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Total Exact Domination in Graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple nontrivial undirected graph. A set $T \subseteq V(G)$ is said to be a total exact dominating set if T is both a total dominating set and an exact dominating set of G. The cardinality of a minimum total exact dominating set is called the total exact domination number and is denoted by $\gamma_{te}(G)$. The total exact domination numbers of various types of special graphs, such as path, cycle, star, complete bipartite, and graphs resulting from binary operations such as join, corona, and lexicographic product are obtained in this study. Furthermore, if G does not have a total exact dominating set, then G is called a $non-\gamma_{te}$-graph. Examples include complete graphs, fan graphs, and wheel graphs with more than two vertices. We also consider some disconnected graphs in the corona and lexicographic product, making the study more interesting. In defining total exact domination, a condition of exact domination was modified because it contradicted the definition of total domination.

2020 Mathematics Subject Classifications: 05C69,05C38, 05C76

Key Words and Phrases: Total domination, exact domination, total exact domination

1. Introduction

Graph domination is a thoroughly researched area in graph theory, which is essential for optimizing networks, managing resources, and modeling social networks. The basic concept of a dominating set and its associated domination number was initially presented by O. Ore in 1962 in his foundational paper on Graph Theory [1]. Since then, various forms of domination have been studied, each offering unique insights into the structural

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properties of graphs. Graph domination is a well-explored topic within graph theory, playing a crucial role in fields such as network optimization, resource management, and social network modeling. Cockayne, Dawes, and Hedetniemi [2] were among the first to formally define total domination in graphs in 1980. In their study, a set $T \subsetneq V(G)$, where $V(G)$ is the vertex set of G, was defined as a total dominating set if every vertex in $V(G)$ is adjacent to at least one vertex in T. Building upon this definition, we explore its intersection with exact domination in this paper.

The idea of exact domination in graphs was formally put forth by Kinsley and Joeshi in 2020, building upon earlier work in domination theory [3]. Where $N(v)$ denotes the set of all vertices adjacent to v, a set $T \subseteq V(G)$ is called an exact dominating set if it satisfies the following conditions:

i $|N(v) \cap T| = 1$ for all $v \in V(G) \setminus T$ and

ii $|N(u) \cap T| \leq 1$ for all $u \in T$.

In this paper, we introduce an integration of total and exact domination. Combining the two domination conditions will modify the second condition for an exact dominating set, changing $|N(u) \cap T| \leq 1$ to $|N(u) \cap T| = 1$, since $|N(u) \cap T| < 1$ for all $u \in T$ contradicts the definition of a total dominating set. Furthermore, not all graphs have a total exact dominating set; we denote them as non-γ_{te}-graphs.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple nontrivial undirected graph where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. The set of neighbors of a vertex u in G is called the *open neighborhood* of u in G and is denoted by $N_G(u) = N(u) = \{v \in V(G) : uv \in E(G)\}$ and the closed neighborhood of u is the set $N[u] = N(u) \cup \{u\}$. The open neighborhood of a subset T of $V(G)$ is the set $N_G(T) = N(T) = \cup_{v \in T} N_G(v)$ and its closed neighborhood is the set $N_G[T] = N[T] = N(T) \cup T$ [4]. A set $T \subseteq V(G)$ is a dominating set (resp. total dominating set) of G if $N[T] = V(G)$ (resp. $N(T) = V(G)$). The domination number $\gamma(G)$ (resp. total domination number $\gamma_t(G)$) of G is the minimum cardinality of a dominating set (resp. total dominating set) [4]. A set that has a cardinality of $\gamma_t(G)$ is a *γ_t-set.* A set $T \subseteq V(G)$ is said to be an *exact dominating set* if (i) $|N(v) \cap T| = 1 \; \forall \; v \in V(G) \setminus T$ and (ii) $|N(u) \cap T| \leq 1 \; \forall \; u \in T$. The minimum cardinality of an exact dominating set is the *exact domination number* of the graph denoted by $\gamma_e(G)$. A set that has a cardinality of $\gamma_e(G)$ is a $\gamma_{\mathbf{e}}$-set [3].

A set $T \subseteq V(G)$ is said to be the *total exact dominating set* if it satisfies (i) $|N(v) \cap T| = 1 \; \forall \; v \in V(G) \setminus T$ and (ii) $|N(u) \cap T| = 1 \; \forall \; u \in T$. Combining the two conditions, we can express it concisely as: $|N(u) \cap T| = 1 \; \forall \; u \in V(G)$. The cardinality of a minimum total exact dominating set is *the total exact domination number*, denoted by $\gamma_{te}(G)$. A set that has a cardinality of $\gamma_{te}(G)$ is a *γ_{te}-set.* Graph G is considered a $non - \gamma_{te} - graph$ if it does not contain a total exact dominating set, following a definition analogous to that of a non-γ_{p0}-graph as presented in [5].

The *join* of two graphs, G and H, denoted by $G+H$, is the graph with vertex set $V(G+H)=V(G)\cup V(H)$ and edge set

$$E(G+H)=E(G)\cup E(H)\cup\{uv : u\in V(G), v\in V(H)\}\ [6].$$

The *corona* $G\circ H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and then forming the join $\langle\{v\}\rangle+H^v=v+H^v$, where H^v is a copy of H, for each $v\in V(G)$ [6].

The *lexicographic product* or *composition* of two graphs G and H is the graph $G[H]$ with vertex set $V(G[H]) = V(G)\times V(H)$ and edge set $E(G[H]) = \{(x,u)(y,v) \mid xy\in E(G) \text{ or } x=y \text{ and } uv\in E(H)\}$. Any subset (C) of $(V(G[H]))$ can be expressed as $C=\bigcup_{x\in S}(\{x\}\times T_x)$, where $S\subseteq V(G)$ and $T_x\subseteq V(H)$ for each $x\in S$. The set S is called the G-projection of C, and $\bigcup_{x\in S}T_x$ is called the H-projection of C [7].

The *path graph* P_n is a tree with two nodes of vertex degree 1, and the other $n-2$ nodes of vertex degree 2 [8]. A *cycle graph* C_n are defined as simple, connected, undirected graphs consisting of a single cycle passing through all n vertices, with each vertex having degree 2 and the graph forming a closed loop [9]. The *complete graph* K_m has every pair of its m points adjacent [7]. The *complement of complete graphs* $\overline{K}_m$ are totally disconnected graphs and are regular of degree 0 [7]. A *complete bipartite graph* $K_{m,n}$ is a graph equivalent to $\overline{K}_m+\overline{K}_n$ such that $n,m\geq 1$ [6]. The *star graph* S_n, of order n, is a tree on n vertices with one vertex of degree $n-1$ and the remaining $n-1$ vertices of degree 1 [7]. A *wheel graph* W_n, of order n, is formed by adding a vertex adjacent to all vertices of a cycle [7]. A *fan graph* F_n is formed by joining a path to a single vertex [7]. The *generalized fan graph* $F_{m,n}$ is defined as the graph join $\overline{K}_m+P_n$, where $\overline{K}_m$ is the *empty graph* on m vertices and P_n is the path graph on n vertices [10]. The generalized wheel graph $W_{m,n}$ is a graph obtained by joining the vertices of $\overline{K}_m$ to every vertex of a cycle C_n. That is, $W_{m,n}=C_n+\overline{K}_m$ [11]. A *windmill graph* W_n^m is a graph formed by connecting m copies of a complete graph K_{n-1} to a single common vertex [9]. The *friendship graph* F_n is a graph consisting of n triangles that share a common central vertex. It is also called the *Dutch windmill graph* [12].

