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A Specific Category Of Meromorphic Functions With Positive Coefficients

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Abstract. This work presents and investigates a new class of meromorphically uniformly convex functions with positive coefficients that a differential operator defines. It also derives properties such as coefficient bounds, distortion properties, δ-neighborhoods, convex linear combination, and convolution properties.

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1. Introduction

Let's say that Ψ represents the class of functions f of this form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1}$$

they have a single pole at the origin with residue 1 and are regular in the domain $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Let the univalent, meromorphically starlike (of order Π), and

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meromorphically convex (of order Π) subclasses of Ψ be denoted by Ψ_s and $\Psi^*(\Pi)$ and $\Psi_k(\Pi), 0 \leq \Pi < 1$, $f(z)$ of the form (1) is analytically contained in $\Psi^*(\Pi)$ if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \Pi, \quad z \in \mathbb{U}^*.$$

Likewise, $f \in \Psi_k(\Pi)$ if and only if $f(z)$ has the form (1) and fulfills

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \Pi, \quad z \in \mathbb{U}^*.$$

It is recognized that, in the case of $\Pi = 1$, the only function that is both $\Psi^*(1)$ and $\Psi_k(1)$ is $f(z) = \frac{1}{z}$. Looking for a subclass of Ψ_s with properties resembling those of $\Psi^*(\Pi)$ makes sense because the work in the meromorphic univalent scenario has partly mirrored that of the regular univalent case. Juneja and Reddy [1] introduced the class Ψ_p of functions of the sort.

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0, \qquad (2)$$
$$\Psi_p^*(\Pi) = \Psi_p \cap \Psi^*(\Pi).$$

A linear operator A_ξ^m is defined for functions $f(z)$ in the class Ψ_p in the following way.

$$A_\xi^0 f(z) = f(z),$$
$$A_\xi^1 f(z) = \left(1+\frac{\mu+\xi}{\alpha+\mu}\right) f(z) + \frac{\mu+\xi}{\alpha+\mu} z f'(z),$$
$$\vdots$$
$$A_\xi^m f(z) = A(A_\xi^{m-1} f(z)) = \frac{1}{z} + \sum_{n=1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m a_n z^n \qquad (3)$$

for $m \in \mathbb{N}_0 = 0, 1, 2, \ldots$.

Definition 1. *Let $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ be the subclass of Ψ_p that consists of the form (2) and satisfies the analytical criteria for $-1 \leq \Pi < 1$, $\xi > 0$ and $\Lambda \geq 1$.*

$$\operatorname{Re}\left\{\frac{A_\xi^{m+1} f(z)}{A_\xi^m f(z)} - \Pi\right\} > \Lambda\left|\frac{A_\xi^{m+1} f(z)}{A_\xi^m f(z)} - 1\right|, \qquad (4)$$

$A_\xi^m f(z)$ is given by (3).

Well-known classes of meromorphic uniformly linear function with positive coefficients are unified by the function class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$.

Differential or integral operators with normalized analytic univalent functions are now popular in the study of Geometric function theory. The classes $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ and various other subclasses of Ψ were studied rather extensively by Clunie [2] and also see ([3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]). Motivated by the works of Madhavi et al. [16], we define the following a new subclass $\varphi_p(\Pi, v, \xi)$. In this paper, we introduce and study the subclass $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ of meromorphic functions with positive coefficients generalization of a differential operator, including the Madhavi et al. operator [16] for functions in $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$.

2. Coefficient Inequalities

The coefficient bounds of function $f(z)$ for the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ are obtained in this section.

