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Inverse Boundary Value Problem for Pseudo Hyperbolic Equation of the Fourth Order With Nonlocal Integral Conditions of the Second Kind

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Abstract. In this paper, we consider a nonlinear inverse boundary value problem for a fourth-order pseudo hyperbolic equation with nonlocal conditions of the integral type. First, we introduce the definition of a classical solution to the problem. The purpose of this paper is to determine the unknown coefficient of the right-hand side and to solve the problem of interest. The problem is considered in a rectangular domain. To study the solvability of the inverse problem, we perform a transformation from the original problem to some auxiliary inverse problem with trivial boundary conditions. Using the principle of contraction mappings, we prove the existence and uniqueness of solutions to the auxiliary problem. Then we again perform a transformation to the problem and as a result obtain the solvability of the inverse problem.

2020 Mathematics Subject Classifications: 35R30, 35L80, 35A01, 35A02, 35A09

Key Words and Phrases: Inverse boundary problem, Pseudo hyperbolic equation, Method Fourier, Classic solution

1. Introduction

The foundations of the theory and practice of studying inverse problems were laid and developed in the pioneering works [1], [2], [3], [4]. Subsequently, the applied significance of inverse problems attracted the attention of many authors, and in recent decades numerous articles and monographs devoted to inverse problems have been published (see, for example, [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19] and the literature cited therein).

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2. Problem Statement on Determining the Unknown Coefficient and the Constant Term

In the rectangle $\Pi_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ we will consider the following problem:

$$\omega_{tt}(x,t) - \omega_{ttxx}(x,t) + \omega_{xxxx}(x,t)$$
$$= \varphi(t)\omega(x,t) + \psi(t)r(x,t) + s(x,t) \quad (x,t) \in \Pi_T, \tag{1}$$

$$\omega(x,0) = \varepsilon(x) + \int_0^T \sigma_1(t)\omega(x,t)dt,$$
$$\omega_t(x,0) = \nu(x) + \int_0^T \sigma_2(t)\omega(x,t)dt \quad (0 \le x \le 1), \tag{2}$$

$$\omega(0,t) = \omega_x(1,t) = \omega_{xx}(0,t) = \omega_{xxx}(1,t) = 0 \quad (0 \le t \le T), \tag{3}$$

$$\omega(x_i,t) = n_i(t) \quad (x_i \in (0,1) \quad i = 1,2; \quad x_1 \ne x_2 \quad 0 \le t \le T), \tag{4}$$

where $r(x,t)$, $s(x,t)$, $\varepsilon(x)$, $\nu(x)$, $\sigma_i(t)$, $n_i(t)$ $(i = 1,2)$ given functions, and $\omega(x,t)$, $\varphi(t)$, $\psi(t)$-sought functions.

Let us designate

$$V^{4,2}(\Pi_T) = \{\omega(x,t) : \omega(x,t) \in C^2(\Pi_T), \omega_{xxxx}(x,t), \omega_{ttxx}(x,t \in C(\Pi_T)\}.$$

Definition 1. *We call the triple of* $\{\omega(x,t),\ \varphi(t),\ \psi(t)\}$ *functions* $\omega(x,t)$, $\varphi(t)$ *and* $\psi(t)$ *a classical solution of problem* (1)-(4) *if:*

(i) $\omega(x,t) \in V^{4,2}(\Pi_T)$;

(ii) $\varphi(t) \in C[0,T]$ *and* $\psi(t) \in C[0,T]$;

(iii) functions $\omega(x,t)$, $\varphi(t)$ *and* $\psi(t)$ *satisfy all conditions* (1)-(4) *in the usual sense.*

Let us consider the following problem: Find a triple of $\{\omega(x,t),\ \varphi(t),\ \psi(t)\}$ *functions* $\omega(x,t) \in V^{4,2}(\Pi_T)$, $\varphi(t) \in C[0,T]$ *and* $\psi(t) \in C[0,T]$ *from relations* (1)-(3), *and*

$$h_i''(t) - \omega_{ttxx}(x_i,t) + \omega_{xxxx}(x_i,t) =$$
$$= \varphi(t)n_i(t) + \psi(t)r(x_i,t) + s(x_i,t) \quad (i = 1,2;\ 0 \le t \le T). \tag{5}$$