Example 1. *Let G be the graph in Figure 1 and let*

$$T=\{t_4,u_1,u_4,u_5,u_8,u_9,r_3,r_4,w_3,w_4,z_1,x_1\}.$$

The vertices in T are colored red, and the vertices in $V(G)\setminus T$ are colored blue. Clearly, each blue vertex is adjacent to exactly one red vertex in T, that is, each vertex in $V(G)\setminus T$ is adjacent to exactly one vertex in T or $|N(v)\cap T|=1\ \forall v\in V(G)\setminus T$. Observe further that each red vertex is adjacent to exactly one red vertex in T, that is, each vertex in T are adjacent to exactly one vertex in T or $|N(u)\cap T|=1\ \forall u\in T$.

Thus, T is a total exact dominating set of G and $\gamma_{te}(G) = |T| = 12$. In Figure 2, clearly, $S = \{u_1, u_4, u_5, u_8, u_9, r_3, w_3, z_1, x_1\}$ is a γ_e-set of G since $|N(u) \cap S| = 1\ \forall v \in V(G) \setminus S$ and $|N(v) \cap S| \leq 1\ \forall v \in S$ with $|N(u_1) \cap S| = |N(r_3) \cap S| = |N(w_3) \cap S| = 0$. Thus, $\gamma_e(G) = |S| = 9$.

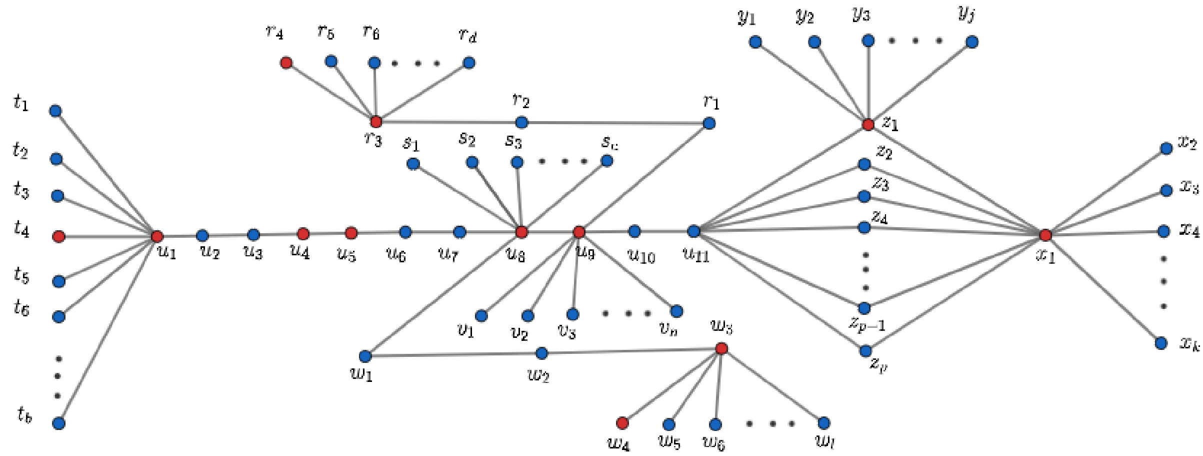


Figure 1: Graph G with $\gamma_{te}(G) = 12$.

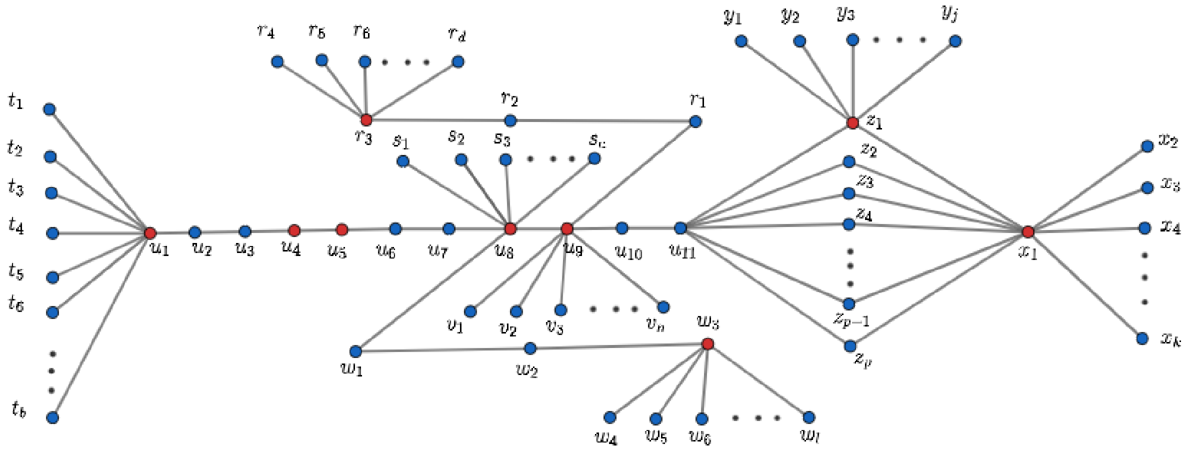


Figure 2: Graph G with $\gamma_e(G) = 9$.

3. Known Results

The following results will be used to prove the main results.

Theorem 1. [2]. *For any connected graph G with $p \geq 3$ vertices, then*

$$\gamma_t(G) \leq \frac{2p}{3}.$$

Proposition 1. [6] *The total domination number of a cycle C_n or a path P_n on $n \geq 3$ vertices are given by*

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod 4, \\ \frac{n+2}{2}, & \text{if } n \equiv 2 \pmod 4, \\ \frac{n+1}{2}, & \text{otherwise.} \end{cases}$$

Theorem 2. [13] *Let G and H be connected graphs. Then $C \subseteq V(G+H)$ is a total dominating set of $G+H$ if and only if it satisfies at least one of the following:*

(i) $C \cap V(G)$ *is a total dominating set of* G.

(ii) $C \cap V(H)$ *is a total dominating set of* H.

(iii) $C \cap V(G) \neq \emptyset$ *and* $C \cap V(H) \neq \emptyset$.

Corollary 1. [13] *Let G and H be connected graphs. Then $C = \{x, y\}$, where $x \in V(G)$ and $y \in V(H)$, is a minimum total dominating set of $G + H$ and $\gamma_t(G + H) = 2$.*

Corollary 2. [14] *Let G be a connected graph of order m and let H be any graph of order n. Then $\gamma_t(G \circ H) = m$.*

4. Main Results

This section contains results corresponding to the total exact domination number of paths, cycles, complete bipartite graphs and graphs resulting from some binary operations. Furthermore, some $non-\gamma_{te}-graphs$ are shown.

Clearly, every total exact dominating set of G is an exact dominating set and a total dominating set. Thus, the next statement is immediate from this observation, definition of total exact dominating set and Theorem 1.

Remark 1. *Let G be any graph such that G has a total exact dominating set with $p \geq 3$ vertices. Then*

$$\gamma_e(G) \leq \gamma_{te}(G) \text{ and } 2 \leq \gamma_{te}(G) \leq \frac{2p}{3}.$$

Theorem 3. *For any positive integers p and q with $1 \leq p \leq q$, there exists a simple graph G such that $\gamma_e(G) = p$ and $\gamma_{te}(G) = q$.*

Proof. Consider the following cases:
Case 1: $p = q$
Let G be the graph shown in Figure 3. Clearly, the set

$$T = \{u_1, u_2, u_3, u_4, u_5, u_6, \ldots, u_{p-1}, u_p\}$$

is both a γ_e-set and a γ_{te}-set of G. Therefore, $\gamma_e(G) = p = q = \gamma_{te}(G)$.

