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Haar Wavelets and D-Stability of Lumped-Parameter Dynamical Systems

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Abstract. D-stability is a well-known mathematical tool used to analyze and characterize dynamical systems. It plays an important role in the stability analysis of dynamical systems, particularly in cases where stability is preserved under various types of perturbation, especially those involving positive diagonal scaling. The analysis of D-stability ensures the stability of dynamical systems. In this paper, we present new results on the characterization of D-stability and strong D-stability for structured matrices of the form $(I_n - A \otimes P^t)$, where I_n is an $n \times n$ identity matrix and the matrices A and P associated with a lumped-parameter dynamical system

$$\begin{cases} x(t) = A\ x(t) + B\ u(t), \quad x(0) = x_0 \\ y(t) = C\ x(t) + D\ u(t). \end{cases}$$

The results on D-stability and strong D-stability are obtained using mathematical tools from linear algebra, matrix analysis, system theory and their interactions with the computation of structured singular values. Furthermore, we present the numerical approximations to singular values and pseudo-spectrum of Haar wavelet matrices associated with a lumped-parameter dynamical system.

2020 Mathematics Subject Classifications: 15A18, 65K05

Key Words and Phrases: Haar wavelets, Structured singular value, block diagonal perturbations, D-stability, pseudo-spectrum

1. Introduction

Haar wavelets are an excellent mathematical tool for studying and analyzinglyzing signal processing and optimal control of linear time-varying systems. Regarding system analysis via Haar wavelets, the classical work was done by [1]. In their classical paper, Chen and Hasiao [1] constructed and analyzed a Haar operation matrix for the integrals of

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Haar wavelets vector. The Haar product matrix was constructed and analyzed by Hasiao [2] to study problems such as state analysis of linear time-delayed systems.

Haar wavelets $h_i(t)$ denote the group of square waves having the magnitude ± 1 in some given intervals and 0, elsewhere. The zeros make Haar transformation faster compared with transformations associated with square functions. The scaling function is a line $h_0(t) = 1$, $0 \leq t < 1$. In general, the Haar wavelets as a family of single square wavelets can be written as

$$h_n(t) = h_1\left(2^i t - k\right);\, n = 2^i + k,\, i \geq 0,\, 0 \leq k < 2^i.$$

The operational matrices used to solve the optimization and identification problems from the dynamic systems were constructed by using orthogonal functions, see [3]. Many operational matrices were constructed using orthogonal functions, such as block pulse [4], Lagurre [5, 6], Legendre [7], Chebyshev [8], and Fourier [9]. In [1], an operational matrix was constructed for integration using Haar wavelets. Furthermore, this operational matrix was used to study and analyze lumped-parameter and distributed-parameter systems.

The structural stability scheme with in-plane forces as the discretized parameters was studied and analyzed in [10–12]. A more general methodology was presented to construct the lumped-parameter force stiffness matrices for elements such as beams, curved beams, shells and circular plates. The numerical experiments were performed to compare the results with exact and various other solutions, for instance, the consistent geometric stiffness matrix solutions. A transfer matrix with in-plane forces in the form of a lumped-parameters was constructed by [13], and [14].

The computation of structured singular values (μ-values) [15] is a well-known mathematical tool for addressing an important problem in the analysis of linear time-invariant systems. The μ-value also quantifies the stability analysis of linear systems subject to structured perturbations. The computation of the μ-value is possible with respect to all kind of perturbations and this includes, real, complex, and a mixture of both. The exact computation of the μ-value is an NP-hard problem [16].

The NP-hard nature of computing μ-value motivates the development of iterative methods and numerical algorithms for computing upper and lower bound. For upper bounds see [17, 18] and the references therein. For lower bounds, see [19, 20] and references therein. For applications of μ-values in various research directions, see [15, 21–28].

The D-stability or diagonal stability was introduced in a classical paper by Arrow and McManus [29], and then by Enthoven and Arrow [30] in the study of equilibrium. dynamics. A given matrix A is D-stable if and only if for every positive diagonal matrix D, the matrix product DA or AD has all eigenvalues in the left half of the complex plane. The characterization of D-stability for various class of matrices have been extensively studied in [31–34].

The concepts of D-stability and μ-values are closely interconnected. In [35], the novel results were analyzed and proposed on the relationship between D-stability of real-valued square matrices and structured singular values. It was shown that a given n-dimensional real-valued matrix is D-stable if and only if its real-valued μ-value is greater than or equal to 0, and strictly less than 1. Furthermore, some new results were presented on conditions to strong D-stability in terms of μ-values, see [36]. Additional results on the relationships between D-stability, strong D-stability and real-valued μ-values were presented in [37]. In [38], new results on the interconnections between H-stable, $D(\alpha)$-stable, semi-stable matrices and structured singular values were analyzed and presented.

Spectra and pseudo-spectra of structured matrices play an important role to study and analyze system of linear equations appearing across various research disciplines. Recently, novel results on the spectral, pseudo-spectrum of D-stable matrices in economic models were presented in [39]. Spectral properties of D-stable matrices in transportation problems were studied and analyzed in [40]. A detailed analysis of stability, D-stability and the pseudo-spectrum for economic models was presented in [41]. In [42], the interconnection between Schur stability and μ-values were analyzed, leading to new results.

In this article, we present new results on D-stability, and strong D-stability for the structured matrix of the form $(I_n - A \otimes P^t)$, where I_n is an $n \times n$ identity matrix and the matrices A and P are from the following lumped-parameter dynamical system

$$\begin{cases} x(t) = A\ x(t) + B\ u(t), \quad x(0) = x_0 \\ y(t) = C\ x(t) + D\ u(t). \end{cases}$$

We use an idea of interconnection between D-stability and μ-values to construct our results. For D-stability, we aim to show that a given matrix is D-stable if its structured singular values belong to $[0, 1)$.

Overview of article: In section 2, we give basic concepts, definitions, and observations on structured singular values, D-stability, and strong D-stability. The problem statement is formulated in section 3 of the article. In section 4, we recall sufficient conditions for D-stability and strong D-stability of an n-dimensional real-valued matrix. We provide new results for D-stable, and strong D-stable matrices in section 5. The main idea to construct and prove new results is based on the computation of eigenvalues, singular values and structured singular values and their interaction with D-stability and strong D-stability. Numerical tests for structured matrices, for instance, Haar matrices and Haar wavelet operational matrices, are presented in section 6, and finally we conclude in section 7.

2. Preliminaries

In this section, we recall the definitions and present well-known results to provide a background on D-stable and strong D-stable matrices and the computation of μ-values. We also recall some existing and fundamental results on the interconnections between D-stable, strong D-stable matrices and structured singular values.

In the μ-theory the uncertainties across the system are presented with the set of block-diagonal matrices. There are three possible types of uncertainties, that is, repeated real scalar blocks, repeated complex scalar blocks, and real or complex full blocks. The following Definition 1 is about the set of block-diagonal matrices.

Definition 1. *The set* $\mathbb{B}_1$ *is the set of block-diagonal matrices and is defined as*

$$\mathbb{B}_1 := \{diag\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_S I_{r_S}; \Delta_1, \Delta_2, \cdots, \Delta_F) : \delta_i \in \mathbb{K},\ \Delta_j \in \mathbb{K}^{m_j, m_j},\ i = 1:S,\ j = 1:F\},$$

where $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$.

The computation of structured singular values or μ-values involve the computation of eigenvalues and singular values. It demands the computation of the largest singular value of an admissible perturbation Δ from the set of block-diagonal matrices such that the modified matrix $I - M\Delta$ for a given system matrix M has atleast one of its eigenvalue to be exactly equal to zero. The following is the definition of μ-value for a given M with respect to set of block-diagonal matrices $\mathbb{B}_1$.

Definition 2. *[15] For a given* $M \in \mathbb{C}^{n,n}$, *the structured singular value is denoted by* $\mu_{\mathbb{B}_1}(M)$ *and is defined by*

$$\mu_{\mathbb{B}_1}(M) := \begin{cases} 0, & if\ det(I - M\Delta) \neq 0,\ \forall \Delta \in \mathbb{B}_1 \\ (min\{||\Delta||_2 :\ det(I - M\Delta) = 0,\ \forall \Delta \in \mathbb{B}_1\})\,,^{-1} & else \end{cases}$$

where ***min*** *is taken over all* $\Delta \in \mathbb{B}_1$.

Remark 1. *[15] The set* $\mathbb{B}_1$ *represents a multi-index of integers. Hence, it does make sense to identify as one of the valid candidate from the set. This implies that the computation of* μ*-value depends on a given matrix and the set of block-diagonal matrices.*

Remark 2. *[15] In the set* $\mathbb{B}_1$, *the full blocks can be taken as the rank-1 matrices, that is, dyads.*

Remark 3. *[15] From the definition of* μ*-value, one can easily verify that for any* $\alpha \in \mathbb{C}$, *we have that* $\mu_{\mathbb{B}_1}(\alpha M) = \alpha \mu_{\mathbb{B}_1}(M)$.

