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Weakly Connected Independence Number of a Graph

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Abstract. Let G be a simple undirected connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. The subgraph $\langle S\rangle_w$ of $S \subseteq V(G)$ is the graph whose vertex set is $N[S]$ and whose edge set E_w consists of edges in $E(G)$ incident to some vertex in S. A subset S of $V(G)$ is a *weakly connected set* of G if $\langle S\rangle_w$ is connected. S is called a *weakly connected independent set* (WCIS) of G if it is both weakly connected and independent. In this paper, we characterize the weakly connected independent sets in the join, corona, and the lexicographic product of two graphs. From these characterizations the weakly connected independence numbers of the corresponding graphs are easily determined. Also, characterization of graphs G with weakly connected independence numbers $\alpha_w(G)$ equal to 1, $n-1$ and n are given. It is also shown that for any non-negative integers k, m, and n with $k > m+1$ and $n \geq k+m+2$, there exists a connected graph G such that $|V(G)| = n$, $\alpha_w(G) = k$ and $\alpha(G) = k+m$.

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Key Words and Phrases: Independent, weakly connected set, weakly connected independence number, join, corona, lexicographic

1. Introduction

The concept of weakly connected domination was introduced by Grossman [1] and was studied by Dunbar et.al [2] where upper and lower bounds for $\gamma_w(G)$ were obtained. This parameter extends domination by ensuring weak connectivity within the dominating set.

Graph theory plays a vital role in modeling networks, where balancing independence and connectivity is crucial. Weakly connected domination has been widely studied, with works by Alzoubi et al. [3] on minimal sets and Bendali et al. [4] on computational

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complexity. Recent studies, such as those by Hamja et al. [5] and Militante and Eballe [6], further explored variations in weakly connected parameters.

Motivated by the growing interest in weak connectivity and its role in network theory, this paper introduces and investigates the weakly connected independence number. The study of this new parameter involves addressing the inherent difficulty of combining independence and connectivity, two properties that often counteract each other in graph structures. Similar to weakly connected domination, we believe that understanding the weakly connected independence number will yield significant contributions to independence theory and stimulate further research in graph operations, and combinatorial optimization.

2. Terminologies and Notations

Let $G = (V(G), E(G))$ be a simple undirected graph. The distance between two vertices $v, w \in V(G)$, denoted $d_G(v, w)$, is the length of a shortest v-w path connecting v and w. Any v-w path of length $d_G(v, w)$ is called a v-w geodesic. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, while its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = S \cup N_G(S)$. Any $v \in V(G)$ with $|N_G(v)| = 0$ is called an *isolated vertex*. Vertex v is a *leaf* or an *endvertex* if $|N_G(v)| = 1$. A vertex w of G is a *support vertex* if $wv \in E(G)$ for some leaf v in G. The sets $I(G)$, $L(G)$, and $S(G)$ will, respectively, denote the sets containing all the isolated vertices, leaves, and support vertices in G.

A subset A of $V(G)$ is an *independent* set if for every pair of distinct vertices in G do not form an edge. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the *independence number* of G. Any independent set with cardinality equal to $\alpha(G)$ is called an α-set in G.

A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. It is a *super dominating set* if for every $v \in V(G) \setminus S$ there exists $w \in S$ such that $N_G(w) \cap [V(G) \setminus S] = \{v\}$. The domination number (super domination number) of G, denoted $\gamma(G)$ (resp. $\gamma_{sp}(G)$) is the minimum cardinality of a dominating (resp. super dominating) set in G. Any dominating set (super dominating set) with cardinality $\gamma(G)$ (resp. $\gamma_{sp}(G)$) is called a γ-set (resp. γ_{sp}-set).