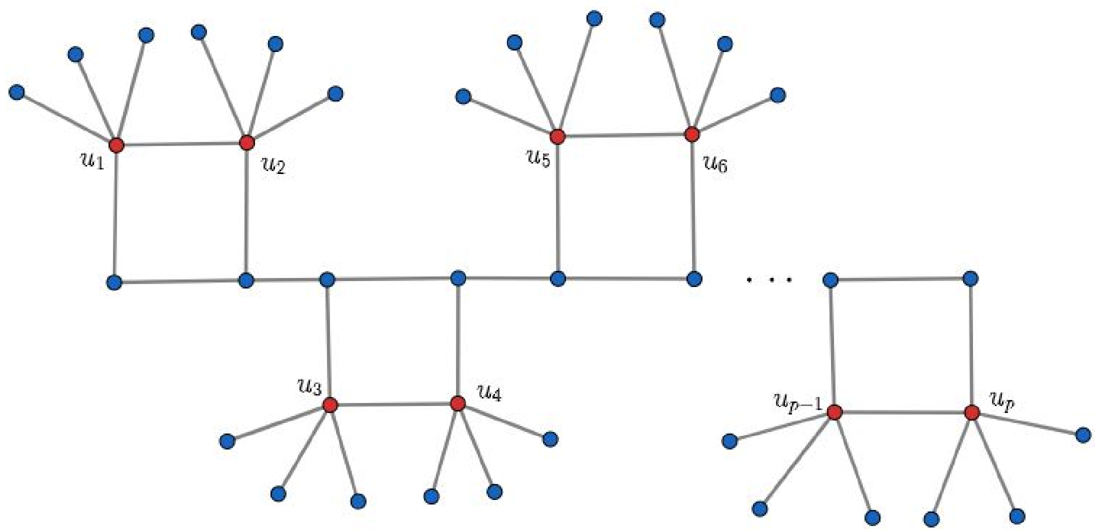


Figure 3: Graph G with $\gamma_{te}(G) = \gamma_e(G) = p$.

Case 2: $p < q$
Let G be the graph shown in Figure 4 or Figure 5. Let $m = q - p$ and $m \in \mathbb{Z}^+$. Observe that

$$S_1 = \{u_1, u_2, u_3, \ldots, u_{p-m}\} \cup \{v_i : i = 1, 2, \ldots, m\}$$

is a γ_e-set of G, and

$$S_2 = \{u_1, u_2, u_3, \ldots, u_{p-m}\} \cup \{v_i : i = 1, 2, \ldots, m\} \cup \{w_i : i = 1, 2, \ldots, m\}$$

is a γ_{te}-set of G. It follows that $\gamma_e(G) = |S_1| = p - m + m = p$ and

$$\gamma_{te}(G) = |S_2| = (p - m) + m + m = p + m = p + (q - p) = q.$$

Therefore, $\gamma_e(G) = p < q = \gamma_{te}(G)$.

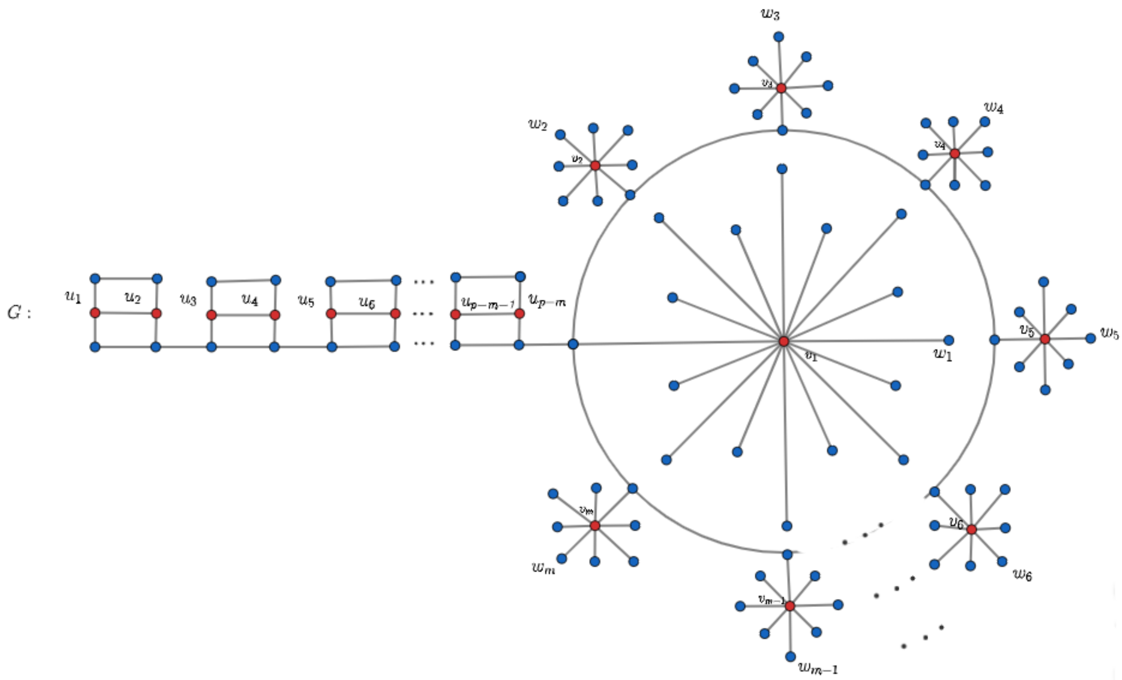


Figure 4: Graph G with $\gamma_e(G) = p$.

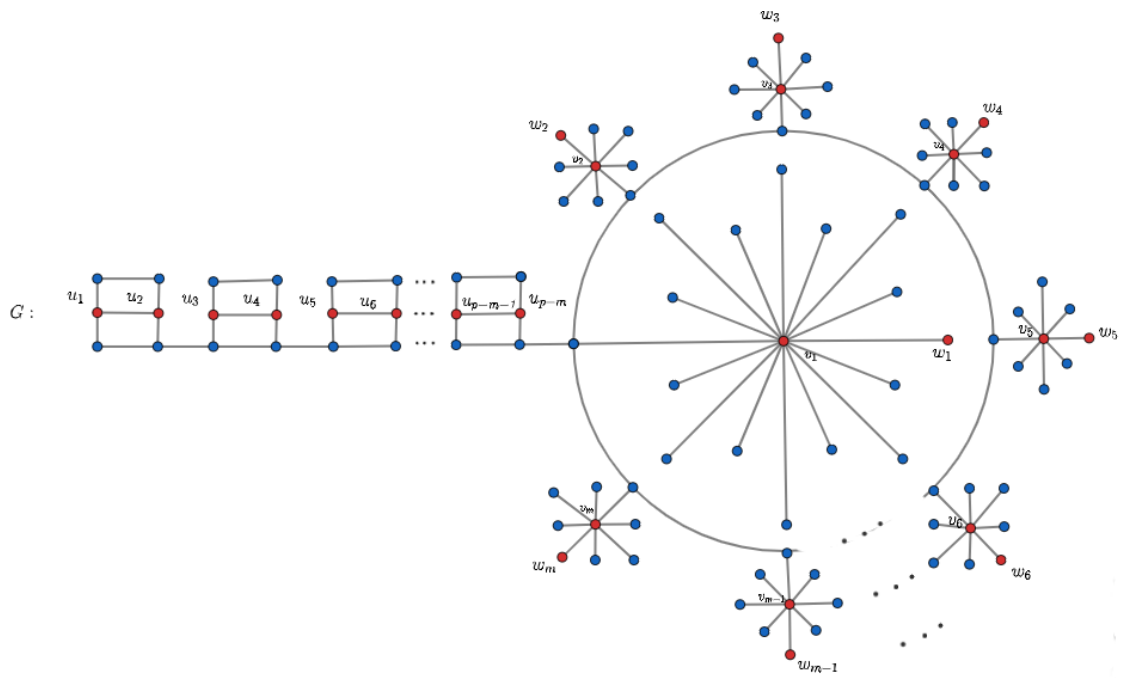


Figure 5: Graph G with $\gamma_{te}(G) = q$.

This confirms the claim. □

The next result follows from Theorem 3.