An alternative expression for the computation of $\mu_{\mathbb{B}_1}(M)$ can be followed from the following Lemma 1.

Lemma 1. *[15] For given* $M \in \mathbb{C}^{n,n}$ *and for all* $\Delta \in \mathbb{B}_1$, *we have* $\mu_{\mathbb{B}_1}(M) = max\ \rho(\Delta M)$, *where* $\rho(\cdot)$ *denotes the spectral radius of a matrix, and* ***max*** *is taken over all* $\Delta \in \mathbb{B}_1$.

The concept of D-stability or some time known as diagonal stability in the literature, of a given matrix is a play an important and role in matrix theory and control theory, particularly when analyzing the stability of linear time invariant dynamical systems.

Definition 3. *[34] A given* n*-dimensional matrix* M *is said to be a* D*-stable matrix if for every positive diagonal matrix* D, *the matrix product* DM *or* MD *has all of its eigenvalues in the left half of the complex plane.*

Remark 4. *The matrix products* DM *and* MD *are the similar matrices and their* D*-stability remains preserved under perturbations subject to both rows and columns.*

The following four observations given in [34] holds true for D-stable matrices.

Observation 1. *The condition which holds true for matrices under consideration that implying stability and remains preserved under positive diagonal multiplication is a sufficient condition for D-stability of matrices.*

Observation 2. *If given $M \in \mathbb{C}^{n,n}$ such that DM is stable for a positive diagonal matrix D, then non of the eigenvalue of M is exactly equal to* 0, *and hence M-1 is invertible, $\hat{D}^T M \hat{D}$, $\hat{D}MD$, M^T are all D-stable matrices, with $\hat{D}$ having a positive diagonal structure.*

Observation 3. *If given $M \in \mathbb{C}^{n,n}$ such that DM is stable for a positive diagonal matrix D, then $k \times k$ principal sub-matrix of M belongs to euclidean closure of $k \times k$ D-stable matrices.*

Observation 4. *Let $M \in \mathbb{C}^{n,n}$ such that DM is stable for a positive diagonal matrix D, then M is a D-stable matrix if and only if $det(M \pm iD) \neq 0$, for all positive diagonal matrices D.*

Definition 4. *[43] A given $M \in \mathbb{C}^{n,n}$ is said to be strongly D-stable if there exists $\gamma > 0$ such that $M + \hat{M}$ is a D-stable matrix for each $\hat{M} \in \mathbb{R}^{n,n}$ with $\sigma_{max}(\hat{M}) < \gamma$.*

Remark 5. *All the* 13 *sufficient conditions to D-stability [34] satisfies are extended to strong D-stability, and the simpler conditions which holds true for strong D-stability are constructed and analyzed in [36] and compare with the one which are presented in [35].*

3. Problem Statement

We consider lumped-parameter dynamical system with $x(t)$ representing the n number of states; $u(t)$, the input data; $y(t)$, the output data. The lumped-parameter linear system, with its state equation and output equation has the following mathematical formulation:

$$\begin{cases} x(t) = A\ x(t) + B\ u(t), \quad x(0) = x_0 \\ y(t) = C\ x(t) + D\ u(t). \end{cases}$$

Assumption 1. *For $0 \leq t < 1$, assume that $u(t)$ is a square integrable function.*

The Haar series expansion of square integrable function $u(t)$ can be written as

$$u(t) = \tilde{A}\ H(t),$$

where $\tilde{A}$ is a structured matrix, and $H(t)$ is the matrix of Haar functions.

Remark 6. *The Haar series of state variable vector is $\frac{dx(t)}{dt} = FH(t)$.*

The integration of $\frac{dx(t)}{dt}$ yields $x(t)$ as,

$$x(t) = \int_0^t \left(\frac{dx(t)}{dt} + x_0 \right) dt \ = \ F \int_0^t H(r)\, dr \ + \ x_0 \ = \ FPH(t) + x_0.$$

In view of $u(t)$, and $x(t)$, one may obtain the following matrix equation,

$$F = (I_n - A \otimes P^t)\, Q,$$

where

$$F = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{m-1} \end{pmatrix}; Q = \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{m-1} \end{pmatrix},$$

and $\otimes$ denotes Kronecker-product, that is,

$$A \otimes P^t = \begin{pmatrix} P_{11}A & P_{12}A & \dots & P_{1m}A \\ P_{11}A & P_{22}A & \dots & P_{2m}A \\ \vdots & \vdots & \ddots & \vdots \\ P_{1m}A & P_{2m}A & \dots & P_{mm}A \end{pmatrix}.$$

In this article, we study and analyze the lumped-parameter linear system by characterizing the spectral properties of $(I_n - A \otimes P^t)$. Furthermore, our results are primiraly based on the analysis of interconnection between μ-theory and theory of matrix stability.

4. Sufficient conditions for D-stability, and strong D-stability

In this section, we provide a number of sufficient conditions for D-stability and strong D-stability of a given n-dimensional real-valued matrix M. These sufficient conditions are provided by C.R. Johnson [34] and W.S. Kafri [44], respectively. One may have a look at these classical papers by Johnson and Kafri to get the proof of each and every sufficient condition for D-stability and strong D-stability.

4.1. Sufficient condition for D-stability:

For a given $M \in \mathbb{R}^{n,n}$, the sufficient conditions for D-stability are:
C_1 : All the eigenvalues $\lambda_i\left(DM + M^t D\right) > 0,\ \forall i$, D is a positive diagonal matrix.
C_2 : Given $M \in \mathbb{R}^{n,n}$ is an M-matrix, that is, all the off-diagonal entries are non-positive and all the principal minors are positive.
C_3 : There exists a positive diagonal matrix D such that $MD = B = (b_{ij})$ which satisfies the condition that

$$Re(b_{ii}) > \sum_{j=1}^{n} |b_{ij}|; \quad i = 1:n,\ j \neq i.$$

C_4 : Given $M \in \mathbb{R}^{n,n}$ is a triangular matrix and the real part of all the off-diagonal entries m_{ii} is strictly positive.
C_5 : Given $M \in \mathbb{R}^{n,n}$ is a sign stable matrix.
C_6 : For given $M \in \mathbb{R}^{n,n}$, each principal minor is positive and M is a tri-diagonal matrix.

C_7 : Given $M \in \mathbb{R}^{n,n}$ is an oscillatory matrix, that is, M is totally non-negative matrix.
C_8 : For each $x \in \mathbb{R}^{n,1}$, $x \neq 0$, there exists a positive diagonal matrix D such that real part of $x^t DMx$ is strictly positive.
C_9 : For given $M \in \mathbb{R}^{n,n}$, the Hadamard product of P and M is a stable matrix for each positive definite matrix P.
C_{10} : For given $M \in \mathbb{R}^{n,n}$ each principal minor is positive and M is strictly sign symmetric matrix.
C_{11} : Given $M \in \mathbb{R}^{n,n}$ such that $M \in \mathbb{R}^{2,2} \cap P_0^+$.
C_{12} : Given $M \in \mathbb{R}^{n,n}$ such that $M \in \mathbb{R}^{3,3} \cap P_0^+$, and $M = \begin{pmatrix} x & a & b \\ \alpha & y & c \\ \beta & \alpha & z \end{pmatrix}$.
C_{13} : Given Given $M \in \mathbb{R}^{n,n}$ such that $M \in \mathbb{R}^{n,n} \cap P_0^+$ satisfies GKK condition with $n \leq 4$.