Similarly ([1]) the following is proved

Theorem 1. *Let* $s(x,t),\ r(x,t) \in C(D_T)$, $\varepsilon(x), v(x) \in C[0,1]$, $n(t) \equiv n_1(t)r(x_2,t) - n_2(t)r(x_1,t) \ne 0$ $(0 \le t \le T)$, $n_i(t) \in C^2[0,T]$ $(i = 1,2)$, $\sigma_i(t) \in C[0,T]$, *and the matching conditions*

$$n_i(0) = \int_0^T \sigma_1(t)n_i(t)dt + \varepsilon(x_i), \quad n_i'(0) = \int_0^T \sigma_2(t)n_i(t)dt + v(x_i), \quad i = 1,2,$$

are satisfied.

Then the following assertions are valid:

i) each classical solution $\{\omega(x,t),\, \varphi(t),\, \psi(t)\}$ *of the problem* (1)-(4) *is a solution of problem* (1)-(3), (5), *as well;*

ii) each solution $\{\omega(x,t),\, \varphi(t),\, \psi(t)\}$ *of the problem* (1)-(3), (5), *if*

$$\left(T \,\| \,\sigma_2(t)\|_{C[0,T]} + \|\sigma_1(t)\,\|_{C[0,T]} + \frac{T}{2}\,\|\varphi(t)\|_{C[0,T]}\right)\, T < 1,$$

is a classical solution of problem (1)-(4).

3. Study of Solvability of the Inverse Problem

It is obvious that component $\omega(x,t)$ of the solution $\{\omega(x,t), \varphi(t), \psi(t)\}$ of (1)-(3), (5) has the form:

$$\omega(x,t) = \sum_{k=1}^{\infty} \omega_k(t)\sin\lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right), \tag{6}$$

where $\omega_k(t) = 2\int_0^1 \omega(x,t)\sin\lambda_k x dx \;\; (k=1,2,...)$-Fourier coefficients of component $\omega(x,t)$ in the field in the $L_2(0,1)$ system $\left(\lambda_k = \frac{\pi}{2}(2k-1)\right)_{k=1}^{\infty}$.

Then applying the formal Fourier scheme, from (1) and (2) we obtain

$$(1+\lambda_k^2)\omega_k''(t) + \lambda_k^4\omega_k(t) = Q_k(t;\omega,\varphi,\psi) \;\; (0 \le t \le T; \;\; k=1,2,...), \tag{7}$$

$$\omega_k(0) = \varepsilon_k + \int_0^T \sigma_1(t)\omega_k(t)dt, \;\; \omega_k'(0) = v_k + \int_0^T \sigma_2(t)\omega_k(t)dt \;\; (k=1,2,...), \tag{8}$$

where

$$Q_k(t;\omega,\varphi,\psi) \;= \varphi(t)\omega_k(t) + \psi(t)r_k(t) + s_k(t),$$

$$s_k(t) = 2\int_0^1 s(x,t)\sin\lambda_k x\, dx, \;\; r_k(t) = 2\int_0^1 r(x,t)\sin\lambda_k x\, dx,$$

$$\varepsilon_k = 2\int_0^1 \varepsilon(x)\sin\lambda_k x dx, \;\;\; v_k = 2\int_0^1 v(x)\sin\lambda_k x dx \;\; (k=1,2,...).$$

It is easy to see that the solution to problem (7)-(8) has the form:

$$\omega_k(t) = \left(\varepsilon_k + \int_0^T \sigma_1(t)\omega_k(t)dt\right)\cos\beta_k t \, + \, \frac{1}{\beta_k}\left(v_k + \int_0^T \sigma_2(t)\omega_k(t)dt\right)\sin\beta_k t +$$

$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t Q_k(\tau;\omega,\varphi,\psi)\sin\beta_k(t-\tau)d\tau \;\; (k=1,2,...), \tag{9}$$

where

$$\beta_k^2 = \frac{\lambda_k^4}{1+\lambda_k^2} \;\; (k=1,2,...).$$

Now, after substituting expression $\omega_k(t)$ $(k = 1, 2, ...)$ to determine $\omega(x, t)$, we have:

$$\omega(x,t) = \sum_{k=1}^{\infty} \left(\varepsilon_k + \int_0^T \sigma_1(t)\omega_k(t)dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(v_k + \int_0^T \sigma_2(t)\omega_k(t)dt\right)\sin\beta_k t +$$

