{n-2}} v^2$$

Then there exists a constant $\beta_1 > 0$ such that

$$Q_+(v) \geqslant \beta_1 \left(\int_{B_\rho^+} |\nabla v|^2 + \bar{\gamma} \int_{B_\rho^+} v^2 \right) \quad \forall v \in E_\mu^+, \quad \text{where}$$
$$E_\mu^+ := \left\{ v \in H^1\left(B_\rho^+\right) : \int_{B_\rho^+} \nabla v \cdot \nabla \psi_j = 0 \quad \forall j \in \{1, \dots, n+1\} \right\}.$$

Proof. Let us introduce the function $\tilde{v}$ defined on $B(0,\rho)$ by for $y := (y', y_n) \in B(0,\rho)$,

$$\tilde{v}(y) := \begin{cases} v(y) & \text{if } \ y_n > 0, \\ v(y', -y_n) & \text{if } \ y_n < 0. \end{cases}$$

Easy Computations imply that

$$\tilde{v} \in H^1\left(B(0,\rho)\right),$$
$$2Q_+(v) = \tilde{Q}_+(\tilde{v}) := \int_{B(0,\rho)} |\nabla \tilde{v}|^2 + \bar{\gamma} \int_{B(0,\rho)} (\tilde{v})^2 - \frac{n+2}{n-2} \int_{B(0,\rho)} \omega_{0,\mu}^{\frac{4}{n-2}} \tilde{v}^2. \tag{77}$$

Notice that the function $\widetilde{Q}_+$ is a positive definite quadratic form on the space

$$E_{0,\mu} := \left\{ v \in H^1\left(B(0,\rho)\right) : \int_{B(0,\rho)} \nabla v \nabla \psi_j = 0 \quad \forall j = 1, \dots, n+2 \right\},$$

(See Proposition 1 of [32] and equation (19) by taking $\Omega = B(0,1)$, $K = \bar{\gamma}$ and $N = 1$). This implies that there exists a constant $\beta_0 > 0$ such that

$$\tilde{Q}_+(w) \geqslant \beta_0 \|w\|_{H^1(B(0,\rho))} \quad \forall w \in E_{0,\mu}. \tag{78}$$

In the following, we will prove that $\tilde{v} \in E_{0,\mu}$. For this aim, for $1 \leqslant j \leqslant n+1$, we compute

$$\int_{B(0,\rho)} \nabla \tilde{v} \nabla \psi_j = 2 \int_{B_\rho^+} \nabla v \nabla \psi_j = 0,$$

since $v \in E_\mu^+$. Now, for $j = n+2$, observe that (by easy computations)

$$-\Delta \psi_{n+2} = \frac{n+2}{n-2} \omega_{0,\mu} \psi_{n+2} \quad \text{in } B(0,\rho) \ ; \quad \frac{\partial \psi_{n+2}}{\partial \nu} = c(\rho,\mu) x_n \quad \text{on } \partial B(0,\rho).$$

Thus, by oddness (with respect the variable x_n), we obtain

$$\int_{B(0,\rho)} \nabla \tilde{v} \nabla \psi_{n+2} = \int_{B(0,\rho)} -\Delta \psi_{n+2} \tilde{v} + \int_{\partial B(0,\rho)} \frac{\partial \psi_{n+2}}{\partial \nu} \tilde{v} = 0,$$

Hence, $v \in E_{0,\mu}$ and the assumptions of Proposition 1 of [32] are satisfied.
Combining (77) and (78) (by taking $w = \widetilde{v}$), we get

$$Q_+(v) \geqslant (\beta_0/2) \, \|\tilde{v}\|^2_{H^1(B(0,\rho))} \geqslant \beta_0 \|v\|^2_{H^1\left(B_\rho^+\right)}.$$

We remark that

$$\begin{aligned} &\int_{B_\rho^+} |\nabla v|^2 + \bar{\gamma} \int_{B_\rho^+} v^2 \leqslant \|v\|^2_{H^1\left(B_\rho^+\right)} \leqslant \frac{1}{\bar{\gamma}} \left(\int_{B_\rho^+} |\nabla v|^2 + \bar{\gamma} \int_{B_\rho^+} v^2 \right) \quad \text{if} \quad \bar{\gamma} \leqslant 1, \\ &\frac{1}{\bar{\gamma}} \left(\int_{B_\rho^+} |\nabla v|^2 + \tilde{\gamma} \int_{B_\rho^+} v^2 \right) \leqslant \|v\|^2_{H^1\left(B_\rho^+\right)} \leqslant \int_{B_\rho^+} |\nabla v|^2 + \bar{\gamma} \int_{B_\rho^+} v^2 \quad \text{if} \quad \bar{\gamma} > 1. \end{aligned} \tag{79}$$

The proof of the lemma is thereby completed.