4.2. Sufficient condition for strong D-stability:

For a given $M \in \mathbb{R}^{n,n}$, the sufficient conditions for the strong D-stability are:
C_1 : For a positive diagonal matrix D, all the eigenvalues $\lambda_i \left(DM + M^t D\right) < 0, \ \forall i$.
C_2 : Given $M \in \mathbb{R}^{n,n}$ is an M-matrix, that is, all the off-diagonal entries are non-positive and all the principal minors are positive.
C_3 : There exists a positive diagonal matrix D such that $MD = B = (b_{ij})$ which satisfies the condition that

$$Re(b_{ii}) < -\sum_{1 \leq j \leq n}^{n} |b_{ij}|; \quad 1 \leq i \leq n, \ j \neq i.$$

C_4 : Given $M \in \mathbb{R}^{n,n}$ is a sign triangular matrix, and $m_{ii} < 0, \ i = 1 : n.$.
C_5 : Given $M \in \mathbb{R}^{n,n}$ is a sign stable matrix without having a any of non-zero entry.
C_6 : For given $M \in \mathbb{R}^{n,n}$ is a jocabi matrix, and each of jth-order principal minor is of sign $(-1)^j$.
C_7 : Given $M \in \mathbb{R}^{n,n}$ is an oscillatory matrix, that is, M is totally non-negative matrix.
C_8 : For each $x \in \mathbb{R}^{n,1}$, $x \neq 0$, there exists a positive diagonal matrix D such that real part of $x^t DMx$ is strictly positive.
C_9 : For given $M \in \mathbb{R}^{n,n}$, the Hadamard product $(H \circ (M + G))$ is Schur stable matrix for each positive definite symmetric matrix H, and a perturbation matrix G such that $||G||_2 < \alpha, \ \alpha \in \mathbb{R}$.
C_{10} : For given $M \in \mathbb{R}^{n,n}$ each jth-order principal minor is of sign $(-1)^j$.
C_{11} : Given $M \in \mathbb{R}^{2,2}$ is strongly D-stable iff its jth-order principal minors are of sign $(-1)^j$.
C_{12} : Given $M \in \mathbb{R}^{3,3}$ with all of its jth-order principal minors are with sign $(-1)^j$, and

$$m_{11}m_{22}m_{33} < \frac{m_{12}m_{23}m_{31} + m_{21}m_{32}m_{13}}{2}.$$

C_{13} : Given Given $M \in \mathbb{R}^{n,n}$ is strongly D-stable matrix if for $n \leq 4$, and M satisfies GKK condition.

5. New Results

In this section, we present new results on D-stability and strong D-stability for structured matrices associated with lumped-parameter dynamical systems, as described in the section on **Problem Statement**. We make use of various mathematical tools from linear algebra, matrix analysis and system theory to construct and present our results. The main ideas involve the computation of the spectrum and the analysis of the interconnections between D-stability and structured singular values.

The characterization of D-stability [35] for a given real-valued n-dimensional matrix M in terms of the real structured singular values is given by the following Theorem 1.

Theorem 1. *Let $M \in \mathbb{R}^{n,n}$ be the given matrix. Then M is a D-stable matrix if and only if it is stable and none of the eigenvalues of $M \pm iD$ is exactly equal to zero, and*

$$0 \leq \mu_{\mathbb{B}_1}\left((iI + M)^{-1}(iI - M)\right) < 1.$$

The following Theorem 2 shows that $(I_n - A \otimes P^t) \in \mathbb{R}^{n,n}$ is a D-stable matrix if it is stable and the structured singular values of $(I_n - A \otimes P^t)^{-1}$ are greater than or equal to zero and strictly less than one.

Theorem 2. *Let $(I_n - A \otimes P^t) \in \mathbb{R}^{n,n}$. Then $(I_n - A \otimes P^t)$ is D-stable if $(I_n - A \otimes P^t)$ is stable, and $0 \leq \mu_{\mathbb{B}_1}\left(\frac{1}{(I_n - A \otimes P^t)^2}\right) < 1$.*

Proof. The matrix $(I_n - A \otimes P^t)$ is D-stable if and only if $(I_n - A \otimes P^t)$ is stable, and satisfy the condition that

$$\prod_i \lambda_i \left(\begin{bmatrix} I_n - A \otimes P^t & -P \\ P & I_n - A \otimes P^t \end{bmatrix}\right) \neq 0 \ \forall i.$$

To show that $0 \leq \mu_{\mathbb{B}_1}\left(\frac{1}{(I_n - A \otimes P^t)^2}\right) < 1$, it is enough to show that

$$\prod_i \lambda_i \left(\begin{bmatrix} I_n - A \otimes P^t & -P \\ P & I_n - A \otimes P^t \end{bmatrix}\right) \neq 0 \ \forall i, \forall P \in \Omega,$$

where $\Omega = \{P \in \mathbb{R}^{n,n} : diag(p_{ii}) > 0 \ \forall i\}$. Since we know that

$$\prod_i \lambda_i \left(\begin{bmatrix} I_n - A \otimes P^t & -P \\ P & I_n - A \otimes P^t \end{bmatrix}\right) \neq 0.$$

In turn this implies that

$$\prod_i \lambda_i \left[\left(I_n - A \otimes P^t\right)^2 - P\left(\frac{1}{(I_n - A \otimes P^t)}\right) P(I_n - A \otimes P^t)\right] \neq 0.$$

Also, $\prod_i \lambda_i \left(I_n - \frac{1}{(I_n - A\otimes P^t)^2}\tilde{P}\right) \neq 0$, where $\tilde{P} = diag(\tilde{P_{ii}}) = P$, a positive diagonal matrix from Ω. Further, we have

$$\prod_i \lambda_i \left(I_n - \frac{1}{(I_n - A\otimes P^t)^2}\tilde{P}\right) \neq 0.$$

Thus, finally we have that $0 \leq \mu_{\mathbb{B}_1}\left(\frac{1}{(I_n - A\otimes P^t)^2}\right) < 1$.

The following Theorem 3 shows D-stability of $(I_n - A\otimes P^t) \in \mathbb{R}^{n,n}$ if the real part of all the eigenvalues of $P(I_n - A\otimes P^t) + (I_n - A\otimes P^t)^t P$ is strictly positive.

Theorem 3. *Let $(I_n - A\otimes P^t) \in \mathbb{R}^{n,n}$. Then $(I_n - A\otimes P^t)$ is D-stable if*

$$Re\left[\lambda_i(P(I_n - A\otimes P^t) + (I_n - A\otimes P^t)^t P)\right] > 0,\ \forall i,\ \forall P \in \Omega,$$

where $\Omega := \{P \in \mathbb{R}^{n,n} : diag(P_{ii}) > 0,\ \forall i\}$, and

$$0 \leq \mu_{\mathbb{B}_1}\left[\left(iI_n + P(I_n - A\otimes P^t) + (I_n - A\otimes P^t)^t\, P\right)^{-1}\left(iI_n - P(I_n - A\otimes P^t) - (I_n - A\otimes P^t)^t\, P\right)\right] < 1.$$

Proof. We aim to show that $(I_n - A\otimes P^t)$ is D-stable matrix if

$$Re\left[\lambda_i\left(P(I_n - A\otimes P^t) + (I_n - A\otimes P^t)^t P\right)\right] > 0,\ \ \forall i,\ \ \forall P \in \Omega.$$

To prove we have to follow all steps of Theorem 1. Next, we aim to prove that $(I_n - A\otimes P^t)$ is D-stable matrix if it's structured singular value is strictly less than 1. For this, we consider $\Delta \in \mathbb{B}_1$, a block diagonal structured matrix. Let

$$\Delta = (i\, I_n - P)(i\, I_n + P)^{-1}.$$

As, we know that for $P \in \Omega$, and for given $(I_n - A\otimes P^t)$, we have that

$$\lambda_i\left[P(I_n - A\otimes P^t) + (I_n - A\otimes P^t)^t P\right] \neq 0.$$

This yields that

$$\lambda_i\left[P(I_n - A\otimes P^t) + (I_n - A\otimes P^t)^t P + iP\right] \neq 0,\ \ \forall i$$

if

$$\lambda_i\left[P(I_n - A\otimes P^t) + (I_n - A\otimes P^t)^t P + i(i\, I_n + \Delta)^{-1}(i\, I_n - \Delta)\right] \neq 0.$$

In turn this implies that

$$\lambda_i\Bigg[\Bigg(i\,I_n{+}P(I_n{-}A{\otimes}P^t){+}(I_n{-}A{\otimes}P^t)^tP\Bigg){-}\Bigg(i\,I_n{-}P(I_n{-}A{\otimes}P^t){-}(I_n{-}A{\otimes}P^t)^tP\Bigg)\Delta\Bigg] \neq 0,\ \ \forall \Delta \in \mathbb{B}_1.$$

Thus final we have that

$$\lambda_i\Bigg[\Bigg(I_n{-}(i\,I_n{+}P(I_n{-}A{\otimes}P^t){+}(I_n{-}A{\otimes}P^t)^t)P\Bigg)\Bigg(i\,I_n{-}P(I_n{-}A{\otimes}P^t){-}(I_n{-}A{\otimes}P^t)^tP\Bigg)\Delta\Bigg] \neq 0,\ \forall\,\Delta \in \mathbb{B}_1.$$

The last expression for $\lambda_i(\cdot)$ implies that

$$0 \leq \mu_{\mathbb{B}_1}\Bigg[\Bigg(i\,I_n{+}P(I_n{-}A{\otimes}P^t){+}(I_n{-}A{\otimes}P^t)^t)P\Bigg)^{-1}\Bigg(i\,I_n{-}P(I_n{-}A{\otimes}P^t){-}(i\,I_n{-}A{\otimes}P^t)^tP\Bigg)\Bigg] < 1.$$

The following Theorem 4 shows that given $(I_n - A \otimes P^t) \in \mathbb{R}^{n,n}$ is a D-stable matrix if it is stable, and $\Bigg(i\,I_n + (I_n - A \otimes P^t)\Bigg)^{-1}(i\,I_n - A \otimes P^t)$ are greater than or equal to zero and strictly less than one.