$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t Q_k(\tau;\omega,\varphi,\psi)\sin\beta_k(t-\tau)d\tau\Bigg\}\sin\lambda_k x. \tag{10}$$

Next, using equation (9), from (5) and (6) we obtain:

$$\varphi(t) = [n(t)]^{-1}\{(n_1''(t) - s(x_1,t))r(x_2,t) - (n_2''(t) - s(x_2,t))r(x_1,t) +$$

$$+\sum_{k=1}^{\infty}\beta_k^2\left[\left(\varepsilon_k + \int_0^T \sigma_1(t)\omega_k(t)dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(v_k + \int_0^T \sigma_2(t)\omega_k(t)dt\right)\sin\beta\right.$$

$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t Q_k(\tau;\omega,\varphi,\psi)\sin\beta_k(t-\tau)d\tau +$$

$$\left.+\frac{1}{\lambda_k^2}Q_k(\tau;\omega,\varphi,\psi)\right]\left(r(x_2,t)\sin\lambda_k x_1 - r(x_1,t)\sin\lambda_k x_2\right)\Bigg\}, \tag{11}$$

$$\psi(t) = [n(t)]^{-1}\{(n_2''(t) - s(x_2,t))n_1(t) - (n_1''(t) - s(x_1,t))n_2(t) +$$

$$+\sum_{k=1}^{\infty}\beta_k^2\left[\left(\varepsilon_k + \int_0^T \sigma_1(t)\omega_k(t)dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(v_k + \int_0^T \sigma_2(t)\omega_k(t)dt\right)\sin\beta +\right.$$

$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t Q_k(\tau;\omega,\varphi,\psi)\sin\beta_k(t-\tau)d\tau +$$

$$\left.+\frac{1}{\lambda_k^2}Q_k(\tau;\omega,\varphi,\psi)\right]\left(r(x_2,t)\sin\lambda_k x_1 - r(x_1,t)\sin\lambda_k x_2\right)\Bigg\} +$$

$$\left.+\frac{1}{\lambda_k^2}Q_k(\tau;\omega,\varphi,\psi)\right]\left(n_1(t)\sin\lambda_k x_2 - n_2(t)\sin\lambda_k x_1\right)\Bigg\}. \tag{12}$$

To study the problem of the uniqueness of the solution of problem (1)-(3), (5), the following lemma plays an important role.

Lemma 1. *If - $\{\omega(x,t),\ \varphi(t),\ \psi(t)\}$ any solution of problem* (1)-(3), (5)*, then the function* $\omega_k(t) = 2\int_0^1 \omega(x,t)\sin\lambda_k x dx$ $(k = 1, 2, ...)$*, i.e. the Fourier coefficients* $\omega(x,t)$ *in the system* $\left(\lambda_k = \frac{\pi}{2}(2k-1)\right)_{k=1}^{\infty}$ *satisfy the* $[0,T]$*system* (9).

This lemma implies the validity of the following

Corollary 1. *Let system* (10), (11), (12) *have a unique solution. Then problem* (1)-(3), (5) *cannot have more than one solution, i.e. if problem* (1)-(3), (5) *has a solution, then it is unique.*

In order to study the problem (1)-(3), (5), we define the following spaces.

Denote by $B_{2,T}^5$ [20], [21] the set of all functions $\omega(x,t)$ of the form

$$\omega(x,t) = \sum_{k=1}^{\infty} \omega_k(t) \sin \lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right),$$

defined on Π_T, where each of the functions $\omega_k(t) \in C[0,T]$ $(k = 1,2,\ldots)$ and

$$J_T(\omega) \equiv \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|\omega_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} < +\infty.$$

The norm in this space is defined as

$$\|\omega(x,t)\|_{B_{2,T}^5} = J(\omega).$$

By E_T^5 we denote the space of the vector functions $\{\omega(x,t),\, \varphi(t),\, \psi(t)\}$ such that $\omega(x,t) \in B_{2,T}^5$, $\varphi(t), \psi(t) \in C[0,T]$, and equip this space by the norm

$$\|\eta\|_{E_T^5} = \|\omega(x,t)\|_{B_{2,T}^5} + \|\varphi(t)\|_{C[0,T]} + \|\psi(t)\|_{C[0,T]}.$$

Clearly, $B_{2,T}^5$ and E_T^5 are Banach spaces.