The subgraph $\langle S \rangle_w$ of $S \subseteq V(G)$ is the graph whose vertex set is $N_G[S]$ and whose edge set E_w consists of edges in $E(G)$ incident to some vertex in S. A subset S of $V(G)$ is a *weakly connected set* of G if $\langle S \rangle_w$ is connected. S is called a *weakly connected independent set* (WCIS) of G if it is both weakly connected and independent. The maximum cardinality of a WCIS of G is called the *weakly connected independence number* of G and is denoted by $\alpha_w(G)$. A WCIS of G having cardinality $\alpha_w(G)$ is called a *maximum* WCIS of G. A set S that is both a WCIS and a dominating set of G is called a *weakly connected independent dominating set* (WCIDS) of G. Note that a minimum WCIDS of G always exists (see Dunbar et al. [2]). Denote by $\iota_c(G)$ the cardinality of a minimum WCIDS (or ι_G-set) of G. Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set

$V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. Denote by H^v the copy of H in $G \circ H$, every vertex of which is adjacent to a unique vertex $v \in G$. The *lexicographic product* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$, and $(u, u')(v, v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$. Observe that any non-empty subset C of $V(G) \times V(H)$ (in fact, any set of ordered pairs) can be written as $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$. Henceforth, we shall use this form to denote any subset C of $V(G) \times V(H)$.

Readers are referred to [7] for other basic definitions that are not given here.

3. Results

It is worth noting that if G is a graph and $S \subseteq V(G)$, then $\langle S \rangle_w$ is simply obtained from the $\langle N[S] \rangle$ by deleting all the edges $e = xy$ in $E(G)$ with $x, y \in N(S) \setminus S$.

The first result is easy and almost follows from the definitions.

Theorem 1. For any graph G of order n, $1 \leq \alpha_w(G) \leq \alpha(G)$. Moreover,

(*i*) $\alpha_w(G) = 1$ if and only if every component H of G is complete; and

(*ii*) $\alpha_w(G) = \alpha(G)$ if and only if $\langle S \rangle_w$ is connected for some α-set Sin G.

Proof. Clearly, $1 \leq \alpha_w(G)$. Since every weakly connected independent set is independent, it follows that $\alpha_w(G) \leq \alpha(G)$.

(*i*) Suppose that $\alpha_w(G) = 1$ and assume on the contrary that G has a component H which is not complete. Then there exist distinct vertices v and w of H such that $d_H(v, w) = d_G(v, w) = 2$. Let $S = \{v, w\}$ and let $u \in N_G(v) \cap N_G(w)$. Then S is independent and $\langle S \rangle_w$ is connected. This implies that $\alpha_w(G) \geq |S| = 2$, contrary to our assumption that $\alpha_w(G) = 1$. Thus, every component H of G is complete.

For the converse, suppose that every component of G is complete. Let S be an α_w-set in G. Since $\langle S \rangle_w$ is connected, $S \subseteq V(H)$ for a unique complete component H of G. Since S is an independent set in H, it follows that $|S| = 1$. Thus, $\alpha_w(G) = |S| = 1$.

(*ii*) Suppose that $\alpha_w(G) = \alpha(G)$, say S is an α_w-set in G. Then S is an independent set and and $\langle S \rangle_w$ is connected. Since $|S| = \alpha(G)$, it follows that S is an α-set in G.

Conversely, suppose $\langle S \rangle_w$ is connected for some α-set S in G. Then S is a weakly connected independent set. Therefore, $\alpha_w(G) = |S| = \alpha(G)$. □

The next result follows from Theorem 1.

Corollary 1. Let G be a connected graph of order n. Then $\alpha_w(G) = 1$ if and only if $G = K_n$.

We now characterize all connected graphs G of order $n \geq 2$ such that $\alpha_w(G) = n - 1$.

Theorem 2. Let G be a connected graph of order $n \geq 2$. Then $\alpha_w(G) = n - 1$ if and only if $G = K_{1,n-1}$.

Proof. Assume that G is a connected graph of order $n \geq 2$ and $\alpha_w(G) = n - 1$. If $n = 2$, then $\alpha_w(G) = 1$. By Corollary 1, $G = K_2 = K_{1,1}$. Suppose $n \geq 3$. Let S be an α_w-set of G, say $S = V(G) \setminus \{w\}$. Since S is independent, it follows that $uv \notin E(G)$ for every pair of vertices $u, v \in S$. Now, since S is weakly connected in G, it follows that $w \in N_G(v)$ for each $v \in S$. Consequently, $G = K_{1,n-1}$.