Theorem 4. *Let $(I_n{-}A{\otimes}P^t) \in \mathbb{R}^{n,n}$. Then $(I_n{-}A{\otimes}P^t)$ is D-stable matrix if $(I_n{-}A{\otimes}P^t)$ is stable, and*

$$0 \leq \mu_{\mathbb{B}_1}\Bigg[\Bigg(i\,I_n + (I_n - A \otimes P^t)\Bigg)^{-1}(i\,I_n - A \otimes P^t)\Bigg] < 1, \quad \forall\, P \in \Omega.$$

Proof.

The matrix $(I_n - A \otimes P^t)$ is D-stable if it is stable and $\lambda_i\Bigg((I_n - A \otimes P^t) + i\,P\Bigg) \neq 0,\ \ \forall\, P \in \Omega.$

We aim to prove that $(I_n - A \otimes P^t)$ is D-stable if it is structured singular value is strictly less than 1. For this we assume that $(I_n - A \otimes P^t)$ is D-stable. Let $\Delta \in \mathbb{B}_1$ with block diagonal structure, $\Delta = (i\,I_n - P)(i\,I_n + P)^{-1},\ \forall\, P \in \Omega$. Then, $P \in \Omega$ in terms of Δ can be re-written as $P = (i\,I_n{+}\Delta)^{-1}(i\,I_n{-}\Delta),\ \ \forall\,\Delta \in \mathbb{B}_1$. Since, $\lambda_i\Bigg((I_n{-}A{\otimes}P^t){+}i\,P\Bigg) \neq 0$, for some $P \in \Omega$.

This yields

$$\lambda_i\Bigg[(I_n - A \otimes P^t) + i(i\,I_n + \Delta)^{-1}(i\,I_n - \Delta)\Bigg] \neq 0, \quad \forall i, \forall \Delta \in \mathbb{B}_1.$$

By making use of singular value decomposition, we have

$$\sigma_i\Bigg[(I_n{-}A{\otimes}P^t){+}i(i\,I_n{+}\Delta)^{-1}(i\,I_n{-}\Delta)\Bigg] = \sigma_i\Bigg[\Bigg(i\,I_n{+}(I_n{-}A{\otimes}P^t)\Bigg){-}\Bigg(i\,I_n{-}(I_n{-}A{\otimes}P^t)\ \Bigg)\Delta\Bigg],\ \forall \Delta \in \mathbb{B}_1.$$

The $\sigma_i(\cdot)$ denotes that number of non-zero singular-value of a matrix. From this, we have

$$\Bigg(i\,I_n{+}(I_n{-}A{\otimes}P^t)\Bigg){-}\Bigg(i\,I_n{-}(I_n{-}A{\otimes}P^t)\ \Bigg)\Delta = \Bigg(I_n{-}(i\,I_n{+}(I_n{-}A{\otimes}P^t)^{-1}(i\,I_n{-}(I_n{-}A{\otimes}P^t))\Delta\Bigg).$$

This further yields

$$\lambda_i\left[I_n - \Big(i\,I_n + (I_n - A\otimes P^t)\Big)^{-1}\Big(i\,I_n - (I_n - A\otimes P^t)\Big)\Delta\right] \neq 0,\ \ \forall\Delta\in\mathbb{B}_1,$$

and hence

$$0 \leq \mu_{\mathbb{B}_1}\left[\Big(i\,I_n + (I_n - A\otimes P^t)\Big)^{-1}(i\,I_n - A\otimes P^t)\right] < 1, \quad \forall\, P\in\Omega.$$

The following Theorem 5 is to give the necessary condition for the D-stability of $(I_n - A\otimes P^t)\in\mathbb{R}^{n,n}$. It is shown that this given matrix is D-stable if we can express it in matrix series for such the structured singular value of $\Big(I_n + (I_n + A + \frac{A^2}{2!} + \ldots)\Big)^{-1}\Big(I_n - (I_n + A + \frac{A^2}{2!} + \ldots)\Big)$ is greater than and equal to zero and strictly less than one.

Theorem 5. *Let $(I_n - A\otimes P^t)\in\mathbb{R}^{n,n}$. The necessary condition for $(I_n - A\otimes P^t)$ to be a D-stable matrix i s that for $A\in\mathbb{R}^{n,n}$, $(I_n - A\otimes P^t)$ can be expressed as $(I_n - A\otimes P^t) = I_n + A + \frac{A^2}{2!} + \ldots$, and*

$$0 \leq \mu_{\mathbb{B}_1}\left[\Big(I_n + (I_n + A + \frac{A^2}{2!} + \ldots)\Big)^{-1}\Big(I_n - (I_n + A + \frac{A^2}{2!} + \ldots)\Big)\right] < 1.$$

Proof. For the necessary condition of $(I_n - A\otimes P^t)$ to be a D-stable matrix, we aim to show that

$$\lambda_i\left[i\,I_n + (I_n + A + \frac{A^2}{2!} + \ldots)P\right] \neq 0,\ \ \forall i,\ \forall P\in\Omega.$$

Let $\Delta\in\mathbb{B}_1$ with a block-diagonal structure, and $\Delta = (I_n - P)(I_n + P)^{-1}$ such that $P = (I_n + \Delta)^{-1}(I_n - \Delta)$. This further yields that

$$\lambda_i\left[i\,I_n + (I_n + A + \frac{A^2}{2!} + \ldots)(I_n + \Delta)^{-1}(I_n - \Delta)\right] \neq 0,\ \ \forall i,\ \forall\Delta\in\mathbb{B}_1.$$

Furthermore,

$$\lambda_i\left[\Big(i\,I_n + (I_n + A + \frac{A^2}{2!} + \ldots)\Big)^{-1}\Big(i\,I_n - (I_n + A + \frac{A^2}{2!} + \ldots)\Big)\Delta\right] \neq 0,\ \ \forall i,\ \forall\Delta\in\mathbb{B}_1.$$

Finally, we conclude that

$$0 \leq \mu_{\mathbb{B}_1}\left[\Big(I_n + (I_n + A + \frac{A^2}{2!} + \ldots)\Big)^{-1}\Big(I_n - (I_n + A + \frac{A^2}{2!} + \ldots)\Big)\right] < 1.$$

5.1. Strong D-stability:

In this subsection, we provide new results on strong D-stability of $(I_n - A \otimes P^t)$, which is a n-dimensional real-valued matrix. The following Theorem 6 shows the strong D-stability and we have made use of the eigenvalue perturbation result to the largest and simple eigenvalue to analyze its behaviour, which in turn helps us to conclude our results for D-stability.

Theorem 6. *Let $(I_n - A \otimes P^t)$ be a n-dimensional real-valued matrix. Then $(I_n - A \otimes P^t)$ is strongly D-stable if for n-dimensional matrices $A_1,\ A_2,\ \cdots,A_r$, the matrix* $\log(I_n - A \otimes P^t) + \left[\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)\right)^t \Delta + \Delta\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r)\right)\right]$ *is a D-stable matrix for all $\Delta \in \mathbb{B}_1$, here $\otimes$ denotes the entry-wise product of matrices, and $\alpha_i \in \mathbb{R},\ \ \alpha_i > 0\ \forall i$.*

Proof. Suppose that $\Delta \in \mathbb{B}_1$ has a block-diagonal structure, whereas the set $\mathbb{B}_1$ can have a matrix of real and complex uncertainties. For $0 < \theta \leq 2\pi$, let $\lambda(t) = |\lambda(t)|e^{i\theta}$ be the simple and largest eigenvalue. Assume that $x(t),\ y(t)$ have structure and size similar to given matrix $(I_n - A \otimes P^t)$, and are the right hand and left hand eigen-vectors. Consider that $\tilde{x}(t)$ of the form

$$\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)\right)^t \Delta + \Delta\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r)\right) y(t)$$

The eigenvalue perturbation result by Kato to $\lambda(t)$ which yields

$$\frac{d}{dt}|\lambda(t)|^2 = 2\epsilon \frac{|\lambda(t)|}{r} Re\left(\tilde{x}^t(t)\dot{\Delta}(t)x(t)\right)\ ;\ \ r = e^{i\theta}\, y^t(t)x(t),\ \epsilon > 0.$$

This further implies that

$$\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)\right)^t \Delta + \Delta\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r)\right) > 0.$$

In turn, this further yields $\log(I_n - A \otimes P^t) + \left[\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)\right)^t \Delta + \Delta\left(\log(I_n - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r)\right)\right]$ is a D-stable matrix.