Notice that, for $a \in \partial\Omega$, a neighborhood of a in Ω is not necessary a half ball. For this reason, we need to take a general case.

Lemma 11. *Let $a \in \partial\Omega$, μ be a large real and ρ be a small radius. Let*

$$Q_{a,\rho}(v) := \int_{B(a,\rho)\cap\Omega} |\nabla v|^2 + \bar{\gamma} \int_{B(a,\rho)\cap\Omega} v^2 - \frac{n+2}{n-2} \int_{B(a,\rho)\cap\Omega} \omega_{a,\mu}^{\frac{4}{n-2}} v^2.$$

Then, there exists a constant $\beta_2 > 0$ such that

$$Q_{a,\rho}(v) \geqslant \beta_2 \left(\int_{B(a,\rho)\cap\Omega} |\nabla v|^2 + \bar{\gamma} \int_{B(a,\rho)\cap\Omega} v^2 \right) + o\left(\|v\|^2_{H^1(\Omega)} \right) \quad \forall v \in F_{a,\mu},$$

where $F_{a,\mu}$ is defined in (4).

Proof. Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$. Without loss of generality, we can assume that $a = 0$ and $\nu_a = -e_n$ (which implies that the tangent space to $\partial\Omega$ at $a = 0$ is $\mathbb{R}^{n-1} \times \{0\}$ and a basis of this tangent space is $(e_1, \ldots, e_{n-1})$).

Since ρ is small and Ω is a regular domain, there exists a smooth function $f : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, satisfying

$$f(0) = 0, \ \nabla f(0) = 0 \text{ and } \Omega \cap B(0,\rho) = \left\{ x := (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ |x| < \rho, \ x_n > f(x') \right\}.$$

Now, we define

$$\varphi : \Omega \cap B(0,\rho) \longrightarrow \mathbb{R}^{n-1} \times \mathbb{R}, \quad \varphi\left(x', x_n\right) = \left(x', x_n - f\left(x'\right)\right). \tag{80}$$

From (80), we remark that there exists a neighborhood $\mathcal{V}$ of 0 in $B\left(0,\rho\right)$ such that φ induces a diffeomorphism between $\mathcal{V} \cap \Omega$ and $B^+ := \{x = (x', x_n) \in B\left(0, \rho/2\right) : \ x_n > 0\}$, that is

$$\varphi(\mathcal{V} \cap \Omega) = B^+. \tag{81}$$

In addition, we have $B\left(0, \rho/4\right) \subset \mathcal{V}$. Furthermore, from the definition of φ in (80), we deduce that

$$\frac{\partial \varphi}{\partial x_i}(x) = e_i - \frac{\partial f}{\partial x_i}\left(x'\right) e_n \text{ for } 1 \leqslant i \leqslant n-1 \quad \text{ and } \quad \frac{\partial \varphi}{\partial x_n}(x) = e_n, \tag{82}$$

which implies that the Jacobian of φ at each point x is 1 ($|\mathrm{Jac}\,\varphi| = 1$). Now, let us define the function v_1 by

$$v_1 : B^+ \longrightarrow \mathbb{R}, \quad v_1 := v \circ \varphi^{-1}. \tag{83}$$