Corollary 3. *The difference $\gamma_{te} - \gamma_e$ can be made arbitrarily large.*

Theorem 4. *If T is a total exact dominating set of a graph G, then $|T|$ is even.*

Proof. Let T be a total exact dominating set with n vertices. By definition, $|N(v) \cap T| = 1 \ \forall \ v \in T$. Thus, for every vertex $v \in T$, there exists exactly one other vertex $x \in T$ such that $N(v) \cap T = \{x\}$ and $N(x) \cap T = \{v\}$.Thus the vertices of T can be paired uniquely into disjoint sets of two. Therefore, $|T|$ must be even. □

Theorem 5. *If T is a γ_t-set of a graph G and $|T|$ is odd, then T is not a γ_{te}-set of G.*

Proof. Let T be a γ_t-set of a graph G and $|T|$ is odd. Then there exists three vertices, say u_i, u_j, u_k such that the vertex u_j is adjacent to u_i and u_k by definition of total dominating set. Hence, $|N(u_j) \cap T| = 2$, a contradiction to the definition of exact dominating set. Therefore, T is not a γ_{te}-set of G. □

Theorem 6. *Let n be a positive integer such that $n \geq 5$. If $n \equiv 1 \pmod 4$, then the path graph P_n is $non - \gamma_{te} - graph$.*

Proof. Let $n = 5$. By Proposition 1, $\gamma_t(P_5) = \frac{5+1}{2} = 3$. Let $T_1 = \{u_2, u_5\}$ and $T_2 = \{u_2, u_4\}$. Clearly, T_1 and T_2 are not total dominating sets since $\gamma_t(P_5) = 3$. Therefore, they are not γ_{te}-sets. It is also clear that $T_3 = \{u_2, u_3, u_4\}$ is the only γ_t-set of P_5 and $|T_3|$ is odd. By Theorem 5, T_3 is not a $\gamma_{te} - set$ of P_5. Let $T_4 = \{u_1, u_2, u_4, u_5\}$. Clearly, $|N(u_i) \cap T_4| = 1$ for $u_i \in V(P_5) \setminus \{u_3\}$ and $|N(u_3) \cap T_4| = 2$. Thus, T_4 is not a $\gamma_{te} - set$ of P_5. Since there is no possible way to create a set that is both total and exact dominating set, P_5 is a $non - \gamma_{te} - graph$.

Now, suppose that $n > 5$. Let $p = \frac{n-1}{4}$ and $j = 1, 2, \ldots, p$. Group the vertices of P_n into p disjoint subsets R_j, such that

$$\begin{aligned} R_1 &= \{u_1, u_2, u_3, u_4\}, \\ R_2 &= \{u_5, u_6, u_7, u_8\}, \\ R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\}, \\ &\vdots \\ R_{p-1} &= \{u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}, \text{ and} \\ R_p &= \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\} \end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1$ and $|R_p| = 5$.
Case 1: Suppose that we pick u_2 and u_3 first to form T. Let

$$T = \{u_2, u_3, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-7}, u_{n-6}, u_{n-3}, u_{n-2}, u_{n-1}\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \dots, p-1$ and 3 vertices in R_p. Thus, $|T| = 2(p-1) + 3 = 2p + 1 = 2\left(\frac{n-1}{4}\right) + 1 = \frac{n+1}{2}$, and clearly, $N(T) = V(P_n)$. By Proposition 1, T is a γ_t-set of P_n. Also, since $|T|$ is odd, by Theorem 5, T is not a γ_{te}-set of P_n, and u_{n-1} must not be in T. Removing u_{n-1} from T or replacing u_{n-1} by u_n in T means that u_n is not adjacent to a vertex in T, that is, $|N(u_n) \cap T| = 0$ and $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_n) \setminus \{u_n\}$. Thus, T is not a total exact dominating set. Hence, it is not possible to form a γ_{te}-set T of P_n.

Case 2: Suppose that we pick u_1 and u_2 first to form T. Let

$$T = \{u_1, u_2, u_5, u_6, u_9, u_{10}, \dots, u_{n-8}, u_{n-7}, u_{n-4}, u_{n-3}, u_{n-1}, u_n\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \dots, p-1$ and 4 vertices in R_p. Clearly, $|T|$ is even and $N(T) = V(P_n)$. Thus, T is a total dominating set of P_n. It is also clear that $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_n) \setminus \{u_{n-2}\}$ since u_{n-2} is adjacent to u_{n-3} and u_{n-1} in T, that is, $|N(u_{n-2}) \cap T| = 2$ and so, T is not an exact dominating set. Therefore, T is not a γ_{te}-set and u_{n-1} must not be in T. Removing u_{n-1} from T means that u_n is not adjacent to a vertex in T, that is, $|N(u_n) \cap T| = 0$ and $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_n) \setminus \{u_n\}$. Thus, T is not a total exact dominating set. Hence, it is not possible to form a γ_{te}-set T of P_n.

In both cases, it is not possible to create a set that is both a total and exact dominating set of P_n. Therefore, P_n is a non-γ_{te}-graph if $n \equiv 1 \pmod 4$. □

Theorem 7. *Let n be a positive integer such that $n \geq 2$. Then the total exact domination number of a path P_n of order n, where $n \not\equiv 1 \pmod 4$, is given by*

$$\gamma_{te}(P_n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+2}{2}, & \text{if } n \equiv 2 \pmod 4 \\ \frac{n+1}{2}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $V(P_n) = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n\}$. Consider the following cases:
Case 1: Suppose that $n \equiv 0 \pmod 4$.
Let $n = 4$. By Proposition 1, $\gamma_t(P_4) = \frac{4}{2} = 2$. Let $T = \{u_2, u_3\}$. Clearly, $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_4)$. Thus, T is a γ_{te}-set of P_4, and $\gamma_{te}(P_4) = |T| = 2$.

Now, suppose that $n > 4$. Let $p = \frac{n}{4}$ and $j = 1, 2, \dots, p-1, p$. Group the vertices of P_n into p disjoint subsets R_j, such that

$$\begin{aligned}
R_1 &= \{u_1, u_2, u_3, u_4\}, \\
R_2 &= \{u_5, u_6, u_7, u_8\}, \\
R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\}, \\
&\vdots \\
R_{p-1} &= \{u_{n-7}, u_{n-6}, u_{n-5}, u_{n-4}\}, \text{ and} \\
R_p &= \{u_{n-3}, u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1, p$.
Let

$$T = \{u_2, u_3, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-6}, u_{n-5}, u_{n-2}, u_{n-1}\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \ldots, p-1, p$. It follows that

$$|T| = 2p = 2\left(\frac{n}{4}\right) = \frac{n}{2}.$$

Also, note that $N(T) = V(P_n)$. Thus, T is a γ_t-set of P_n by Proposition 1. It is also clear that
$|N(u_i) \cap T| = 1$ for all $u_i \in V(P_n)$. Therefore, T is a γ_{te}-set. This implies that

$$\gamma_{te}(P_n) = |T| = \frac{n}{2}.$$

Case 2: Suppose that $n \equiv 2 \pmod 4$.

When $n = 2$, $S = \{u_1, u_2\}$ is a γ_{te}-set of P_2 since $|N(u_1) \cap S| = 1$ and $|N(u_2) \cap S| = 1$. Thus, $\gamma_{te}(P_2) = |S| = 2$. Let $n = 6$. By Proposition 1, $\gamma_t(P_6) = \frac{6+2}{2} = 2$. Let $T = \{u_1, u_2, u_5, u_6\}$. Clearly, $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_6)$. Thus, T is a γ_{te}-set of P_6, and $\gamma_{te}(P_6) = |T| = 4$.