In Theorem 7, we show the D-stability of a 2-dimensional real-valued matrix, that is, $\big(I_2 - (A \otimes P^t)\big) \in \mathbb{R}^{2,2}$. We again make use of the eigenvalue perturbation result for the largest and simple eigenvalue to analyze its behaviour, which in turn helps us to conclude our results for D-stability.

Theorem 7. *Let* $\big(I_2 - (A \otimes P^t)\big) \in \mathbb{R}^{2,2}$ *such that*

$$I_2 - (A \otimes P^t) = \cos(A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r) + i\, \sin(A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)$$

where $A_1, A_2, \cdots, A_r$ *are 2-dimensional matrices. Then* $(I_2 - A \otimes P^t)$ *is strongly D-stable matrix if it is stable, and for some* $\tilde{\alpha} > 0$, $(I_2 - A \otimes P^t) + M$ *is a D-stable matrix with* $||M|| < \gamma$, *where*

$$M := \Bigg((I_2 - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)\Bigg)^t \Delta + \Delta\Bigg(\log(I_2 - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r)\Bigg).$$

Proof. Let $\Delta \in \mathbb{B}_1,\ \lambda(t),\ x(t),\ y(t)$ be same as described in the proof of Theorem-5. Let $\tilde{x}(t) = M^t y(t)$. We use eigenvalue perturbation result by Kato on simple and largest eigenvalue $\lambda(t)$ to have

$$\frac{d}{dt}|\lambda(t)|^2 = 2\epsilon \frac{|\lambda(t)|}{r} Re\left(\tilde{x}^t(t)\dot{\Delta}(t)x(t)\right)\ ;\ \ r = e^{i\theta}\, y^t(t)x(t),\ \epsilon > 0.$$

Since, we know that $Re\left(\tilde{x}^t(t)\dot{\Delta}(t)x(t)\right) > 0$, thus

$$\Bigg((I_2 - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)\Bigg)^t \Delta(t) + \Delta(t)\Bigg((I_2 - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r)\Bigg)$$

is such that all of its eigenvalues are strictly positive. Next, for $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$; $d_1,\ d_2 > 0$, the matrix

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \cos(A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r) + i\, \sin(A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r) + M$$

allows us to have

$$\log(I_2 - A \otimes P^t) + M = \log(I_2 - A \otimes P^t) + \Bigg(\log(I_2 - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_r\, A_r)\Bigg)^t \Delta +$$

$$\Delta\Bigg(\log(I_2 - A \otimes P^t) \otimes (A_1 + \alpha_2\, A_2 + \cdots + \alpha_2\, A_r)\Bigg)$$

as a D-stable matrix.

Theorem 8 shows that given $(I_n - A \otimes P^t) \in \mathbb{R}^{n,n}$ is a D-stable matrix iff it is a stable matrix and the structured singular value of $\tilde{A}$ is greater than or equal to zero and strictly less than one.

Theorem 8. *Let* $(I_n - A \otimes P^t) \in \mathbb{R}^{n,n}$. *Then* $(I_n - A \otimes P^t)$ *is strongly D-stable if and only if* $(I_n - A \otimes P^t)$ *is stable and* $\exists\, \epsilon > 0$ *such that* $0 \leq \mu_{\mathbb{B}_1}(\tilde{A}) < 1$, *where*

$$\tilde{A} := \begin{pmatrix} \Big(i\, I_n + (I_n - A \otimes P^t)\Big)^{-1}\Big(i\, I_n - (I_n - A \otimes P^t)\Big) & 2i\Big(i\, I_n + (I_n - A \otimes P^t)\Big)^{-1} \\ \epsilon\Big(i\, I_n + (I_n - A \otimes P^t)\Big)^{-1} & -\epsilon\Big(i\, I_n + (I_n - A \otimes P^t)\Big)^{-1} \end{pmatrix}.$$

Proof. As $(I_n - A \otimes P^t)$ is strongly D-stable if and only if $(I_n - A \otimes P^t)$ is stable and $\exists\, \epsilon > 0$ such that $\Delta(I_n - A \otimes P^t) \in \mathbb{R}^{n,n}$ having $\sigma_{max}\Big(\Delta(I_n - A \otimes P^t)\Big) < \epsilon$, the inequality

$$0 \leq \mu_{\mathbb{B}_1}\left[\Big(i\, I_n + (I_n - A\otimes P^t) + \Delta(I_n - A\otimes P^t)\Big)^{-1}\Big(i\, I_n - (I_n - A\otimes P^t) - \Delta(I_n - A\otimes P^t)\Big)\right] < 1$$

holds true. Consider,

$$\tilde{A}\Big(\Delta(I_n - A\otimes P^t)\Big) = \Big(i\, I_n + (I_n - A\otimes P^t) + \Delta(I_n - A\otimes P^t)\Big)^{-1}\Big(i\, I_n - (I_n - A\otimes P^t) - \Delta(I_n - A\otimes P^t)\Big)$$

$$= 2i\Big(I_n + (I_n - A \otimes P^t) + \Delta(I_n - A \otimes P^t)\Big)^{-1} - I_n$$

$$= \Big(i\, I_n + (I_n - A \otimes P^t)\Big)^{-1}\Big(i\, I_n - (I_n - A \otimes P^t)\Big) - 2i\Big(i\, I_n + (I_n - A \otimes P^t)\Big)^{-1}$$

where

$$X = \frac{\Delta(I_n - A \otimes P^t)}{\epsilon}\left[I_n + \epsilon\Big(i\, I_n + (I_n - A\otimes P^t)\Big)^{-1}\frac{\Delta(I_n - A \otimes P^t)}{\epsilon}\right]^{-1}\epsilon\Big(i\, I_n + (I_n - A\otimes P^t)\Big)^{-1}.$$

From [35], it follows that $0 \leq \mu_{\mathbb{B}_1}\left[\tilde{A}\Big(\Delta(I_n - A \otimes P^t)\right] < 1,\ \forall \Delta \in \mathbb{B}_1$ and for all

$$\sigma_{max}\Big(\Delta(I_n - A \otimes P^t)\Big) < \epsilon$$

if and only if

$$0 \leq \mu_{\mathbb{B}_1}\left[\Big(i\, I_n + (I_n - A \otimes P^t)\Big)^{-1}\Big(i\, I_n - (I_n - A \otimes P^t)\Big)\right] < 1,$$

and thus implying that $0 \leq \mu_{\mathbb{B}_1}(\tilde{A}) < 1$.

6. Numerical Experimentation

This section is about the numerical experimentation for the approximation and visualization of eigenvalues, singular values, structured singular values, and pseudo-spectra for Haar matrices and structured matrices corresponding to lumped-parameter dynamical systems. For pseudo-spectrum in the complex plane, we display the level sets corresponding to resolvent norm $||(A - zI_n)^{-1}||$, for a given matrix A.

Example 1. For the family of Haar wavelets, the scaling function $h_1(x)$ is defined as

$$h_1(x) = \begin{cases} 1 & for \;\; x \in [0,1) \\ 0, & else. \end{cases}$$

The Haar wavelets for $[0,1)$ maybe defined as

$$h_i(x) = \begin{cases} 1 & for \;\; x \in [\alpha,\beta) \\ -1, & for \;\; x \in [\beta,\gamma) \\ 0, & else, \end{cases}$$

where $\alpha = \frac{k}{m},\ \beta = \frac{k+5}{m},\ \gamma = \frac{k+1}{m},\ m = 2^l,\ l = 1 : j,\ k = 0 : m-1$. Here, l and k are the level of resolution and translation parameters, respectively. For $j = 3$, and $n = 16$ (the size of matrix), the Haar matrix $H = H(i,j) = h_i(x_j)$ taken from [45] is:

$$H = H(i,j) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}.$$

The spectral properties such as the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of $H(i,j)$ are presented in Figure 1. In Figure 2, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level shows absolute value of eigenmode being plotted at a log scale. Further it shows how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

In Figure 2, we plot the value of the inverse of the resolvent norm. We show the real part of the pseudomode in magenta. The right singular vector corresponding to the

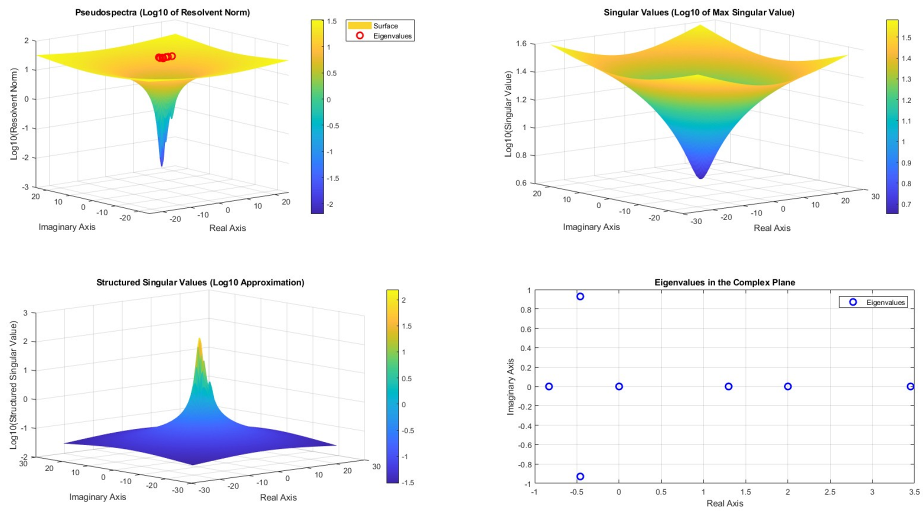

Figure 1: Spectral properties of matrix $H = H(i,j)$ in Example-1.

smallest singular value of the matrix $(zI_{16} - H(i,j))$, is shown in pseudomode.