For the converse, suppose that $G = K_{1,n-1}$. Let w be the central vertex of G. Then, clearly, $S = V(G) \setminus \{w\}$ is a weakly connected independent set in G. Since S is also an α-set in G, it follows from Theorem 1(ii) that $\alpha_w(G) = n - 1$. □

Theorem 3. Let k, m, and n be non-negative integers with $k > m+1$ and $n \geq k+m+2$. Then there exists a connected graph G such that $|V(G)| = n$, $\alpha_w(G) = k$ and $\alpha(G) = k + m$.

Proof. Let us consider the following cases:
Case 1. Suppose that $n = k + m + 2$.

Suppose first that $m = 0$. Let $H_1 = K_{1,k}$, where u is the central vertex of H_1 and $v_1, v_2, \ldots, v_k$ are the remaining vertices (see Figure 1). Let G be the graph obtained from H_1 by adding the vertex x and the edges xu and xv_k. Clearly, the set $S_1 = \{v_1, v_2, \ldots, v_k\}$ is both an α-set and an α_w-set of G. Thus, $|V(G)| = k + 2 = n$, $\alpha_w(G) = \alpha(G) = k$.

Figure 1: Graph G with $\alpha_w(G) = \alpha(G) = k$

Next, suppose that $m > 0$. Let H_2 be the union of $K_{1,k-1}$ and $K_{1,m+1}$, where u is the central vertex of $K_{1,k-1}$ and $x_1, x_2, \ldots, x_{k-1}$ are its remaining vertices, v is the central vertex of $K_{1,m+1}$ and $y_1, y_2, \ldots, y_{m+1}$ are its remaining vertices. Let G be the graph obtained from H_2 by adding the edge uv (see Figure 2). The set $S_2 = \{x_1, x_2, \ldots, x_{k-1}\} \cup \{y_1, y_2, \ldots, y_{m+1}\}$ is the unique α-set of G and $S_3 = \{x_1, x_2, \ldots, x_{k-1}, v\}$ is an α_w-set of G. Hence, $|V(G)| = k + m + 2 = n$, $\alpha_w(G) = |S_3| = k$ and $\alpha(G) = |S_2| = k + m$.

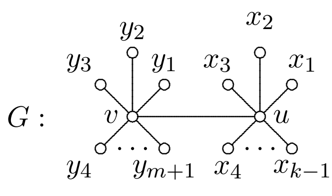

Figure 2: Graph G with $\alpha_w(G) = k$ and $\alpha(G) = k + m$

Case 2. Suppose that $n > k + m + 2$.

Consider the graphs G_1 and G_2 in Figure 3. Suppose that $m = 0$.

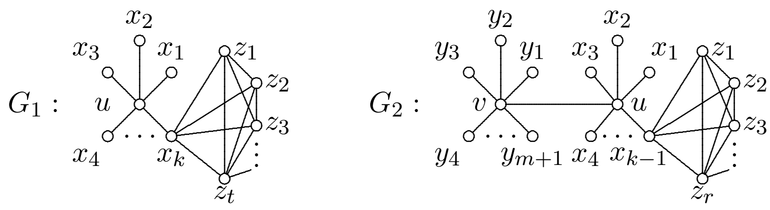

Figure 3: Graph G with $\alpha_w(G) = k$ and $\alpha(G) = k + m$

Let $t = n - k - 1$ and take $G = G_1$. Then, clearly, $S_4 = \{x_1, x_2, \ldots, x_k\}$ is a both an α-set and an α_w-set of G. Thus, $|V(G)| = k + t + 1 = n$, $\alpha_w(G) = \alpha(G) = k$. If $m > 0$, then set $r = n - k - m - 2$ and take $G = G_2$. It can easily be verified that $|V(G)| = (k - 1) + (m + 1) + r + 2 = n$, $\alpha_w(G) = k$ and $\alpha(G) = k + m$. □

The next result is a consequence of Theorem 3.

Corollary 2. The difference $\alpha(G) - \alpha_w(G)$ can be made arbitrarily large.