Now, suppose that $n > 6$, let $p = \frac{n-2}{4}$ and $j = 1, 2, \ldots, p-1, p$. Group the vertices of P_n into p disjoint subsets R_j, such that

$$\begin{aligned}
R_1 &= \{u_1, u_2, u_3, u_4\}, \\
R_2 &= \{u_5, u_6, u_7, u_8\}, \\
R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\}, \\
&\vdots \\
R_{p-1} &= \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}\}, \text{ and} \\
R_p &= \{u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1$ and $|R_p| = 6$.
Let

$$T = \{u_1, u_2, u_5, u_6, u_9, u_{10}, \ldots, u_{n-9}, u_{n-8}, u_{n-5}, u_{n-4}, u_{n-1}, u_n\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \ldots, p-1$ and 4 vertices in R_p. It follows that

$$|T| = 2(p-1) + 4 = 2p + 2 = 2\left(\frac{n-2}{4}\right) + 2 = \frac{n+2}{2}.$$

Also, note that $N(T) = V(P_n)$. Thus, T is a γ_t-set of P_n by Proposition 1. It is also clear that $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_n)$. Therefore, T is a γ_{te}-set and

$$\gamma_{te}(P_n) = |T| = \frac{n+2}{2}.$$

Case 3: Suppose that $n \equiv 3 \pmod 4$.
When $n = 3$, $S = \{u_1, u_2\}$ is a γ_{te}-set of P_3 since $|N(u_1) \cap S| = 1$, $|N(u_2) \cap S| = 1$ and $|N(u_3) \cap S| = 1$. Thus, $\gamma_{te}(P_3) = |S| = 2$. Let $n = 7$. By Proposition 1, $\gamma_t(P_7) = \frac{7+1}{2} = 4$. Let $T = \{u_1, u_2, u_5, u_6\}$. Clearly, $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_7)$. Thus, T is a γ_{te}-set of P_7, and $\gamma_{te}(P_7) = |T| = 4$. Also, $R = \{u_2, u_3, u_6, u_7\}$ is another γ_{te}-set of P_7 since $|N(u_i) \cap R| = 1$ for all $u_i \in V(P_7)$.

Now, suppose that $n > 7$, let $p = \frac{n-3}{4}$ and $j = 1, 2, \ldots, p-1, p$. Group the vertices of P_n into p disjoint subsets R_j, such that

$$\begin{aligned}
R_1 &= \{u_1, u_2, u_3, u_4\},\\
R_2 &= \{u_5, u_6, u_7, u_8\},\\
R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\},\\
&\vdots\\
R_{p-1} &= \{u_{n-10}, u_{n-9}, u_{n-8}, u_{n-7}\}, \text{ and}\\
R_p &= \{u_{n-6}, u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}
\end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1$ and $|R_p| = 7$.
Let

$$T = \{u_1, u_2, u_5, u_6, u_9, u_{10}, \ldots, u_{n-10}, u_{n-9}, u_{n-6}, u_{n-5}, u_{n-2}, u_{n-1}\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \ldots, p-1$ and 4 vertices in R_p. It follows that

$$|T| = 2(p-1) + 4 = 2p + 2 = 2\left(\frac{n-3}{4}\right) + 2 = \frac{n+1}{2}.$$

Also, note that $N(T) = V(P_n)$. Thus, T is a γ_t-set of P_n by Proposition 1. It is also clear that $|N(u_i) \cap T| = 1$ for all $u_i \in V(P_n)$. Therefore, T is a γ_{te}-set. This implies that

$$\gamma_{te}(P_n) = |T| = \frac{n+1}{2}.$$

□

Theorem 8. *Let n be a positive integer such that $n \geq 3$. If $n \not\equiv 0 \pmod 4$, then the cycle graph C_n is a $non - \gamma_{te} - graph$.*

Proof. Let $V(C_n) = \{u_1, u_2, u_3, \ldots, u_{n-1}, u_n\}$ with $\deg(u_i) = 2$ for all $u_i \in V(C_n)$. Consider the following cases:
Case 1: $n \equiv 1 \pmod 4$

Let $n = 5$. By Proposition 1, $\gamma_t(C_5) = 3$. Clearly, $T_1 = \{u_1, u_2, u_3\}$, $T_2 = \{u_2, u_3, u_4\}$, $T_3 = \{u_3, u_4, u_5\}$, $T_4 = \{u_4, u_5, u_1\}$, and $T_5 = \{u_5, u_1, u_2\}$ are the only γ_t-sets of C_5. For $i = 1, 2, 3, ..., 5$, $|T_i|$ is odd and so, by Theorem 5, T_i is not a $\gamma_{te} - set$. Suppose that T has 4 vertices. Then there exists one vertex, say u_i, such that u_i is adjacent to 2 vertices of C_5, that is, $|N(u_i) \cap T| = 2$. Thus, T is not a $\gamma_{te} - set$. Since there is no possible way to create a set that is both total and exact dominating set, C_5 is a $non - \gamma_{te} - graph$.

Now, suppose that $n > 5$. Let $p = \frac{n-1}{4}$ and $j = 1, 2, \ldots, p-1, p$. Group the vertices of C_n into p disjoint subsets R_j, such that

$$\begin{aligned} R_1 &= \{u_1, u_2, u_3, u_4\}, \\ R_2 &= \{u_5, u_6, u_7, u_8\}, \\ R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\}, \\ &\vdots \\ R_{p-1} &= \{u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}, \text{ and} \\ R_p &= \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\} \end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1$ and $|R_p| = 5$.
Let

$$T = \{u_2, u_3, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-7}, u_{n-6}, u_{n-3}, u_{n-2}, u_{n-1}\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \ldots, p-1$ and 3 vertices in R_p. It follows that

$$|T| = 2(p-1) + 3 = 2p + 1 = 2\left(\frac{n-1}{4}\right) + 1 = \frac{n+1}{2}.$$

Also, it is clear that $N(T) = V(C_n)$ and so, T is a γ_t-set by Proposition 1. Note that $|T|$ is odd. By Theorem 5, T is not a γ_{te}-set. Clearly, $|N(u_i) \cap T| = 1$ for all $u_i \in V(C_n) \setminus \{u_{n-2}\}$ and $|N(u_{n-2}) \cap T| = 2$ and so, u_{n-1} must not be in T. Note that u_1 must not be in T also since otherwise, $|N(u_2) \cap T|$ would become 2. Thus, the only option left is to replace u_{n-1} with u_n; however, $|T|$ remains odd and so T is not a γ_{te}-set. Since T is arbitrarily chosen, there is no possible way to create a set that is both a total and exact dominating set of C_n. Thus,

$$C_n \text{ is a non-}\gamma_{te}\text{-graph if } n \equiv 1 \pmod 4.$$

Case 2: $n \equiv 2 \pmod 4$

Let $n = 6$. By Proposition 1, $\gamma_t(C_6) = \frac{6+2}{2} = 4$. Clearly, $T_1 = \{u_1, u_2, u_3, u_4\}$, $T_2 = \{u_1, u_2, u_4, u_5\}$, $T_3 = \{u_1, u_2, u_5, u_6\}, T_4 = \{u_2, u_3, u_4, u_5\}, T_5 = \{u_2, u_3, u_5, u_6\}$,

$T_6 = \{u_2, u_3, u_6, u_1\}$, $T_7 = \{u_3, u_4, u_5, u_6\}, T_8 = \{u_3, u_4, u_6, u_1\}$, and $T_9 = \{u_4, u_5, u_6, u_1\}$ are the only γ_t-sets of C_6. It is also clear that for $i = 1, 2, 3, ..., 9$, there always exists a vertex u_k in $V(C_6)$ such that $|N(u_k) \cap T_i| = 2$; that is, u_2 is associated with T_1, u_3 with T_2, u_1 with T_3, u_3 with T_4, u_4 with T_5, u_2 with T_6, u_4 with T_7, u_2 with T_8, and u_5 with T_9. It follows that for $i = 1, 2, 3, ..., 9$, T_i is not a γ_{te}-set of C_6. Hence, there is no possible way to create a set that is both a total and exact dominating set of C_6 and so, C_6 is a non-γ_{te}-graph.