Example 2. The integration of $H_m(t) = [h_o(t), h_1(t), \cdots, h_{m-1}(t)]^t$ maybe approximated as

$$\int_0^t H_m(\tau)d\tau \approx QH_m(t),$$

where Q is Haar wavelet operational matrix with order n. The Haar wavelet operational matrix of fractional order integration Q^α and is given by

$$Q^\alpha H_m(t) = J^\alpha H_m(t) = [Qh_0(t), Qh_0(t), \cdots, Qh_{m-1}(t)].^t$$

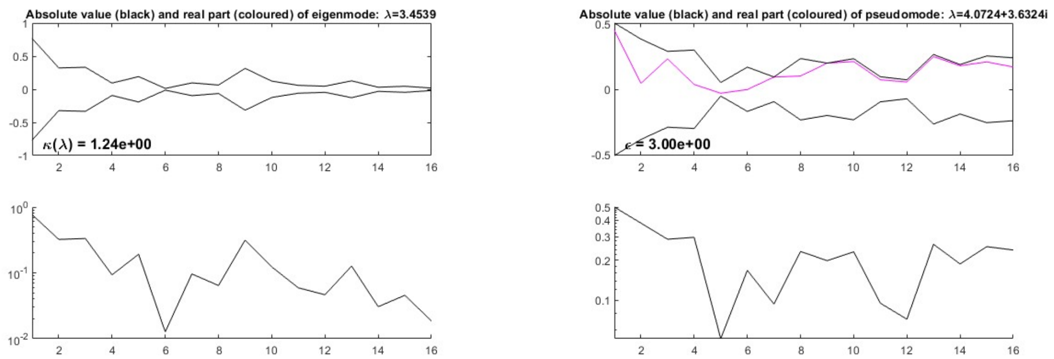

Figure 2: Eigenmode (left) and inverse of resolvent norm (right) of matrix H in Example-1

Here, $Qh_0(t) = \frac{1}{\sqrt{m}} \frac{t^{\alpha}}{\Gamma(1+\alpha)}$,

$$Qh_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 0, & 0 \le t < \frac{k-1}{2^j} \\ 2^{\frac{j}{2}} \phi_1(t), & \frac{k-1}{2^j} \le t < \frac{k-0.5}{2^j} \\ 2^{\frac{j}{2}} \phi_2(t), & \frac{k-0.5}{2^j} \le t < \frac{k}{2^j} \\ 2^{\frac{j}{2}} \phi_3(t), & \frac{k}{2^j} \le t < 1, \end{cases}$$

$\phi_1(t) = \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{k-1}{2^j}\right)^{\alpha}$,
$\phi_2(t) = \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{k-1}{2^j}\right)^{\alpha} - \frac{2}{\Gamma(\alpha+1)} \left(t - \frac{k-0.5}{2^j}\right)^{\alpha}$,
$\phi_3(t) = \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{k-1}{2^j}\right)^{\alpha} - \frac{2}{\Gamma(\alpha+1)} \left(t - \frac{k-0.5}{2^j}\right)^{\alpha} + \frac{1}{\Gamma(\alpha+1)} (t - \frac{k}{2^j})^{\alpha}$.

For $\alpha = 1.5$ and $m = 8$, the Haar wavelet operational matrix [46] is:

$$Q^{\alpha} H_8 = \begin{bmatrix} 0.0042 & 0.0216 & 0.0465 & 0.0770 & 0.1122 & 0.1516 & 0.1948 & 0.2414 \\ 0.0042 & 0.0216 & 0.0465 & 0.0770 & 0.1039 & 0.1084 & 0.1019 & 0.0875 \\ 0.0059 & 0.0305 & 0.0540 & 0.0478 & 0.0331 & 0.0273 & 0.0238 & 0.0214 \\ 0 & 0 & 0 & 0 & 0.0059 & 0.0305 & 0.0540 & 0.0478 \\ 0.0083 & 0.0266 & 0.0149 & 0.0113 & 0.0095 & 0.0083 & 0.0075 & 0.0069 \\ 0 & 0 & 0.0083 & 0.0266 & 0.0149 & 0.0113 & 0.0095 & 0.0083 \\ 0 & 0 & 0 & 0 & 0.0083 & 0.0266 & 0.0149 & 0.0113 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0083 & 0.0266 \end{bmatrix}.$$

The spectral properties such as the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of $H(i, j)$ are presented in Figure 3. In Figure 4, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level shows the absolute value of eigenmode being plotted at a log scale. Further, it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. A large condition number means that the eigenvalue is sensitive to perturbations.

In Figure 4, we plot the value of the inverse of the resolvent norm. We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value to the matrix $(zI_{16} - H(i, j))$, is shown in pseudomode.

Example 3. We consider $50, 100$ and 200 dimensional Haar matrices which are generated by MATLAB command **haarmtx(n)**. The spectral properties like the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of $50, 100$ and 200 dimensional Haar matrices are presented in Figure 5.

In Figures 6-8, we plot the eigenmode corresponding to the eigenvalues. The top plot in each figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The

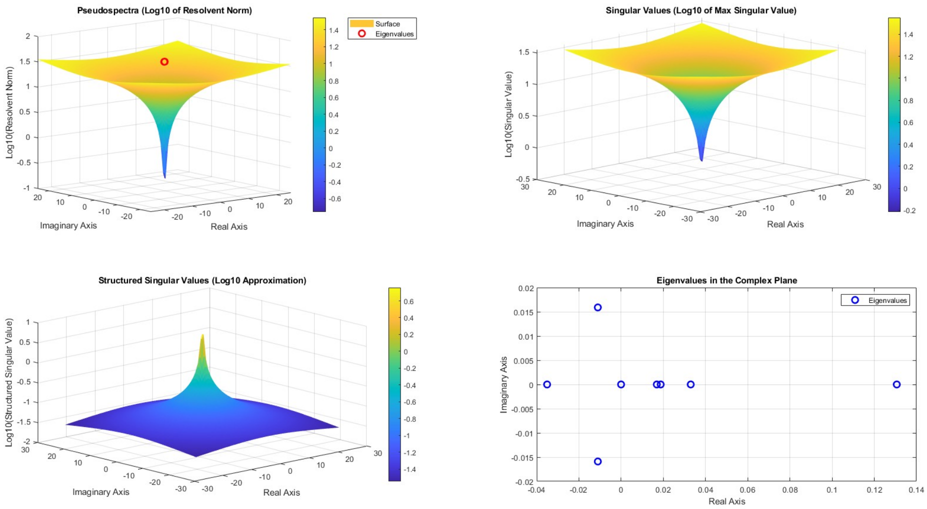

Figure 3: Spectral properties of matrix $Q^{\alpha}H_8$ in Example-2.

plot at the bottom level in each figure show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations. Further, we present the plot of the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value corresponding to matrix $zI - M$, is shown in pseudomde.