Theorem 1. *A funcion $f(z)$ of the form (2) is in $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ if*

$$\sum_{n=1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m\left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]|a_n| \leq (1-\Pi).$$

Proof. It suffices to demonstrate that

$$\Lambda\left|\frac{A_\xi^{m+1}f(z)}{A_\xi^m f(z)}-1\right|-\mathrm{Re}\left\{\frac{A_\xi^{m+1}f(z)}{A_\xi^m f(z)}-1\right\}\leq(1-\Pi).$$

We have

$$\Lambda\left|\frac{A_\xi^{m+1}f(z)}{A_\xi^m f(z)}-1\right|-\mathrm{Re}\left\{\frac{A_\xi^{m+1}f(z)}{A_\xi^m f(z)}-1\right\}\leq(\Lambda+1)\left|\frac{A_\xi^{m+1}f(z)}{A_\xi^m f(z)}-1\right|$$

$$\leq\frac{(\Lambda+1)\sum_{n=-1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)|a_n|\,|z_n|}{\frac{1}{|z|}-\sum_{n=-1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m|a_n|\,|z_n|}.$$

By letting $z\to 1$ move along the real axis, we can get

$$\frac{(\Lambda+1)\sum_{n=-1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)|a_n|}{1-\sum_{n=-1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m|a_n|}.$$

The boundary of the above formula is $(1-\Pi)$ if

$$\sum_{n=1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m\left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]|a_n| \leq (1-\Pi).$$

This completes the theorem.

Corollary 1. *Let the function $f(z)$ defined by (2) be in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. Then*

$$a_n \leq \frac{(1-\Pi)}{\sum_{n=1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m\left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}, n \geq 1. \tag{5}$$

Equality holds for the function of the form

$$f_n(z) = \frac{1}{z} + \frac{(1-\Pi)}{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m\left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]} z^n. \tag{6}$$

Remark 1. *For the choice of $\alpha, \xi = 1$ and $\mu = 0$, in Theorem 1 and Corollary 1, we observed that the coefficient estimates for the functions of the class,*

$$|a_n| \leq \frac{(1-\Pi)}{[n+2]^m[(1+\Lambda)(n+1)+1-\Pi]}$$

is coincide with [16].

3. Distortion Theorems

In this section, we obtain the sharp for the distortion theorems of the form (2).

Theorem 2. *Let the function $f(z)$ defined by (2) be in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. Then for $0 < |z| = r < 1$,*

$$\begin{aligned} &\frac{1}{r} - \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]} r \\ &\leq |f(z)| \leq \frac{1}{r} + \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]} r, \end{aligned} \tag{7}$$

with equality for the function,

$$f(z) = \frac{1}{z} + \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]} z, at\ z = r, ir. \tag{8}$$

Proof. Suppose $f(z)$ is in $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. In view of Theorem 1, we have

$$\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]\sum_{n=1}^{\infty} a_n$$

$$\leq \sum_{n=1}^{\infty}\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m\left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right] \leq (1-\Pi)$$

which evidently yields

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}.$$

Consequently, we obtain

$$f(z) = \left|\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n\right| \leq \left|\frac{1}{z}\right| + \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n$$

$$\leq \frac{1}{r} + \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]} r.$$

Also,

$$f(z) = \left|\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n\right| \geq \left|\frac{1}{z}\right| - \sum_{n=1}^{\infty} a_n |z|^n \geq \frac{1}{r} - r\sum_{n=1}^{\infty} a_n$$

$$\geq \frac{1}{r} - \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]} r.$$

Hence the result (7) follows.

Theorem 3. *Let the function $f(z)$ defined by (2) be in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}$$

$$\leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}.$$

The outcome is sharp, with the shape of the extremal function being (1).

Proof. From Theorem 1, we have

$$\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right] \sum_{n=1}^{\infty} n a_n$$

$$\leq \sum_{n=1}^{\infty} \left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right] \leq (1-\Pi)$$

which evidently yields

$$\sum_{n=1}^{\infty} n a_n \leq \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}.$$

Consequently, we obtain

$$\left|f^{'}(z)\right| \leq \left|\frac{1}{r^2}+\sum_{n=1}^{\infty} n a_n r^{n-1}\right| \leq \frac{1}{r^2}+\sum_{n=1}^{\infty} n a_n$$

$$\leq \frac{1}{r^2}+\frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}.$$

Also,

$$\left|f^{'}(z)\right| \geq \left|\frac{1}{z}-\sum_{n=1}^{\infty} n a_n r^{n-1}\right| \geq \frac{1}{r^2}-\sum_{n=1}^{\infty} n a_n$$

$$\geq \frac{1}{r^2}-\frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}.$$

This completes the proof.