Now we consider in E_T^5 the operator

$$H(\omega,\varphi,\psi) = \{H_1(\omega,\varphi,\psi), H_2(\omega,\varphi,\psi), H_3(\omega,\varphi,\psi)\},$$

where $H_1(\omega,\varphi,\psi) = \tilde{\omega}(x,t) \equiv \sum\limits_{k=1}^{\infty} \tilde{\omega}_k(t) \sin \lambda_k x,\ H_2(\omega,\varphi,\psi) = \tilde{\varphi}(t)$,
$H_3(\omega,\varphi,\psi) = \tilde{\psi}(t)$,$\tilde{\omega}_k(t$ ($k = 1,2,...$), $\tilde{\varphi}(t)$ and $\tilde{\psi}(t)$ are the right hand sides of (9) and (13), (14) correspondingly.

Now, let the given problems satisfy the following conditions:

1.$\varepsilon(x) \in C^4[0,1]$, $\varepsilon^{(5)}(x) \in L_2(0,1)$,
$\varepsilon(0) = \varepsilon'(1) = \varepsilon''(0) = \varepsilon'''(1) = \varepsilon^{(4)}(0) = 0$;
2.$v(x) \in C^2[0,1]$, $v^{(4)}(x) \in L_2(0,1)$, $v(0) = v'(1) = v'''(0) = v'''(1) = 0$;
3.$s(x,t), s_x(x,t) \in C(D_T)$, $s_{xx}(x,t) \in L_2(D_T)$, $s(0,t) = s_x(1,t) = 0$ $(0 \le t \le T)$;
4.$r(x,t), r_x(x,t) \in C(D_T)$, $r_{xx}(x,t) \in L_2(D_T)$, $r(0,t) = r_x(1,t) = 0$ $(0 \le t \le T)$;
5.$\sigma_i(t) \in C[0,T]$, $n_i(t) \in C^2[0,T]$ $(i = 1,2)$,
$n(t) \equiv n_1(t) r(x_2,t) - n_2(t) r(x_1,t) \neq 0$ $(0 \le t \le T)$.

Now, from (15)-(17) we find:

$$\|\tilde{\omega}(x,t)\|_{B_{2,T}^5} \le A_1(T) + B_1(T) \|\varphi(t)\|_{C[0,T]} \|\omega(x,t)\|_{B_{2,T}^5} +$$

$$+C_1(T)\left\|\omega(x,t)\right\|_{B^5_{2,T}}+D_1(T)\left\|\psi(t)\right\|_{C[0,T]}, \tag{13}$$

$$\left\|\tilde{\varphi}(t)\right\|_{C[0,T]}\leq A_2(T)+B_2(T)\left\|\varphi(t)\right\|_{C[0,T]}\left\|\omega(x,t)\right\|_{B^5_{2,T}}+$$
$$+C_2(T)\left\|\omega(x,t)\right\|_{B^5_{2,T}}+D_2(T)\left\|\psi(t)\right\|_{C[0,T]}, \tag{14}$$

$$\left\|\tilde{\psi}(t)\right\|_{C[0,T]}\leq A_3(T)+B_3(T)\left\|\varphi(t)\right\|_{C[0,T]}\left\|\omega(x,t)\right\|_{B^5_{2,T}}+$$
$$+C_3(T)\left\|\omega(x,t)\right\|_{B^5_{2,T}}+D_3(T)\left\|\psi(t)\right\|_{C[0,T]}, \tag{15}$$

where

$$A_1(T)=\sqrt{7}\left\|\varepsilon^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{14}\left\|v^{(4)}(x)\right\|_{L_2(0,1)}+$$
$$+\sqrt{10T}\left\|s_{xx}(x,t)\right\|_{L_2(D_T)},\ B_1(T)=\sqrt{14}T,$$