Now, suppose that $n > 6$. Let $p = \frac{n-2}{4}$ and $j = 1, 2, \ldots, p$. Group the vertices of C_n into p disjoint subsets R_j, such that

$$\begin{aligned} R_1 &= \{u_1, u_2, u_3, u_4\}, \\ R_2 &= \{u_5, u_6, u_7, u_8\}, \\ R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\}, \\ &\vdots \\ R_{p-1} &= \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}\}, \text{ and} \\ R_p &= \{u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\} \end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1$ and $|R_p| = 6$.
Let

$$T = \{u_2, u_3, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-8}, u_{n-7}, u_{n-4}, u_{n-3}, u_{n-1}, u_n\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \ldots, p-1$ and 4 vertices in R_p. It follows that

$$|T| = 2(p-1) + 4 = 2p + 2 = 2\left(\frac{n-2}{4}\right) + 2 = \frac{n+2}{2}.$$

Also, it is clear that $N(T) = V(C_n)$ and so, T is a γ_t-set by Proposition 1. Clearly, $|N(u_i) \cap T| = 1$ for all $u_i \in V(C_n) \setminus \{u_{n-2}, u_1\}$, $|N(u_{n-2}) \cap T| = 2$ and $|N(u_1) \cap T| = 2$. Therefore, T is not a γ_{te}-set, and both u_{n-1} and u_n must not be in T so that $|N(u_{n-2}) \cap T| = 1$ and $|N(u_1) \cap T| = 1$. Note also that u_{n-2} and u_1 must not be in T since otherwise, $|N(u_{n-3}) \cap T|$ and $|N(u_2) \cap T|$ would become 2. Thus, no two vertices can replace both u_{n-1} and u_n in T. Since T is arbitrarily chosen, there is no possible way to create a set that is both a total and exact dominating set of C_n. Thus,

$$C_n \text{ is a non-}\gamma_{te}\text{-graph if } n \equiv 2 \pmod 4.$$

Case 3: $n \equiv 3 \pmod 4$

Let $n = 7$. By Proposition 1, $\gamma_t(C_7) = 4$. Clearly, $T_1 = \{u_1, u_2, u_4, u_5\}$, $T_2 = \{u_1, u_2, u_5, u_6\}$, $T_3 = \{u_2, u_3, u_5, u_6\}$, $T_4 = \{u_2, u_3, u_6, u_7\}$, $T_5 = \{u_3, u_4, u_6, u_7\}$, $T_6 = \{u_3, u_4, u_7, u_1\}$, and $T_7 = \{u_4, u_5, u_7, u_1\}$ are the only γ_t-sets of C_7. It is also clear that for $i = 1, 2, 3, ..., 7$, there always exists a vertex u_k in $V(C_7)$ such that $|N(u_k) \cap T_i| = 2$; that is, u_3 is associated with T_1, u_7 with T_2, u_4 with T_3, u_1 with T_4, u_5 with T_5, u_2 with

T_6, and u_6 with T_7. It follows that for $i = 1, 2, 3, ..., 7$, T_i is not a γ_{te}-set of C_7. Hence, there is no possible way to create a set that is both a total and exact dominating set of C_7 and so, C_7 is a non-γ_{te}-graph.

Now, suppose that $n > 7$. Let $p = \frac{n-3}{4}$ and $j = 1, 2, \ldots, p$. Group the vertices of C_n into p disjoint subsets R_j, such that

$$\begin{aligned} R_1 &= \{u_1, u_2, u_3, u_4\}, \\ R_2 &= \{u_5, u_6, u_7, u_8\}, \\ R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\}, \\ &\vdots \\ R_{p-1} &= \{u_{n-10}, u_{n-9}, u_{n-8}, u_{n-7}\}, \text{ and} \\ R_p &= \{u_{n-6}, u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}, \end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1$ and $|R_p| = 7$.
Let

$$T = \{u_2, u_3, u_6, u_7, u_{10}, u_{11}, \ldots u_{n-9}, u_{n-8}, u_{n-5}, u_{n-4}, u_{n-1}, u_n\},$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \ldots, p-1$ and 4 vertices in R_p. It follows that

$$|T| = 2(p-1) + 4 = 2p + 2 = 2\left(\frac{n-3}{4}\right) + 2 = \frac{n+1}{2}.$$

Also, it is clear that $N(T) = V(C_n)$ and so, T is a γ_t-set by Proposition 1. Clearly, $|N(u_i) \cap T| = 1$ for all $u_i \in V(C_n) \setminus \{u_1\}$ and $|N(u_1) \cap T| = 2$. Therefore, T is not a γ_{te}-set. Note that $|N(u_1) \cap T| = 1$ if either u_2 or u_n, but not both, is not in T. If $u_2 \notin T$ and $u_n \in T$, $|N(u_3) \cap T| = 0$. If $u_n \notin T$ and $u_2 \in T$, $|N(u_{n-1}) \cap T| = 0$. In either case, T is not a γ_{te}-set. Since T is arbitrarily chosen, there is no possible way to create a set that is both a total and exact dominating set of C_n. Thus,

$$C_n \text{ is a non-}\gamma_{te}\text{-graph if } n \equiv 3 \pmod 4.$$

Therefore, in any case, C_n is a non-γ_{te}-graph if $n \not\equiv 0 \pmod 4$. □

Theorem 9. *Let n be a positive integer such that $n \geq 4$. Then the total exact domination number of a cycle C_n of order n, where $n \equiv 0 \pmod 4$, is given by*

$$\gamma_{te}(C_n) = \frac{n}{2}.$$

Proof. Suppose that $n \equiv 0 \pmod 4$. Let $n = 4$. By Proposition 1, $\gamma_t(C_4) = \frac{4}{2} = 2$. Clearly, $T_1 = \{u_1, u_2\}$, $T_2 = \{u_2, u_3\}$, $T_3 = \{u_3, u_4\}$, and $T_4 = \{u_4, u_1\}$ are the only γ_t-sets of C_4. It is also clear that for $i = 1, 2, 3, 4$, $|N(u_k) \cap T_i| = 1$ for all vertices u_k in $V(C_4)$. Thus, for $i = 1, 2, 3, 4$, T_i is a γ_{te}-set of C_4 and so, $\gamma_{te}(C_4) = |T_i| = 2$.

Now, suppose that $n > 4$. Let $p = n/4$ and $j = 1, 2, \ldots, p$. Group the vertices of C_n into p disjoint subsets R_j, such that

$$\begin{aligned} R_1 &= \{u_1, u_2, u_3, u_4\}, \\ R_2 &= \{u_5, u_6, u_7, u_8\}, \\ R_3 &= \{u_9, u_{10}, u_{11}, u_{12}\}, \\ &\vdots \\ R_{p-1} &= \{u_{n-7}, u_{n-6}, u_{n-5}, u_{n-4}\}, \text{ and} \\ R_p &= \{u_{n-3}, u_{n-2}, u_{n-1}, u_n\}. \end{aligned}$$

where $|R_j| = 4$ for $j = 1, 2, \ldots, p-1, p$.
Let

$$T = \{u_2, u_3, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-6}, u_{n-5}, u_{n-2}, u_{n-1}\}$$

where T is formed by getting 2 vertices in each R_j for $j = 1, 2, \ldots, p-1, p$. It follows that

$$|T| = 2p = 2\left(\frac{n}{4}\right) = \frac{n}{2}.$$

Also, it is clear that $N(T) = V(C_n)$ and so, T is a γ_t-set by Proposition 1. Clearly, $|N(u_i) \cap T| = 1$ for all $u_i \in V(C_n)$. Therefore, T is a γ_{te}-set. This implies that

$$\gamma_{te}(C_n) = |T| = \frac{n}{2}.$$

□

Theorem 10. *The total exact domination number of a complete bipartite graph $K_{m,n}$ is given by*

$$\gamma_{te}(K_{m,n}) = 2$$

.