Using (82), easy computations imply that

$$|\nabla v(x)|^2 = \left|\left(\nabla v_1\right)\left(\varphi(x)\right)\right|^2 + O\left(\rho\left|\left(\nabla v_1\right)\left(\varphi(x)\right)\right|^2\right),$$

which implies that, by using (81),

$$\int_{\mathcal{V}\cap\Omega} |\nabla v(x)|^2\, dx = \int_{B^+} |\nabla v_1(z)|^2\, dz + O\left(\rho \int_{B^+} |\nabla v_1(z)|^2\, dz\right),$$

and therefore

$$\int_{\mathcal{V}\cap\Omega} |\nabla v|^2 + \bar{\gamma} \int_{\mathcal{V}\cap\Omega} |v|^2 = \int_{B^+} |\nabla v_1|^2 + \bar{\gamma} \int_{B^+} (v_1)^2 + O\left(\rho \int_{B^+} |\nabla v_1|^2\right). \tag{84}$$

Concerning the last integral in the definition of $Q_{a,\rho}$, we have

$$\int_{\mathcal{V}\cap\Omega} \omega_{0,\mu}^{\frac{4}{n-2}}(x) v^2(x) dx = \int_{\mathcal{V}\cap\Omega} \omega_{a,\mu}^{\frac{4}{n-2}}(x) v_1^2(\varphi(x)) dx = \int_{B^+} \omega_{0,\mu}^{\frac{4}{n-2}}\left(\varphi^{-1}(z)\right) v_1^2(z) dz. \tag{85}$$

Observe that equation (C. 25) of [27] gives us

$$\begin{aligned}\left(1+\mu^2|x|^2\right)^\gamma = \left(1+\mu^2\left|\varphi^{-1}(z)\right|^2\right)^\gamma &= \left(1+\mu^2|z|^2\right)^\gamma + O\left(\left(1+\mu^2|z|^2\right)^{\gamma-1}\mu^2|z|^2\rho\right)\\ &= \left(1+\mu^2|z|^2\right)^\gamma + O\left(\left(1+\mu^2|z|^2\right)^\gamma\rho\right).\end{aligned} \tag{86}$$

Hence we obtain

$$\begin{aligned}\int_{B^+} \omega_{0,\mu}^{\frac{4}{n-2}}\left(\varphi^{-1}(z)\right) v_1^2(z) dz &= \int_{B^+} \omega_{0,\mu}^{\frac{4}{n-2}} v_1^2(z) dz + O\left(\rho \int_{B^+} \omega_{0,\mu}^{\frac{4}{n-2}} v_1^2(z) dz\right)\\ &= \int_{B^+} \omega_{0,\mu}^{\frac{4}{n-2}} v_1^2(z) dz + O\left(\rho \left\|v_1\right\|^2_{L^{2n/(n-2)}(B^+)}\right).\end{aligned} \tag{87}$$

Combining (84), (85) and (87), we get

$$Q_{a,\rho}(v) = \int_{(B(0,\rho)\cap\Omega)\setminus\mathcal{V}} \left(|\nabla v|^2 + \bar{\gamma} v^2\right) - \frac{n+2}{n-2} \int_{(B(0,\rho)\cap\Omega)\setminus\mathcal{V}} \omega_{0,\mu}^{\frac{4}{n-2}} v^2$$

$$+\int_{B^+}|\nabla v_1|^2+\bar{\gamma}\int_{B^+}(v_1)^2-\frac{n+2}{n-2}\int_{B^+}\omega_{0,\mu}^{\frac{4}{n-2}}v_1^2+O\left(\rho\,\|v_1\|_{H^1(B^+)}\right), \tag{88}$$

Observe that, since $B(0,\rho/4)\subset\mathcal{V}$, we deduce that

$$\int_{(B(0,\rho)\cap\Omega\setminus\mathcal{V})}\omega_{0,\mu}^{\frac{4}{n-2}}v^2\leqslant c\|v\|^2_{L^2(B(0,\rho))}\left(\int_{\mathbb{R}^n\setminus B(0,\rho/4)}\omega_{0,\mu}^{\frac{2n}{n-2}}\right)^{2/n}\leqslant\frac{c}{(\mu\rho)^2}\|v\|^2_{L^2(B(0,R)\cap\Omega)}.$$

Thus (88) becomes

$$Q_{a,\rho}(v)=\int_{(B(0,\rho)\cap\Omega)\setminus\mathcal{V}}\left(|\nabla v|^2+\bar{\gamma}v^2\right)+Q_+(v_1)+O\left(\rho\|v\|_{H^1(B(0,\rho)\cap\Omega)}\right). \tag{89}$$

At this step, we need to apply Lemma 10 to the function v_1, defined by (83), but $v_1\notin E_\mu^+$. For this reason, we decompose v_1 as follows:

$$v_1=\sum_{j=1}^{n+1}\sigma_j\psi_j+v_1^{\perp}\quad\text{with}\quad v_1^{\perp}\in E_\mu^+,$$

where the ψ_j's are defined in (76).