$$C_1(T)=\sqrt{14}T\left(\left\|\sigma_1(t)\right\|_{C[0,T]}+\left\|\sigma_2(t)\right\|_{C[0,T]}\right),\ D_1(T)=\sqrt{10T}\left\|r_{xx}(x,t)\right\|_{L_2(D_T)},$$

$$A_2(T)=\left\|[n(t)]^{-1}\right\|_{C[0,T]}\{\left\|(n_1''(t)-s(x_1,t))r(x_2,t)-(n_2''(t)-s(x_2,t))r(x_1,t)\right\|_{C[0,T]}+$$
$$+2\left\||r(x_2,t)|+|r(x_1,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\varepsilon^{(5)}(x)\right\|_{L_2(0,1)}+\right.$$
$$\left.\left.+\sqrt{2}\left\|v^{(4)}(x)\right\|_{L_2(0,1)}+\sqrt{2T}\left\|s_{xx}(x,t)\right\|_{L_2(D_T)}+\left\|\left\|s_{xx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0.1)}\right]\right\},$$

$$B_2(T)=2\left\|[n(t)]^{-1}\right\|_{C[0,T]}\left\||r(x_2,t)|+|r(x_1,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(T+1),$$

$$C_2(T)=2\left\|[n(t)]^{-1}\right\|_{C[0,T]}\left\||r(x_2,t)|+|r(x_1,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$
$$\times T\left(\left\|\sigma_1(t)\right\|_{C[0,T]}+\left\|\sigma_2(t)\right\|_{C[0,T]}\right),$$

$$D_2(T)=2\left\|[n(t)]^{-1}\right\|_{C[0,T]}\left\||r(x_2,t)|+|r(x_1,t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$
$$\times\left(\sqrt{2T}\left\|r_{xx}(x,t)\right\|_{L_2(D_T)}+\left\|\left\|r_{xx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0.1)}\right),$$

$$A_3(T)=\left\|[n(t)]^{-1}\right\|_{C[0,T]}\{\left\|(n_2''(t)-s(x_2,t))n_1(t)-(n_1''(t)-s(x_1,t))n_2(t)\right\|_{C[0,T]}+$$
$$+2\left\||n_2(t)|+|n_1(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\varepsilon^{(5)}(x)\right\|_{L_2(0,1)}+\sqrt{2}\left\|v^{(4)}(x)\right\|_{L_2(0,1)}+\right.$$

$$+\sqrt{3T}\left\|s_{xx}(x,t)\right\|_{L_2(D_T)}+\left\|\left\|s_{xx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0.1)}\Bigg]\Bigg\},$$

$$B_3(T)=2\left\|[n(t)]^{-1}\right\|_{C[0,T]}\left\||n_2(t)|+|n_1(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(T+1),$$

$$C_2(T)=2\left\|[n(t)]^{-1}\right\|_{C[0,T]}\left\||n_2(t)|+|n_1(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\times T\left(\left\|\sigma_1(t)\right\|_{C[0,T]}+\left\|\sigma_2(t)\right\|_{C[0,T]}\right),$$

$$D_3(T)=2\left\|[n(t)]^{-1}\right\|_{C[0,T]}\left\||n_2(t)|+|n_1(t)|\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\times\left(\sqrt{2T}\left\|r_{xx}(x,t)\right\|_{L_2(D_T)}+\left\|\left\|r_{xx}(x,t)\right\|_{C[0,T]}\right\|_{L_2(0.1)}\right).$$

From inequalities (18)-(20) we conclude

$$\left\|\tilde{\omega}(x,t)\right\|_{B^{5,3}_{2,T}}+\left\|\tilde{\varphi}(t)\right\|_{C[0,T]}+\left\|\tilde{\psi}(t)\right\|_{C[0,T]}+$$

$$+B(T)\left\|\varphi(t)\right\|_{C[0,T]}\left\|\omega(x,t)\right\|_{B^{5}_{2,T}}+C(T)\left\|\omega(x,t)\right\|_{B^{5}_{2,T}}+D(T)\left\|\psi(t)\right\|_{C[0,T]}, \tag{16}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),\quad B(T)=B_1(T)+B_2(T)+B_3(T),$$
$$C(T)=C_1(T)+C_2(T)+C_3(T),\quad D(T)=D_1(T)+D_2(T)+D_3(T).$$

So, we can prove the following theorem.