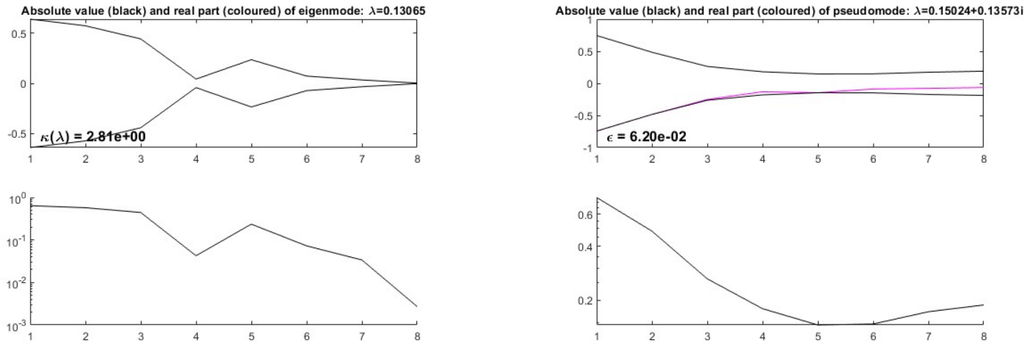

Figure 4: Eigenmode (left) and inverse of resolvent norm (right) of matrix $Q^{\alpha}H_8$ in Example-2

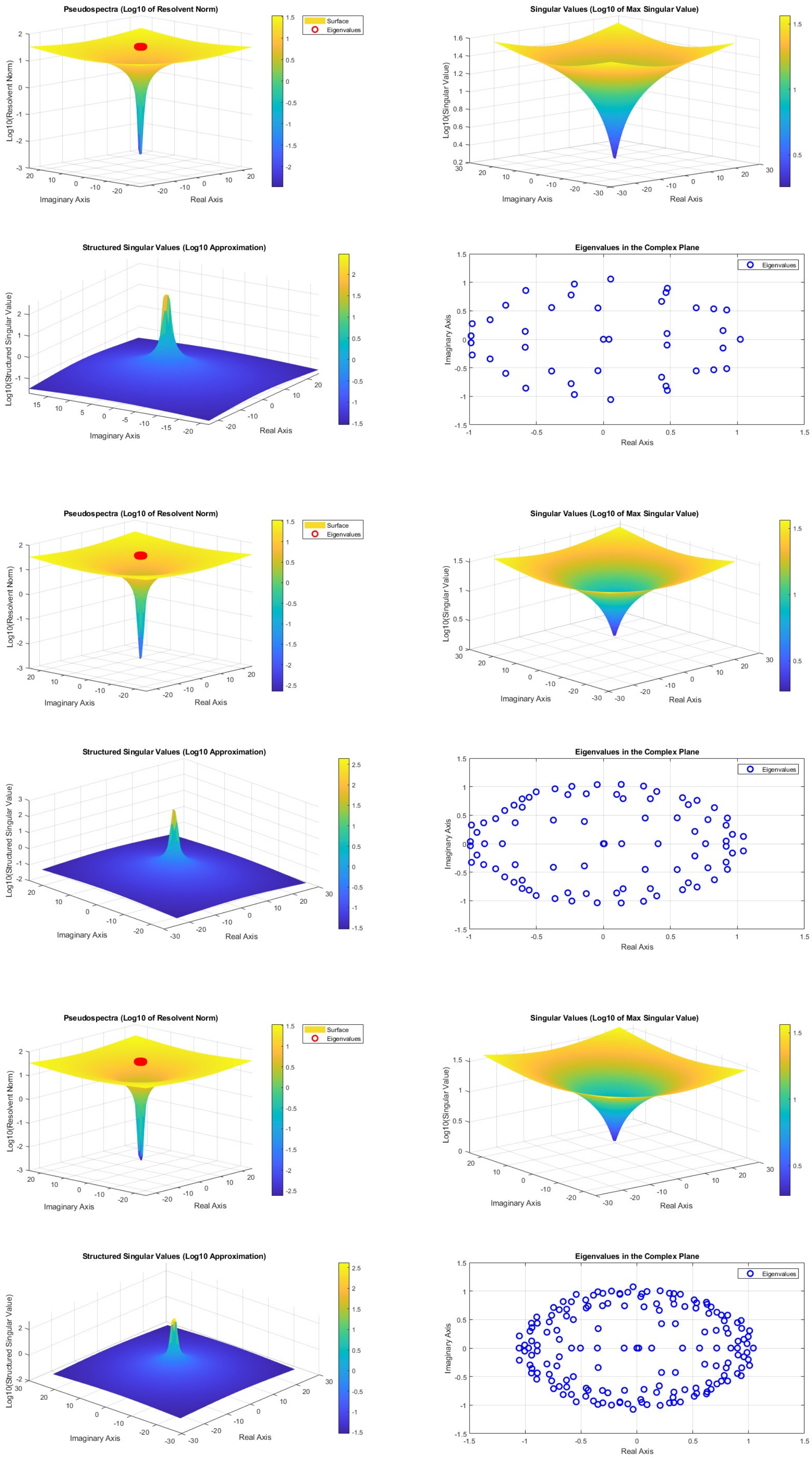

Figure 5: Spectral properties of 50, 100 and 200 Haar matrices in Example-3.

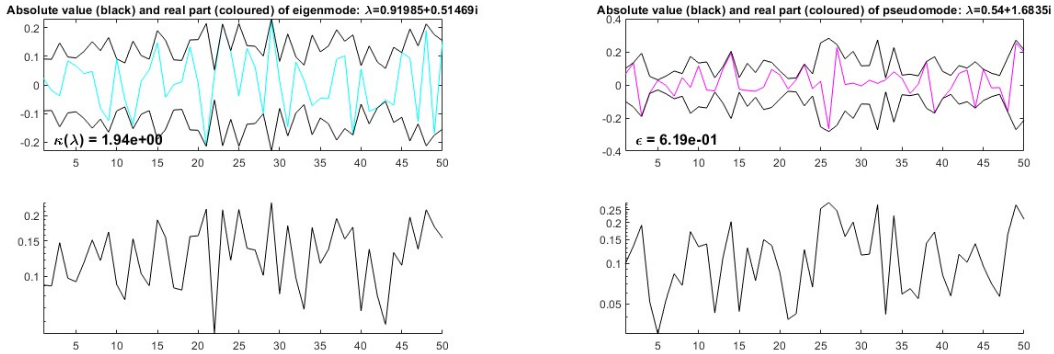

Figure 6: Eigenmode (left) and inverse of resolvent norm (right) of 50 dimensional Haar matrix in Example-3

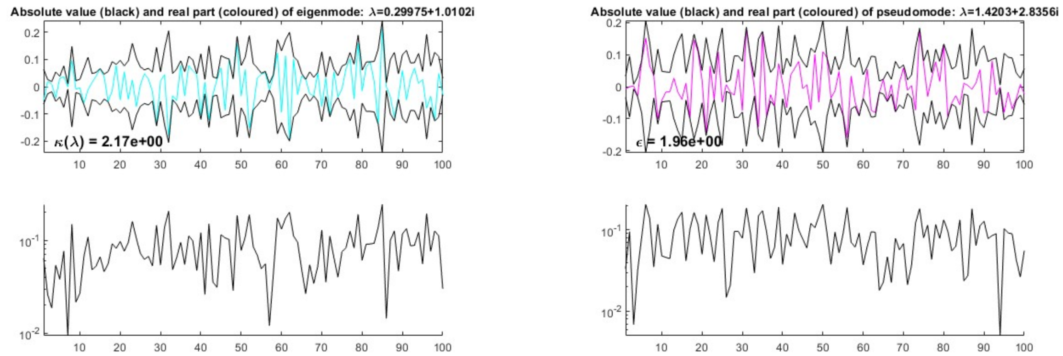

Figure 7: Eigenmode (left) and inverse of resolvent norm (right) of 100 dimensional Haar matrix in Example-3

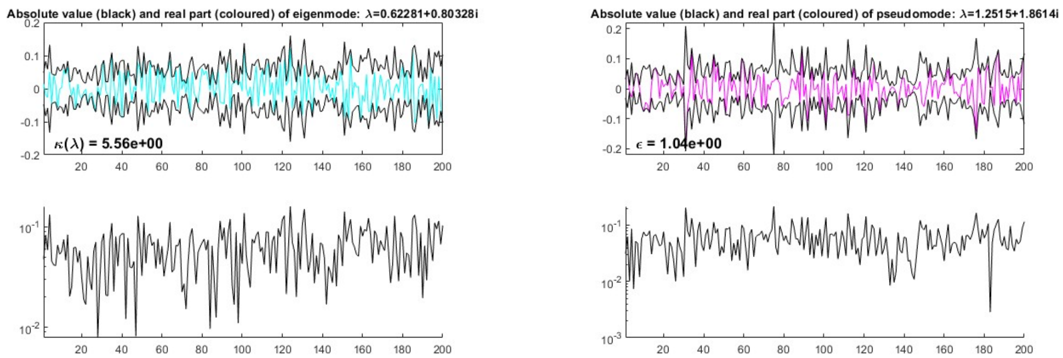

Figure 8: Eigenmode (left) and inverse of resolvent norm (right) of 200 dimensional Haar matrix in Example-3

7. Conclusion

In this article, we have presented new results on D-stability and strong D-stability for the structured matrix of the form $(I_n - A \otimes P^t)$, where I_n is an $n \times n$ identity matrix and the matrices A and P correspond to the following lumped-parameter dynamical system. Our proposed methodology is based on the collection of various tools from linear algebra, matrix analysis and system theory. The analytical and numerical results on D-stability, strong D-stability, spectrum and pseudo-spectrum are obtained by interconnecting the concepts from D-stability theory and μ-theory. In order to further advance the understanding of D-stability analysis for the lumped parameter dynamical systems, our future research aim is to develop a comprehensive lumped-parameter models for the stability, H-stable, $D(\alpha)$-stable, and D-semistable matrices.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Chi Fan Chen and Chi-Huang Hsiao. Haar wavelet method for solving lumped and distributed-parameter systems. *IEE Proceedings-Control Theory and Applications*, 144(1):87–94, 1997.