Proof. The vertex set of $K_{m,n}$ can be partitioned into two disjoint sets, U and V, where each vertex in U is adjacent to every vertex in V and and there are no edges within U and within V. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. Consider $T \subseteq V(K_{m,n})$ containing exactly one vertex $u_p \in U$ and one vertex $v_q \in V$, that is, $T = \{u_p, v_q\}$. Clearly, $|N(u_i) \cap T| = |\{v_q\}| = 1$ for all $u_i \in U$ and $|N(v_j) \cap T| = |\{u_p\}| = 1$ for all $v_j \in V$. Hence, T is a γ_{te}-set of $K_{m,n}$ and so,

$$\gamma_{te}(K_{m,n}) = |T| = 2.$$

□

4.1. Total Exact Dominating Sets in the Join of Graphs

This section contains results when the join $G + H$ has either a $\gamma_{te} - set$ or has no $\gamma_{te} - set$ and its total exact domination number.

Theorem 11. *Let G and H be any graphs. Then $G + H$ is a non-γ_{te}-graph if and only if it satisfies one of the following:*
(i) both G and H have no isolated vertices; or
(ii) exactly one of the graphs G and H has at least one isolated vertex, while the other has none.

Proof. Suppose that both G and H have at least one isolated vertex. Let u and v be isolated vertices in G and H, respectively. Clearly, $T = \{u, v\}$ is a total dominating set of $G + H$ by Theorem 2(iii). Clearly, $|N(u) \cap T| = |\{v\}| = 1$ and $|N(v) \cap T| = |\{u\}| = 1$. Since u is an isolated vertex in G, any vertex $p \neq u$ in G is adjacent to the vertex v in $G + H$. That is, in $G + H$, $|N(p) \cap T| = |\{v\}| = 1$ for all $p \in V(G)$. Also, since v is an isolated vertex in H, any vertex $q \neq v$ in H is adjacent to the vertex u in $G + H$. That is, in $G + H$, $|N(q) \cap T| = |\{u\}| = 1$ for all $q \in V(H)$. Therefore, T is a $\gamma_{te} - set$ and so, $G + H$ is not a non-γ_{te}-graph.

Suppose that (i) holds, that is, both G and H have no isolated vertices.
By Corollary 1, $\gamma_t(G + H) = 2$. Consider the following cases:

Case 1: T is a total dominating set that contains two vertices in $V(G)$ or $V(H)$. Without loss of generality, let $T = \{p, q\}$ where $p, q \in V(G)$. Note that for any vertex $r \in V(H)$, r is adjacent to both p and q in $G+H$, that is, $|N(r) \cap T| = 2$ for all $r \in V(H)$. Thus, T is not a total exact dominating set. Similarly, if T is a total dominating set that contains two vertices in $V(H)$, then T is not a total exact dominating set.

Case 2: T is a total dominating set that contains one vertex in $V(G)$ and one vertex in $V(H)$. Let $T = \{s, t\}$ where $s \in V(G)$ and $t \in V(H)$. Note that s and t are adjacent in $G + H$. Since G has no isolated vertices, there exists at least one vertex $w \in V(G)$ that is adjacent to the vertex s also. Note that w is also adjacent to the vertex t in $G + H$. Thus, $|N(w) \cap T| = 2$. Hence, T is also not a total exact dominating set.

Suppose that (ii) holds. Without loss of generality, suppose that G has no isolated vertices, and H has at least one isolated vertex. Let u be an isolated vertex in H, and let $v \in V(G)$. Pick $T = \{u, v\}$. Then there exists at least one vertex $z \in V(G)$ such that z is adjacent to v since G has no isolated vertices. Note that z is also adjacent to u in $G + H$, that is , $|N(z) \cap T| = 2$. Hence, T is also not a total exact dominating set. Thus, a total exact dominating set cannot exist.

Therefore, under conditions (i) and (ii), $G + H$ is a non-γ_{te}-graph. □

Theorem 12. *Let G and H be any graphs. Then a subset T of $V(G+H)$ is a γ_{te}-set of $G+H$ if and only if $T=\{u,v\}$, where u is an isolated vertex in $V(G)$ and v is an isolated vertex in $V(H)$.*

Proof. Let $T \subseteq V(G+H)$ be a γ_{te}-set of $G+H$. By the proof of Theorem 11, $G+H$ has a total exact dominating set if both G and H have at least one isolated vertex, say u and v, respectively. Clearly, the vertices u and v are adjacent in $G+H$ and set $\{u,v\}$ is a total dominating set of $G+H$ by Theorem 2 (iii). Note that in the graph $G+H$, all vertices in $V(G)\setminus\{u\}$ is adjacent to the vertex v but not in u while all vertices in $V(H)\setminus\{v\}$ is adjacent to the vertex u but not in v. Thus, $|N(r)\cap\{u,v\}| = 1$ for all $r \in V(G+H)$. Therefore, $T=\{u,v\}$.

The converse is clear. □

The next results follow directly from Theorems 11 and 12.

Corollary 4. *If both graphs G and H have at least one isolated vertex, then $\gamma_{te}(G+H)=2$.*

Example 2. *The following example illustrates the validity of Corollary 4.*

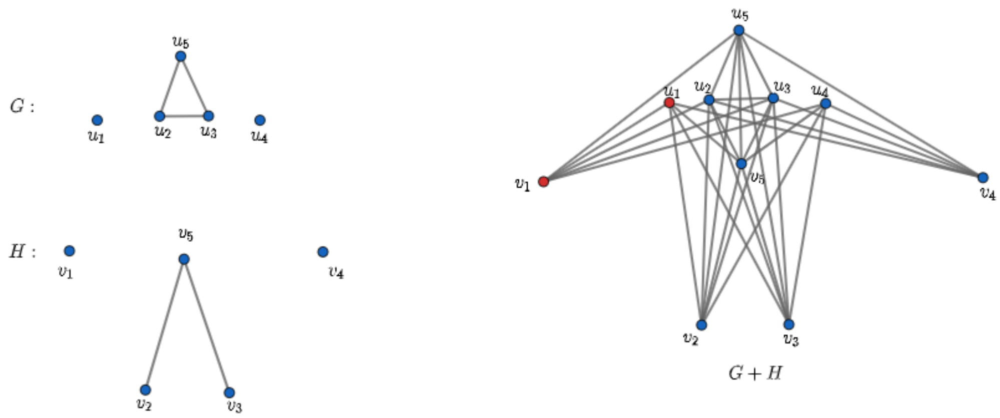


Figure 6: Join of graphs G and H with $\gamma_{te}(G+H)=2$.

Corollary 5. *Let G and H be nontrivial connected graphs, and let u and v be isolated vertices, then $((G\cup H)+\{u,v\})$ is a $non-\gamma_{te}-graph$ and $\gamma_{te}((G\cup\{u\})+(H\cup\{v\}))=2$.*

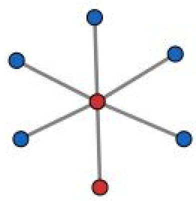
Corollary 6. *If either G or H or both have no isolated vertex, then $G+H$ is a non-γ_{te} graph.*

Corollary 7. *The following are graphs having $\gamma_{te}(G)=2$.*

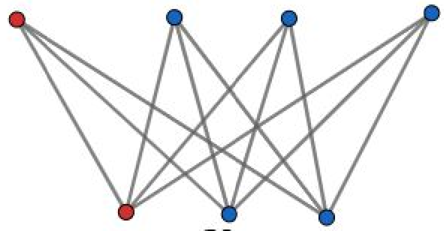
(i) Star graph $S_n = K_1+\overline{K}_n$, $n \geq 1$

(ii) Complete bipartite graph $K_{m,n} = \overline{K}_m+\overline{K}_n$ $m \geq 2$ and $n \geq 1$.