Remark 2. *For the choice of* $\alpha, \xi = 1$ *and* $\mu = 0$, *in Theorems 2 and 3, we observed that the sharp for the distortion theorems for the functions of the class are coincide with [16].*

4. The class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha, \gamma)$ and its neighborhoods

In this section, we obtain neighborhoods from class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha, \gamma)$.

Definition 2. *A function* $f \in \Psi_p$ *is said to in the class* $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha, \gamma)$ *if there exists a function* $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ *such that*

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\gamma, z \in \mathbb{U}^*, (0 \leq \gamma < 1).$$

Following the earlier works on neighborhoods of analytic functions by [17] univalent and [18], we define the δ-neighborhood of a function $f \in \Psi_p$ by

$$N_\delta(f) := \left\{g \in \Psi_p : g(z) = \frac{1}{z}+\sum_{n=1}^{\infty} b_n z^n : \sum_{n=1}^{\infty} n\left|a_n-b_n\right| \leq \delta\right\} \tag{9}$$

Theorem 4. *If* $g \in \varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ *and*

$$\gamma = 1-\frac{\delta\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]-1+\Pi} \tag{10}$$

Then

$$N_\delta(g) \subset \varphi_p(\Pi, \Lambda, \xi, \mu, \alpha, \gamma).$$

Proof. Let $f \in N_\delta(g)$. Then we find from (9) that

$$\sum_{n=1}^{\infty} n\,|a_n - b_n| \leq \delta$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, (n \in N).$$

Since $g \in \varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$, we have

$$\sum_{n=1}^{\infty} b_n \leq \frac{(1-\Pi)}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}.$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1-\sum_{n=1}^{\infty} b_n} \leq \frac{\delta\left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m}{\left[1+\frac{2(\mu+\xi)}{\alpha+\mu}\right]^m \left[\left(\frac{2(\mu+\xi)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]-1+\Pi} = 1-\gamma$$

provided γ is given by (10). Hence, by Definition 2 , $f \in \varphi_p(\Pi, \Lambda, \xi, \mu, \alpha, \gamma)$ for γ given by (10), which completes the proof.

5. Convex linear combinations and convolution properties

In this section, we obtain sharp for $f(z)$ is meromorphically convex of order δ and necessary and sufficient condition for $f(z)$ is in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. And also proved that convolution is in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$.

Theorem 5. *If the function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *is in* $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ *then* $f(z)$ *is meromorphically convex of order* $\delta (0 \leq \delta < 1)$ *in* $|z| < r = r(\Pi, \Lambda, \delta, \mu, \alpha)$*, where*

$$r(\Pi, \Lambda, \delta, \mu, \alpha) = \inf_{m \geq 1} \left\{ \frac{(1-\delta)\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{(1-\Pi)n(n+2-\delta)} \right\}^{\frac{1}{n+1}}.$$

The result is sharp.

Proof. Let $f(z)$ *be in* $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$*. Then, by Theorem 1 , we have*

$$\sum_{n=1}^{\infty} \left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]|a_n| \leq (1-\Pi). \qquad (11)$$

It is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right| \leq (1-\delta),$$

for $|z|<r=r(\Pi,\Lambda,\delta,\xi,\mu,\alpha)$, where $r(\Pi,\Lambda,\delta,\xi,\mu,\alpha)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{\frac{-1}{z^2}+\sum_{n=1}^{\infty} n a_n z^{n-1}}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_m |z|^{n+1}}{1-\sum_{n=1}^{\infty} n a_n |z|^{n+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n |z|^{n+1} \leq 1. \tag{12}$$

By (11), it follows that (12) is true if

$$\begin{aligned} \frac{n(n+2-\delta)}{1-\delta}|z|^{n+1} &\leq \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi}|a_n|, n\geq 1 \\ \text{or } |z| &\leq \left\{\frac{(1-\delta)\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{(1-\Pi)n(n+2-\delta)}\right\}^{\frac{1}{n+1}}. \end{aligned} \tag{13}$$