[2] Chun-Hui Hsiao. State analysis of linear time delayed systems via haar wavelets. *Mathematics and Computers in Simulation*, 44(5):457–470, 1997.

[3] CF Chen and CH Hsiao. A state-space approach to walsh series solution of linear systems. *International Journal of Systems Science*, 6(9):833–858, 1975.

[4] CF Cheng, YT Tsay, and TT Wu. Walsh operational matrices for fractional calculus and their application to distributed systems. *Journal of the Franklin Institute*, 303(3):267–284, 1977.

[5] CHYI HWANG and YEN-PING SHIH. Laguerre operational matrices for fractional calculus and applications. *International Journal of Control*, 34(3):577–584, 1981.

[6] RE King and PN Paraskevopoulos. Parametric identification of discrete-time siso systems. *International Journal of Control*, 30(6):1023–1029, 1979.

[7] RONG-YEU CHANG and MAW-LING WANG. Legendre polynomials approximation to dynamic linear state equations with initial or boundary value conditions. *International Journal of Control*, 40(1):215–232, 1984.

[8] PN Paraskevopoulos. Chebyshev series approach to system identification, analysis and optimal control. *Journal of the Franklin Institute*, 316(2):135–157, 1983.

[9] PN Paraskevopoulos, PD Sparis, and SG Mouroutsos. The fourier series operational matrix of integration. *International journal of systems science*, 16(2):171–176, 1985.

[10] Sudhir J Shah and Walter D Pilkey. Lumped-parameter approach to stability analysis. *Journal of engineering mechanics*, 119(10):2109–2129, 1993.

[11] Farah M Al-Askar, Clemente Cesarano, and Wael W Mohammed. Multiplicative brownian motion stabilizes the exact stochastic solutions of the davey–stewartson equations. *Symmetry*, 14(10):2176, 2022.

[12] Wael W Mohammed, Clemente Cesarano, and Farah M Al-Askar. Solutions to the (4+ 1)-dimensional time-fractional fokas equation with m-truncated derivative. *Mathematics*, 11(1):194, 2022.

[13] Walter D Pilkey and Kevin J O'Connor. Lumped parameter model for stability analysis. *Journal of the Structural Division*, 99(7):1702–1707, 1973.

[14] George Johnson and Walter Pilkey. Lumped parameter circular plate stability analysis. *Journal of the Structural Division*, 102(5):1135–1140, 1976.

[15] Andrew Packard and John Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.

[16] Richard P Braatz, Peter Michael Young, John C Doyle, and Manfred Morari. Computational complexity of/spl mu/calculation. *IEEE Transactions on Automatic Control*, 39(5):1000–1002, 1994.

[17] Michael KH Fan, André L Tits, and John C Doyle. Robustness in the presence of joint parametric uncertainty and unmodeled dynamics. In *1988 American control conference*, pages 1195–1200. IEEE, 1988.

[18] Peter M Young, Matthew P Newlin, and John C Doyle. Practical computation of the mixed μ problem. In *1992 American Control Conference*, pages 2190–2194. IEEE, 1992.

[19] Andy Packard, Michael KH Fan, and John Doyle. A power method for the structured singular value. In *IEEE Conf. on Decision and Control*, pages 2132–2137, 1988.

[20] Peter M Young. Theoretical and computational aspects of the structured singular value. *Systems, control and information*, 38(3):p129–138, 1994.

[21] Bo Bernhardsson, Anders Rantzer, and Li Qiu. Real perturbation values and real quadratic forms in a complex vector space. *Linear algebra and its applications*, 270(1-3):131–154, 1998.

[22] Jie Chen, Michael KH Fan, and Carl N Nett. Structured singular values with non-diagonal structures. i. characterizations. *IEEE transactions on automatic control*, 41(10):1507–1511, 1996.

[23] Jie Chen, Michael KH Fan, and Carl N Nett. Structured singular values with nondiagonal structures. ii. computation. *IEEE transactions on automatic control*, 41(10):1511–1516, 1996.

[24] Diederich Hinrichsen and Anthony J Pritchard. Texts in applied mathematics. 2005.

[25] Michael Karow. μ-values and spectral value sets for linear perturbation classes defined by a scalar product. *SIAM journal on matrix analysis and applications*, 32(3):845–865, 2011.

[26] Michael Karow, Diederich Hinrichsen, and Anthony J Pritchard. Interconnected systems with uncertain couplings: Explicit formulae for mu-values, spectral value sets, and stability radii. *SIAM journal on control and optimization*, 45(3):856–884, 2006.

[27] Li Qiu, Bo Bernhardsson, Anders Rantzer, Edward J Davison, and JC Doyle. A formula for computation of the real stability radius. 1993.

[28] IS Khalil, JC Doyle, and K Glover. *Robust and optimal control*, volume 2. Prentice hall New York, 1996.

[29] Kenneth J Arrow and Maurice McManus. A note on dynamic stability. *Econometrica: Journal of the Econometric Society*, pages 448–454, 1958.

[30] David Carlson. A class of positive stable matrices. *J. Res. Nat. Bur. Standards Sect. B*, 78:1–2, 1974.

[31] Christina A Bahl and Bryan E Cain. The inertia of diagonal multiples of $3\times$ 3 real matrices. *Linear Algebra and its Applications*, 18(3):267–280, 1977.

[32] Bryan E Cain. Real, $3\times$ 3, d-stable matrices. *J. Res. Nat. Bur. Standards Sect. B*, 80:75–77, 1976.

[33] Daniel Hershkowitz. Recent directions in matrix stability. *Linear Algebra and its Applications*, 171:161–186, 1992.

[34] Charles R Johnson. Sufficient conditions for d-stability. *Journal of Economic Theory*, 9(1):53–62, 1974.

[35] Jie Chen, Michael KH Fan, and Cheng-Ching Yu. On d-stability and structured singular values. *Systems & control letters*, 24(1):19–24, 1995.

[36] Jietae Lee and Thomas F Edgar. Real structured singular value conditions for the strong d-stability. *Systems & control letters*, 44(4):273–277, 2001.

[37] Mutti-Ur Rehman, Tulkin H Rasulov, and Fouzia Amir. D-stability, strong d-stability and-values. *Lobachevskii Journal of Mathematics*, 45(3):1227–1233, 2024.

[38] Alisher Shadiyev Mutti-Ur Rehman. Interconnection between h-stable, $d(alpha)$-stable, d-semistable matrices, and μ-values. *Asia Pac. J. Math*, (11):102.

[39] Mutti-Ur Rehman et al. Spectrum and pseudspectrum of d-stable matrices of economy models. *J. Math. Computer Sci*, 38:298–312, 2025.

[40] Mutti-Ur Rehman, Behkzod Aminov, Mohammed N Alshehri, Mustafa M Mohammed, Arafa O Mustafa, Nhla A Abdalrahman, Mona Magzoub, Hala S Mahgoub, Sakeena EM Hamed, Runda AA Bashir, et al. Spectral properties of structured matrices in transportation problems. *European Journal of Pure and Applied Mathematics*, 18(1):5637–5637, 2025.

[41] Mutti-Ur Rehman, Sakeena EM Hamed, Nidal E Taha, Arafa O Mustafa, Khurshidbek Dilmurodov, Hala S Mahgoub, Mona Magzoub, Runda AA Bashir, Mustafa M Mohammed, and Awad A Bakery. Analysis of stability, d-stability, and pseudospectra in economic modeling. *European Journal of Pure and Applied Mathematics*, 18(1):5657–5657, 2025.

[42] M Rehman, J Alzabut, M Tayyab, and F Amir. Interconnection between schur stability and structured singular values. *Contemporary Mathematics*, pages 63–72, 2025.

[43] Eyad H Abed. Strong d-stability. *Systems & control letters*, 7(3):207–212, 1986.

[44] Wasfi Kafri. Robust d-stability. 2001.

[45] Siddu Shiralasetti et al. Some results on haar wavelets matrix through linear algebra. *Wavelet and Linear Algebra*, 4(2):49–59, 2017.

[46] Firdous A Shah and R Abbas. Haar wavelet operational matrix method for the numerical solution of fractional order differential equations. *Nonlinear Engineering*,

4(4):203–213, 2015.