Proof. Let m, n, and k be positive integers such that $k > m+1$ and $n = k+m+2$. By Theorem 3, there exists a connected graph with $|V(G)| = n$, $\alpha_w(G) = k$ and $\alpha(G) = k+m$. Therefore, $\alpha(G) - \alpha_w(G) = m$. □

Next, we give the weakly connected independence number of paths and cycles.

Theorem 4. (i) $\alpha_w(P_n) = \left\lceil \frac{n}{2} \right\rceil = \alpha(P_n)$ for all $n \geq 1$.

(ii) $\alpha_w(C_n) = \left\lfloor \frac{n}{2} \right\rfloor = \alpha(C_n)$ for all $n \geq 3$.

Proof. (i) Let $P_n = [v_1, v_2, \ldots, v_n]$. If n is even, then $S_E = \{v_i \in V(P_n) : i \text{ is even}\}$ and $S_O = \{v_j \in V(P_n) : j \text{ is odd}\}$ are α-sets in P_n. Since $\langle N_G[S_E]\rangle = \langle N_G[S_O]\rangle = P_n$, it follows that S_E and S_O are weakly connected independent sets in P_n. By Theorem 1(ii), we have $\alpha_w(P_n) = \alpha(P_n) = |S_E| = \frac{n}{2}$. If n is odd, then $S = \{v_j \in V(P_n) : j \text{ is odd}\}$ is the unique α-set in P_n. Again, since $\langle N_G[S]\rangle = P_n$, it follows that S is a weakly connected independent set in P_n. By Theorem 1(ii), we have $\alpha_w(P_n) = |S| = \frac{n+1}{2}$.

(ii) Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$. If n is even, the $S_1 = \{v_1, v_3, \cdots, v_{n-1}\}$ is an α-set in C_n. Since $\langle N_G[S_1]\rangle = C_n$, it follows that S_1 is a weakly connected independent set in C_n. By Theorem 1(ii), we have $\alpha_w(C_n) = \alpha(C_n) = |S_1| = \frac{n}{2}$. If n is odd, then $S_2 = \{v_1, v_3, \cdots, v_{n-2}\}$ is an α-set in C_n. Since $\langle N_G[S]\rangle = P_n$, it follows that S_2 is a weakly connected independent set in C_n. By Theorem 1(ii), we have $\alpha_w(C_n) = \alpha(C_n) = |S| = \frac{n-1}{2}$. □

In what follows, we characterize the WCIS in $G + H$ and determine the weakly connected independent number of $G + H$.

Theorem 5. Let G and H be any two graphs. Then S is a WCIS in $G+H$ if and only if either S is an independent set in G or S is an independent set in H.

Proof. Assume that S is a WCIS in $G+H$. Since S is an independent set in $G+H$, it follows that either $S \subseteq V(G)$ or $S \subseteq V(H)$. Thus, S is an independent set in G or an independent set in H.

Conversely, let S be an independent set in G. Clearly, S is an independent set of $G+H$. Since $V(H) \subseteq N_{G+H}(x)$ for every $x \in N_G[S]$, it follows that $\langle N_{G+H}[S]\rangle$ is connected. This implies that $\langle S\rangle_w$ is weakly connected in $G+H$. Similarly, $\langle S\rangle_w$ is weakly connected in $G+H$ if S be an independent set in G. □

Corollary 3. Let G and H be graphs. Then

$$\alpha_w(G+H) = \max\{\alpha(G), \alpha(H)\} = \alpha(G+H).$$

Example 1. Let G be any graph and m be a positive integer. Then

(*i*) $\alpha_w(\overline{K}_m + G) = \max\{m, \alpha(G)\}$,

(*ii*) $\alpha_w(K_m + G) = \alpha(G)$,

(*iii*) $\alpha_w(K_{m,n}) = \alpha_w(\overline{K}_m + \overline{K}_n) = \max\{m, n\}$.

The next result characterizes the WCIS of $G \circ H$.

Theorem 6. Let G be a connected graph and H be any graph. A subset S of $V(G \circ H)$ is a WCIS in $G \circ H$ if and only if one of the following holds:

(*i*) S is an independent set in H^v for some $v \in V(G)$.