Now, we need to estimate the parameters σ_j's. Observe that, on one hand we have:

$$\int_{B^+}\nabla v_1\nabla\psi_1=\sigma_1\int_{B^+}|\nabla\psi_1|^2+\sum_{j\neq 1}\int_{B^+}\nabla\psi_1\nabla\psi_j=c\sigma_1+o\left(\sum|\sigma_j|\right). \tag{90}$$

On the other hand, we have:

$$\begin{aligned}\int_{B^+}\nabla v_1\nabla\psi_1=\int_{B^+}\nabla v_1\nabla\omega_{0,\mu}&=\int_{B^+}(-\Delta\omega_{0,\mu})\,v_1+\int_{\partial B^+}\left(\frac{\partial}{\partial\nu}\omega_{0,\mu}\right)v_1\\&=\int_{B^+}\omega_{0,\mu}^{\frac{n+2}{n-2}}v_1+\int_{\partial B^+}\left(\frac{\partial}{\partial\nu}\omega_{0,\mu}\right)v_1.\end{aligned} \tag{91}$$

Let $\Gamma_1:=\{x=(x',x_n):|x|=\rho \text{ and } x_n\geqslant 0\}$ and $\Gamma_2:=\{x=(x,0):|x|\leqslant\rho\}$. It is easy to see that

$$\partial B^+=\Gamma_1 U\Gamma_2,\quad\frac{\partial}{\partial\nu}\omega_{0,\mu}=0\quad\text{on}\quad\Gamma_2\quad\text{and}\quad\frac{\partial}{\partial\nu}\omega_{0,\mu}=O\left(\frac{1}{\mu^{(n-2)/2}}\right)\quad\text{on}\quad\Gamma_1.$$

Thus

$$\left|\int_{\partial B^+}\left(\frac{\partial}{\partial\nu}\omega_{0,\mu}\right)v_1\right|\leqslant\frac{c}{\mu^{(n-2)/2}}\int_{\Gamma_1}|v_1|\leqslant\frac{c}{\mu^{(n-2)/2}}\|v_1\|_{H^1(B^+)}. \tag{92}$$

For the other integral, we get

$$\int_{B^+}\omega_{0,\mu}^{\frac{n+2}{n-2}}v_1=\int_{B^+}\omega_{0,\mu}^{\frac{n+2}{n-2}}(z)v\left(\varphi^{-1}(z)\right)dz=\int_{V\cap\Omega}\omega_{0,\mu}^{\frac{n+2}{n-2}}(\varphi(x))v(x)dx. \tag{93}$$

Using (86), (93) becomes

$$\begin{aligned}\int_{B^+}\omega_{0,\mu}^{\frac{n+2}{n-2}}v_1&=\int_{\mathcal{V}\cap\Omega}\omega_{0,\mu}^{\frac{n+2}{n-2}}(x)v(x)dx+O\left(\rho\int_{\mathcal{V}\cap\Omega}\omega_{0,\mu}^{\frac{n+2}{n-2}}(x)\,|v(x)|\,dx\right)\\&=\int_\Omega\omega_{0,\mu}^{\frac{n+2}{n-2}}v-\int_{\Omega\setminus\mathcal{V}}\omega_{0,\mu}^{\frac{n+2}{n-2}}v+O\left(\rho\|v\|_{L^{2n/n-2}(V\cap\Omega)}\right)\\&=\int_\Omega\nabla\omega_{0,\mu}\nabla v-\int_{\partial\Omega}\left(\frac{\partial}{\partial\nu}\omega_{0,\mu}\right)v+O\left(\|v\|_{L^{2n/n-2}(\Omega)}\left[\rho+\frac{1}{(\mu\rho)^{(n+2)/2}}\right]\right).\end{aligned}$$