Setting $|z|=r(\Pi,\Lambda,\delta,\xi,\mu,\alpha)$ in (13), the result follows. The result is sharp for the function.

$$f_n(z)=\frac{1}{z}+\frac{(1-\Pi)}{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}z^n, n\geq 1.$$

Theorem 6. *Let* $f_0(z)=\frac{1}{z}$ *and*

$$f_n(z)=\frac{1}{z}+\frac{(1-\Pi)}{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}z^n, n\geq 1.$$

Then $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n$ *is in the class* $\varphi_p(\Pi,\Lambda,\xi,\mu,\alpha)$ *if and only if it can be expressed in the form*

$$f(z)=\vartheta_0 f_0(z)+\sum_{n=1}^{\infty}\vartheta_n f_n(z),$$

where $\vartheta_0\geq 0, \vartheta_n\geq 0, n\geq 1$ *and* $\vartheta_0+\sum_{n=1}^{\infty}\vartheta_n=1.$

Proof. Let $f(z) = \vartheta_0 f_0(z) + \sum_{n=1}^{\infty} \vartheta_n f_n(z)$ with $\vartheta_0 \geq 0, \vartheta_n \geq 0, n \geq 1$ and

$$\vartheta_0 + \sum_{n=1}^{\infty} \vartheta_n = 1.$$

Then

$$f(z) = \vartheta_0 f_0(z) + \sum_{n=1}^{\infty} \vartheta_n f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \vartheta_n \frac{(1-\Pi)}{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]} z^n.$$

Since

$$\sum_{n=1}^{\infty} \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi}$$

$$\vartheta_n \frac{(1-\Pi)}{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}$$

$$= \sum_{n=1}^{\infty} \vartheta_n = 1 - \vartheta_0 \leq 1.$$

By Theorem 1, $f(z)$ is in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. Conversely suppose that the function $f(z)$ is in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$, since

$$a_n \leq \frac{(1-\Pi)}{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}, n \geq 1.$$

$$\vartheta_n = \sum_{n=1}^{\infty} \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi} a_n,$$

and $\vartheta_0 = 1 - \sum_{n=1}^{\infty} \vartheta_n$, it follows that $f(z) = \vartheta_0 f_0(z) + \sum_{n=1}^{\infty} \vartheta_n f_n(z)$. This completes the proof of the theorem.

For the functions $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ belongs to Ψ_p, we denoted by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ and defined as

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$$

Theorem 7. *If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$ then $(f * g)(z)$ is in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$.*

Proof. Suppose $f(z)$ and $g(z)$ are in $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. By Theorem 1, we have

$$\sum_{n=1}^{\infty} \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi} a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi} b_n \leq 1.$$

Since $f(z)$ and $g(z)$ are regular are in $\mathbb{U}^*$, so is $(f*g)(z)$. Furthermore

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi} a_n b_n \\
&\leq \sum_{n=1}^{\infty} \left\{ \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi} \right\}^2 a_n b_n \\
&\leq \left(\sum_{n=1}^{\infty} \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi} a_n \right) \\
&\left(\sum_{n=1}^{\infty} \frac{\left[1+\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right]^m \left[\left(\frac{(\mu+\xi)(1+n)}{\alpha+\mu}\right)(1+\Lambda)+1-\Pi\right]}{1-\Pi} b_n \right) \\
&\leq 1.
\end{aligned}$$

Hence, by Theorem 1, $(f*g)(z)$ is in the class $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$.

Remark 3. *For the choice of* $\alpha, \xi = 1$ *and* $\mu = 0$, *in Theorems 5, 6 and 7, we observed that the the results are coincide with [16].*

6. Summary

By utilizing the new differential operator $A_{\xi}^{m} f(z)$ for meromorphic functions, we introduced a new subclass $\varphi_p(\Pi, \Lambda, \xi, \mu, \alpha)$. The important results for these subclasses include coefficient bounds, distortion properties, δ-neighbourhoods, convex linear combinations, and convolution properties. Furthermore, this study develops the classes utilised in [19–31] by using the new operator for future studies.

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