(*ii*) $S = C \cup (\cup_{v \in N_G(C)} S_v)$, where

  (*a*) C is a WCIS in G, and

  (*b*) S_v is an independent set (may be empty) in H^v for each $v \in N_G(C)$.

Proof. Suppose S is a WCIS in $G \circ H$. Then S is an independent set in $G \circ H$ and $\langle S\rangle_w$ is connected. Consider the following cases:

Case 1: $V(G) \cap S = \varnothing$.
Then $S \subseteq \cup_{u \in V(G)} V(H^u)$. Since $\langle S\rangle_w$ is connected, it follows that S is in exactly one of the components of $\langle \cup_{u \in V(G)} V(H^u)\rangle$, that is, $S \subseteq V(H^v)$ for a unique vertex $v \in V(G)$. Consequently, S is an independent subset of $V(H^v)$. This shows that (*i*) holds.

Case 2: $V(G) \cap S \neq \varnothing$.
Let $C = V(G) \cap S$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Since S is a WCIS in $G \circ H$, C is a WCIS in G. Let $v \in V(G)$ such that $S_v \neq \varnothing$. Then S_v is an independent set in H^v

and $v \notin C$ because S is an independent set in $G \circ H$. Moreover, since $\langle S \rangle_w$ is connected, $v \in N_G(C)$. Thus, $S = C \cup (\cup_{v \in} S_v)$ and (a) and (b) hold.

For the converse, suppose first that (i) holds, that is, suppose that S is an independent set of H^v for a unique $v \in V(G)$. Then, clearly, S is a WCIS of $G \circ H$. Next, suppose that (ii) holds, i.e., $S = C \cup (\cup_{v \in X \subseteq N_G(C)} S_v)$ and satisfies (a) and (b). Since $C \cap N_G(C) = \varnothing$ and C and S_v and are independent sets, S is an independent set in $G \circ H$. Moreover, since $\langle C \rangle_w$ is connected in G and $v \in N_G(C)$ for each non-empty set S_v, it follows that $\langle S \rangle_w$ is connected in $G \circ H$. Thus, S is a WCIS in $G \circ H$.

$$\alpha_w(G \circ H) \geq |S'| = \leq \alpha(H)|V(G)| + (1 - \alpha(H))\iota_c(G).$$

□

Corollary 4. Let G be a connected graph and H be any graph. Then $\alpha_w(G \circ H) = \alpha(H)$ if $G = K_1$. Otherwise,

$$\alpha_w(G \circ H) = \alpha(H)(|V(G)| + (1 - \alpha(H))\iota_c(G)).$$

Proof. Clearly, $\alpha_w(K_1 \circ H) = \alpha_w(K_1 + H) = \alpha(H)$ (see Example 1(ii)). . So suppose $G \neq K_1$ and let S be an α_w-set in $G \circ H$. Then $S = C \cup (\cup_{v \in N_G(C)} S_v)$, where C is a WCIS in G and S_v is an independent set of H^v for each $v \in N_G(C)$ by Theorem 6. Suppose C is not a dominating set of G. Then $V(G) \setminus N_G[C] \neq \varnothing$. Choose $w \in V(G) \setminus N_G[C]$ such that $wx \in E(G)$ for some $x \in N_G(C)$. Then $C^* = C \cup \{w\}$ is a WCIS in G and $N_G(C^*) = N_G(C) \cup N_G(w)$. Let L_v be an α-set in H^v for each $v \in N_G(C^*)$. Then, by Theorem 6, $S^* = C^* \cup (\cup_{v \in N_G(C^*)} L_v)$ is a WCIS of $G \circ H$. This implies that $\alpha_w(G \circ H) = |S| < |S^*|$ which is not possible. Therefore, C is a WCIDS of G. From this and the fact that $1 - \alpha(H) \leq 0$, we have

$$\begin{aligned} \alpha_w(G \circ H) = |S| &\leq |C| + \alpha(H)(|V(G)| - |C|) \\ &= \alpha(H)|V(G)| + (1 - \alpha(H))|C| \\ &\leq \alpha(H)|V(G)| + (1 - \alpha(H))\iota_c(G). \end{aligned}$$