Using the fact that $v \in \mathcal{F}_{a,\mu}$ and equation (19), we obtain

$$\int_{B^+} \omega_{0,\mu}^{\frac{n+2}{n-2}} v_1 = O\left(\|v\|_{L^{2n/n-2}(\Omega)} \left[\frac{1}{\mu} + \rho + \frac{1}{(\mu\rho)^{(n+2)/2}}\right]\right) = o\left(\|v\|_{L^{2n/(n-2)}(\Omega)}\right). \tag{94}$$

Combining (90), (91), (92) and (94), we get

$$\sigma_1 = o\left(\sum |\sigma_j|\right) + o\left(\|v\|_{H^1(\Omega)}\right).$$

In the same way, we get the estimate of σ_i for $i \geqslant 2$ and therefore we obtain

$$\sigma_i = o\left(\sum |\sigma_j|\right) + o\left(\|v\|_{H^1(\Omega)}\right) \quad \forall i = 1, \dots, n+1,$$

which implies that

$$\sigma_i = o\left(\|v\|_{H^1(\Omega)}\right) \quad \forall i = 1, \dots, n+1.$$

Hence we deduce that

$$v_1 - v_1^{\perp} = o\left(\|v\|\omega_{0,\mu}\right) \quad \text{and} \quad \nabla\left(v_1 - v_1^{\perp}\right) = O\left(\|v\| \sum |\nabla\psi_j|\right). \tag{95}$$

This implies that (by using $v_2^{\perp} \in E_{\mu}^{+}$)

$$\begin{aligned} Q_+(v_1) &= \int_{B^+} |\nabla v_1|^2 + \bar{\gamma}\int_{B^+} v_1^2 - \frac{n+2}{n-2}\int_{B^+} \omega_{0,\mu}^{\frac{4}{n-2}} v_1^2 \\ &= \int_{B^+} \left|\nabla v_1^{\perp}\right|^2 + \int_{B^+} \left|\nabla\left(v_1 - v_1^{\perp}\right)\right|^2 + \bar{\gamma}\int_{B^+} \left(v_1^{\perp}\right)^2 + 2\bar{\gamma}\int_{B^+} \left(v_1^{\perp}\right)\left(v_1 - v_1^{\perp}\right) \\ &+ \bar{\gamma}\int_{B^+} \left(v_1 - v_1^{\perp}\right)^2 - \frac{n+2}{n-2}\int_{B^+} \omega_{0,\mu}^{\frac{4}{n-2}} \left[\left(v_1^{\perp}\right)^2 + 2v_1^{\perp}\left(v_1 - v_1^{\perp}\right) + \left(v_1 - v_1^{\perp}\right)^2\right] \\ &= Q_+\left(v_1^{\perp}\right) + o\left(\|v_1^{\perp}\|^2 + \|v\|^2\right) \\ &\geqslant \frac{1}{2}\beta_1\left(\int_{B^+} \left|\nabla v_1^{\perp}\right|^2 + \bar{\gamma}\int_{B^+} \left(v_1^{\perp}\right)^2\right) + o\left(\|v\|^2\right), \end{aligned} \tag{96}$$

by using Lemma 10, Eq. (79) and the fact that $v_1^{\perp} \in E_{\mu}^{+}$.

Combining (96), (95), (89) and (84), the proof of Lemma 11 follows.

Now, we are ready to prove Proposition 1.

Proof of Proposition 1 Let ρ be a small radius and let $B_i := B(a_i, \rho) \cap \Omega$. Since $|a_i - a_j| \geqslant c > 0$ for $i \neq j$, it follows that $B_i \cap B_j = \emptyset$ for each $i \neq j$. Thus we get

$$\begin{aligned} Q(v) = &\sum_{i=1}^{q}\left(\int_{B_i} |\nabla v|^2 + \int_{B_i} V v^2 - \frac{n+2}{n-2}\int_{B_i} \omega_{a_i,\mu_i}^{\frac{4}{n-2}} v^2\right) \\ &+ \int_{\Omega\setminus(UB_i)} |\nabla v|^2 + \int_{\Omega\setminus(UB_i)} V v^2 - \frac{n+2}{n-2}\sum_{i=1}^{q}\int_{\Omega\setminus B_i} \omega_{a_i,\mu_i}^{\frac{4}{n-2}} v^2. \end{aligned}$$