Theorem 2. *Let conditions 1-5 be satisfied and*

$$(A(T)+2)(B(T)(A(T)+2)+C(T)+D(T))<1. \tag{17}$$

The problem (1)-(3), (5) has a unique solution in the ball $K=K_R(||\eta||_{E^5_T}\leq R=A(T)+2)$ *of the space* E^5_T.

Proof. In the space E^5_T consider the equation

$$\eta=H\eta, \tag{18}$$

where $\eta=\{\omega,\varphi,\psi\}$, the components $H_i(\omega,\varphi,\psi)\ \ (i=1,2,3)$ of the operator $H(\omega,\varphi,\psi)$ are defined by the right hand sides of equations (10), (11) and (12).

Consider the operator $H(\omega,\varphi,\psi)$ in the ball $K=K_R$ from E^5_T. Similarly to (18) we obtain that the estimations

$$\left\|H\eta\right\|_{E^5_T}\leq A(T)+B(T)\left\|\varphi(t)\right\|_{C[0,T]}\left\|\omega(x,t)\right\|_{B^5_{2,T}}+C(T)\left\|\omega(x,t)\right\|_{B^5_{2,T}}+$$

$$+D(T)\,\|\psi(t)\|_{C[0,T]} \leq A(T) + (A(T)+2)(B(T)(A(T)+2) + C(T) + D(T)), \tag{19}$$

$$\|H\eta_1 - H\eta_2\|_{E_T^5} \leq B(T)(A(T)+2)\left(\|\omega_1(x,t)-\omega_2(x,t)\|_{B_{2,T}^5} + \|\varphi_1(t)-\varphi_2(t)\|_{C[0,T]}\right)$$
$$+C(T)\,\|\omega_1(x,t)-\omega_2(x,t)\|_{B_{2,T}^5} + D(T)\,\|\psi_1(t)-\psi_2(t)\|_{C[0,T]}\,, \tag{20}$$

for the arbitrary η, η_1, $\eta_2 \in K_R$. Taking into account (17), from estimates (19), (20) it follows that the operator H acts in the ball $K = K_R$ and is contracting. Therefore in the ball $K = K_R$ the operator H has a single fixed point $\{\omega, \varphi, \psi\}$ which is a unique solution to equation (18) in the ball $K = K_R$, i.e. $\{\omega, \varphi, \psi\}$ is a unique solution to system (10), (11) and (12) in the ball $K = K_R$.

The function $\omega(x,t)$ as an element of the space $B_{2,T}^5$ has continuous derivatives $\omega_x(x,t)$, $\omega_{xx}(x,t)$, $\omega_{xxx}(x,t)$, $\omega_{xxxx}(x,t)$, in Π_T.

Similarly, [17] it can be shown that $\omega_{tt}(x,t)$, $\omega_{ttx}(x,t)$, $\omega_{ttxx}(x,t)$, are continuous in Π_T.

It is easy to verify that equation (1) and conditions (2), (3) and (5) are satisfied in the usual sense. Therefore, $\{\omega(x,t),\, \varphi(t),\, \psi(t)\}$ is a solution to problem (1)-(3), (5), and, by virtue of the corollary of Lemma 1, it is unique in the ball $K = K_R$. The theorem is proved.

Using Theorem 1, we prove the following

Theorem 3. *Let all conditions of Theorem 2 be satisfied and*

$$n_i(0) = \int_0^T \sigma_1(t) n_i(t)dt + \varepsilon(x_i), \quad n_i'(0) = \int_0^T \sigma_2(t) n_i(t)dt + v(x_i), \quad i = 1, 2,$$

$$\left(T\,\|\,\sigma_2(t)\|_{C[0,T]} + \|\sigma_1(t)\,\|_{C[0,T]} + \frac{T}{2}(A(T)+2)\right) T < 1.$$

Then problem (1)-(4) *has unique classical solution in the ball* $K = K_R(||\eta||_{E_T^5} \leq R = A(T)+2)$ *from* E_T^5.

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