Next, let C_0 be a minimum WCIDS of G and let S_v be an α-set in H^v for each $v \in N_G(C_0)$. Then, by Theorem 6, $S' = C_0 \cup (\cup_{v \in N_G(C_0)} S_v)$ is a WCIS of $G \circ H$. Hence,

$$\alpha_w(G \circ H) \geq |C| = \leq \alpha(H)|V(G)| + (1 - \alpha(H))\iota_c(G).$$

This proves the desired equality. □

4. Lexicographic product of graphs

Sandueta and Canoy characterized the weakly connected sets in the lexicographic product of graphs.

Theorem 7. *[8]* Let G and H be connected non-trivial graphs and let $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$. Then C is weakly connected in $G[H]$ if and only if S is a weakly connected in G

The next result characterizes WCIS in $G[H]$.

Theorem 8. Let G and H be connected non-trivial graphs and let $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$. Then C is WCIS in $G[H]$ if and only if S is a WCIS of G and T_x is an independent set of H for each $x \in S$.

Proof. Suppose $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$ and that C is a WCIS in $G[H]$. By Theorem 7, S is weakly connected in G. Next, let $x, y \in S$ such that $x \neq y$. Pick $a \in T_x$ and $b \in T_y$. Then $(x, a), (y, b) \in C$ and $(x, a) \neq (y, b)$. Since C is independent, $(x, a)(y, b) \notin E(G[H])$. This implies that $xy \notin E(G)$. Thus, S is independent in G. Hence, S a WCIS in G. Now, let $x \in S$ and let $c, d \in T_x$ such that $c \neq d$. Then $(x, c), (x, d) \in C$ and $(x, c) \neq (x, d)$. Since C is an independent set of $G[H]$, we have $(x, c)(x, d) \notin E(G[H])$. This implies that $cd \notin E(H)$. Thus, T_x is an independent set of H.

Conversely, assume that S is WCIS in G and T_x is independent in H for all $x \in S$. By Theorem 7, C is a weakly connected set in $G[H]$. Let $(x, a), (y, b) \in C$ such that $(x, a) \neq (y, b)$. Suppose $x \neq y$. Since S is an independent set in G, $xy \notin E(G)$. Thus, $(x, a)(y, b) \notin E(G[H])$. Now, assume $x = y$. Then, $a, b \in T_x$ and $a \neq b$. Since T_x is an independent set of H, $ab \notin E(H)$. Hence, $(x, a)(y, b) \notin E(G[H])$. Thus, C is an independent set of $G[H]$. Therefore, C is a weakly connected independent set in $G[H]$. □

Corollary 5. Let G and H be connected non-trivial graphs. Then $\alpha_w(G[H]) = \alpha_w(G)\alpha(H)$.

Proof. Let $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, be an α_w-set in $G[H]$. By Theorem 8, S is a WCIS in G and T_x is an independent set in H for every $x \in S$. Hence,

$$\alpha_w(G[H]) = |C| = |\Sigma_{x\in S}(\{x\} \times T_x)| \leq \alpha_w(G)\alpha(H).$$

Next, let S be an α_w-set in G and A be an α-set in H. For each $x \in S$, let $T_x = A$. By Theorem 8, $C = \cup_{x\in S}(\{x\} \times T_x)$ is a WCIS in $G[H]$. Consequently,

$$\alpha_w(G[H]) \geq |C| = |\Sigma_{x\in S}(\{x\} \times T_x)| = \alpha_w(G)\alpha(H).$$

Therefore, $\alpha_w(G[H]) = \alpha_w(G)\alpha(H)$. □

5. Conclusion

The concept of weakly connected independent set as well as the parameter weakly connected independence number were introduced and initially investigated in this study.

Graphs for which the weakly connected independence number and independence number are equal were characterized. It was shown that the difference between these two parameters can be made arbitrarily large. We also characterized the weakly connected independent sets in the join, corona, and lexicographic product of graphs and determined their respective weakly connected independence number. The newly defined parameter may be explored further for many other graphs. Interested readers may also investigate the complexity of the WCIS problem.

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