Observe that, for each $i \in \{1, \dots, q\}$, we have

$$\int_{\Omega\setminus B_i} \omega_{a_i,\mu_i}^{\frac{4}{n-2}} v^2 \leqslant \left(\int_{\Omega\setminus B_i} v^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \left(\int_{\Omega\setminus B_i} \omega_{a_i,\mu_i}^{\frac{2n}{n-2}}\right)^{2/n} \leqslant \frac{c}{(\mu_i\rho)^2}\|v\|_{H^1(\Omega)}^2. \tag{97}$$

In addition, let $\bar{\gamma} = \min V > 0$, using (97) and Lemma 11, we derive that

$$\begin{aligned} Q(v) &\geqslant \sum_{i=1}^{q} Q_{a_i,\rho}(v) + \int_{\Omega\setminus(\cup B_i)} |\nabla v|^2 + \int_{\Omega\setminus(\cup B_i)} V v^2 + \sum O\left(\frac{\|v\|^2}{(\mu_i\rho)^2}\right) \\ &\geqslant \sum_{i=1}^{q} \beta_2 \left(\int_{B_i} |\nabla v|^2 + \bar{\gamma}\int_{B_i} v^2\right) + \int_{\Omega\setminus\cup B_i} |\nabla v|^2 + \int_{\Omega\setminus\cup B_i} V v^2 + \sum O\left(\frac{\|v\|^2}{(\mu_i\rho)^2}\right) \\ &\geqslant \beta_3 \|v\|^2, \end{aligned}$$

for some positive constant β_3 (since ρ is fixed and the μ_i's are large). This completes the proof. ■

Acknowledgements

The authors gratefully acknowledge Qassim University, represented by the Deanship of Graduate Studies and Scientific Research, on the financial support for this research under the number (QU-J-PG-2-2025-53906) during the academic year 1446 AH / 2024 AD.

Author Contributions: R.A and M.B.A.: conceptualization, methodology, investigation, writing original draft, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research, Qassim University, grant number project QU-J-PG-2-2025-53906.

Data Availability Statement: No data to report in this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399–415, 1970.

[2] R. Schaaf. Stationary solutions of chemotaxis systems. *Trans. Amer. Math. Soc.*, 292:531–556, 1985.

[3] T. Hillen and K.J. Painter. A user's guide to pde models for chemotaxis. *J. Math. Biol.*, 58:183–217, 2009.

[4] M. Bezerra, C. Cuevas, C. Silva, and H. Soto. On the fractional doubly parabolic keller-segel system modelling chemotaxis. *Science China Mathematics*, 65:1827–1874, 2022.

[5] L. Almeida, F. Bubba, B. Perthame, and C. Pouchol. Energy and implicit discretization of the fokker-planck and keller-segel type equations. *Networks and Heterogeneous Media*, 14:23–41, 2019.

[6] C.S. Lin, W.M. Ni, and I. Takagi. Large amplitude stationary solutions to a chemotaxis system. *J. Differential Equations*, 72:1–27, 1988.

[7] W.M. Ni and I. Takagi. On the shape of least-energy solutions to a semi-linear neumann problem. *Comm. Pure Appl. Math.*, 44:819–851, 1991.

[8] W.M. Ni and I. Takagi. Locating the peaks of least-energy solutions to a semi-linear neumann problem. *Duke Math. J.*, 70:247–281, 1993.

[9] M. Del Pino and P. Felmer. Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting. *Indiana Univ. Math. J.*, 48:883–898, 1999.

[10] E. N. Dancer and S. Yan. Multipeak solutions for a singularly perturbed neumann problem. *Pacific J. Math.*, 189(2):241–262, 1999.

[11] M. Del Pino, P. Felmer, and J. Wei. On the role of mean curvature in some singularly perturbed neumann problems. *SIAM J. Math. Anal.*, 31:63–79, 2000.

[12] M. Grossi, A. Pistoia, and J. Wei. Existence of multipeak solutions for a semilinear neumann problem via nonsmooth critical point theory. *Calc. Var. PDE*, 11(2):143–175, 2000.

[13] C. Gui and J. Wei. Multiple interior peak solutions for some singularly perturbed neumann problems. *J. Differential Equations*, 158:1–27, 1999.

[14] J. Wei, B. Xu, and W. Yang. On lin-ni's conjecture in dimensions four and six. *Science in China: Mathematics*, 49(2):281–306, 2019.

[15] Adimurthi and S.L. Yadava. Existence and nonexistence of positive radial solutions of neumann problems with critical sobolev exponents. *Arch. Rat. Mech. Anal.*, 115:275–296, 1991.

[16] O. Rey and J. Wei. Arbitrary number of positive solutions for elliptic problem with critical nonlinearity. *J. Eur. Math. Soc.*, 7:449–476, 2005.

[17] Adimurthi, F. Pacella, and S.L. Yadava. Interaction between the geometry of the boundary and positive solutions of a semilinear neumann problem with critical nonlinearity. *J. Funct. Anal.*, 113:318–350, 1993.

[18] W.M. Ni, X.B. Pan, and I. Takagi. Singular behavior of least-energy solutions of a semi-linear neumann problem involving critical sobolev exponents. *Duke Math. J.*, 67:1–20, 1992.

[19] L. Caffarelli, B. Gidas, and J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth. *Comm. Pure Appl. Math.*, 42:271–297, 1989.

[20] Adimurthi and G. Mancini. The neumann problem for elliptic equations with critical nonlinearity. In *A tribute in honour of G. Prodi*, pages 9–25. Scuola Norm. Sup. Pisa, 1991.

[21] O. Druet, F. Robert, and J. Wei. The lin-nis problem for mean convex domains. *Mem. Amer. Math. Soc.*, 218(1027), 2012.

[22] N. Ghoussoub and C. Gui. Multi-peak solutions for a semilinear neumann problem involving the critical sobolev exponent. *Math. Z.*, 229:443–474, 1998.

[23] C. Gui and C.-S. Lin. Estimates for boundary-bubbling solutions to an elliptic neumann problem. *J. Reine Angew. Math.*, 546:201–235, 2002.

[24] L. Wang, J. Wei, and S. Yan. A neumann problem with critical exponent in nonconvex domains and lin-ni's conjecture. *Trans. Amer. Math. Soc.*, 362(9):4581–4615, 2010.

[25] Z.-Q. Wang. Construction of multi-peaked solutions for a nonlinear neumann problem with critical exponent in symmetric domains. *Nonlinear Anal.*, 27(11):1281–1306, 1996.

[26] J. Wei and S. Yan. Arbitrary many boundary peak solutions for an elliptic neumann problem with critical growth. *J. Math. Pures Appl. (9)*, 88(4):350–378, 2007.

[27] O. Rey. Boundary effect for an elliptic neumann problem with critical nonlinearity. *Comm. Partial Differential Equations*, 22:1055–1139, 1997.

[28] O. Rey. The question of interior blow-up points for an elliptic neumann problem: The critical case. *J. Math. Pures Appl.*, 81:655–696, 2002.

[29] O. Rey and J. Wei. Blow-up solutions for an elliptic neumann problem with sub- or supercritical nonlinearity, ii: $n \geq 4$. *Ann. Inst. H. Poincaré, Anal. non-lin*, 22(4):459–484, 2005.

[30] O. Rey and J. Wei. Blow-up solutions for an elliptic neumann problem with sub- or supercritical nonlinearity, i: $n = 3$. *J. Funct. Anal.*, 212:472–499, 2004.

[31] M. Ben Ayed and K. El Mehdi. Non-existence of interior bubbling solutions for slightly supercritical elliptic problems. *Boundary Value Problems*, 2023(90), 2023.

[32] K. El Mehdi and F. Mohamed Salem. Interior bubbling solutions for an elliptic equation with

slightly subcritical nonlinearity. *Mathematics*, 11(6):1471, 2023.

[33] M. Ben Ayed, K. El Mehdi, and F. Mohamed Salem. Interior multi-peak solution for a slightly subcritical nonlinear neumann equation. *Symmetry*, 16(291), 2024.

[34] M. Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187:511–517, 1984.

[35] A. Bahri, Y.Y. Li, and O. Rey. On a variational problem with lack of compactness: the topological effect of the critical points at infinity. *Calculus of Variations and Partial Differential Equations*, 3:67–94, 1995.