Example 3. *The following examples verify the results of Corollary 7.*



S_6



$K_{3,4}$

Figure 7: Graph S_6 with $\gamma_{te}(S_6) = 2$ and graph $K_{3,4}$ with $\gamma_{te}(K_{3,4}) = 2$.

Corollary 8. *The following are non-γ_{te}-graphs.*

(i) *Fan graph $F_n = K_1 + P_n$, $n \geq 2$*

(ii) *Wheel graph $W_n = K_1 + C_n$, $n \geq 3$*

(iii) *Friendship graph $F_n = K_1 + nP_2$, $n \geq 2$*

(iv) *Windmill graph $W_n^m = K_1 + mK_{n-1}$, $n \geq 3$ and $m \geq 2$.*

(v) *Generalized fan graph $F_{m,n} = \overline{K}_m + P_n$, $m \geq 2$ and $n \geq 2$.*

(vi) *Generalized wheel graph $W_{m,n} = \overline{K}_m + C_n$, $m \geq 2$ and $n \geq 3$.*

4.2. Total Exact Dominating Set in the Corona of Graphs

This section contains results when the corona $G \circ H$ has either a $\gamma_{te} - set$ or has no $\gamma_{te} - set$ and its total exact domination number.

Theorem 13. *Let G be a connected graph, and H be any graph. Then $G \circ H$ is a non-γ_{te}-graph if and only if either of the following is satisfied:*

(i) $|V(G)| \geq 3$,

(ii) $|V(G)| = 1$ *and H has no isolated vertices.*

Proof. Suppose that $|V(G)| = 2$. By Corollary 2, $\gamma_t(G \circ H) = 2$. Thus, $V(G) = \{u, v\}$ is a γ_t-set of $G \circ H$. Take $T = V(G)$. Clearly, in $G \circ H$, $|N(u) \cap T| = |N(v) \cap T| = 1$, $|N(a_u) \cap T| = |\{u\}| = 1$ for all vertices $a_u \in V(H^u)$ and $|N(a_v) \cap T| = |\{v\}| = 1$ for all vertices $a_v \in V(H^v)$. Hence, T is a γ_{te}-set of $G \circ H$. Therefore, $G \circ H$ is not a non-γ_{te}-graph.

Suppose that $|V(G)| = 1$ and H has at least one isolated vertex, say r. Let $V(G) = \{s\}$ and so, r_s is an isolated vertex in H^s. Take $T = \{s, r_s\}$. Since r_s and s are adjacent in $G \circ H$, $|N(s) \cap T| = |N(r_s) \cap T| = 1$. Also, in $G \circ H$, it is clear that $|N(t_s) \cap T| = |\{s\}| = 1$ for all vertices $t_s \in V(H^s)$. Hence, T is a γ_{te}-set of $G \circ H$. Therefore, $G \circ H$ is not a non-γ_{te}-graph.

Suppose that (i) holds, that is, G is of order $m \geq 3$. By Corollary 2, $\gamma_t(G \circ H) = m$. Note that $V(G)$ is a γ_t-set of $G \circ H$ since $N(V(G)) = V(G \circ H)$ and $|V(G)| = m$. Let T be a total exact dominating set of $G \circ H$.

Suppose that $T \neq V(G)$, that is, not all vertices in $V(G)$ are in T, say $z \in V(G) \setminus T$. Then there exist at least two adjacent vertices in $V(H^z)$, say a_z and b_z such that the vertices a_z and b_z must be in T and $|N(a_z) \cap T| = |N(b_z) \cap T| = 1$. Note that z is adjacent to both a_z and b_z in $G \circ H$, that is, $|N(z) \cap T| = |\{a_z, b_z\}| = 2$. This is a contradiction to the definition of a total exact dominating set. Thus, for all $z \in V(G)$, no two vertices in $V(H^z)$ can be chosen to form a total exact dominating set of $G \circ H$ and $T = V(G)$.

Now, suppose that $T = V(G)$, that is, all vertices in G are in T. Since G is connected and $|V(G)| \geq 3$, there exists at least one vertex $u \in V(G)$ that is adjacent to two vertices in G, and so, $|N(u) \cap T| = 2$ for some $u \in V(G)$, a contradiction. Hence, $T \neq V(G)$ and T is not a total exact dominating set of $G \circ H$.

Suppose that (ii) holds, that is, $|V(G)| = 1$ and H has no isolated vertices. Let $V(G) = \{p\}$. By the previous argument, no two vertices in $V(H^p)$ can be chosen to form a total exact dominating set of $G \circ H$. Take $T = \{p, q_p\}$ where q_p is a vertex in $V(H^p)$. Clearly, T is a total dominating set of $G \circ H$ since $N(T) = V(G \circ H)$. Since H has no isolated vertices, there exists at least one vertex c_p that is adjacent to q_p. Note that c_p is also adjacent to p in $G \circ H$, that is, $|N(c_p) \cap T| = |\{p, q_p\}| = 2$, a contradiction. Hence, T is not a total exact dominating set of $G \circ H$.

Thus, it is not possible to create a total exact dominating set of $G \circ H$. Therefore, $G \circ H$ is a non-γ_{te}-graph. □

Theorem 14. *Let G be a connected graph, and H be any graph. Then the set $T \subseteq V(G \circ H)$ is a γ_{te}-set of $G \circ H$ if and only if either of the following conditions is satisfied:*

(i) *$G \cong P_1$ and $T = V(P_1) \cup \{u\}$, where u is an isolated vertex in H or*

(ii) *$G \cong P_2$ and $T = V(P_2)$.*

Proof. Suppose that $T \subseteq V(G \circ H)$ is a γ_{te}-set of $G \circ H$. Then by using the same argument from Theorem 13, the graph $G \circ H$ will have a γ_{te}-set if and only if either (i) $|V(G)| = 1$ and H has at least one isolated vertex, or (ii) $|V(G)| = 2$.

Suppose that $|V(G)| = 1$ and H has at least one isolated vertex, say u. Clearly, $G \cong P_1$. Since both G and H have isolated vertices and $P_1 \circ H = P_1 + H$, by Theorem 12, $T = \{x, u\}$ where $x \in V(P_1)$ and u is an isolated vertex in $V(H)$. Therefore, $T = V(P_1) \cup \{u\}$ where u is an isolated vertex in $V(H)$.

Suppose that $|V(G)| = 2$. Then $G \cong P_2$. By Corollary 2, $\gamma_t(G \circ H) = 2$. Clearly, $V(P_2)$ is a γ_{te}-set of $G \circ H$ since $|N(v) \cap V(P_2)| = 1$ for all $v \in G \circ H$. Thus, $T = V(P_2)$.

The converse is easy.

□

The next results follow directly from Theorems 13 and 14:

Corollary 9. *Let $G \cong P_1$ graph and H be any graph with an isolated vertex. Then*

$$\gamma_{te}(G \circ H) = 2.$$

Corollary 10. *Let $G \cong P_2$ graph and H be any graph. Then*

$$\gamma_{te}(G \circ H) = 2.$$

Example 4. Consider the graphs $G_1 \circ H_1$ and $G_2 \circ H_2$ in Figure 8. Let $T_1 = \{a, u_a\}$ and $T_2 = \{a, b\}$. It is clear that $|N(r_a) \cap T_1| = 1$ for all vertices $r_a \in V(G_1 \circ H_1)$ and $|N(s_a) \cap T_1| = |N(s_b) \cap T_1| = 1$ for all vertices $s_a, s_b \in V(G_2 \circ H_2)$. Hence, T_1 and T_2 are γ_{te}-sets of $G_1 \circ H_1$ and $G_2 \circ H_2$, respectively. It follows that $\gamma_{te}(G_1 \circ H_1) = 2$ and $\gamma_{te}(G_2 \circ H_2) = 2$.

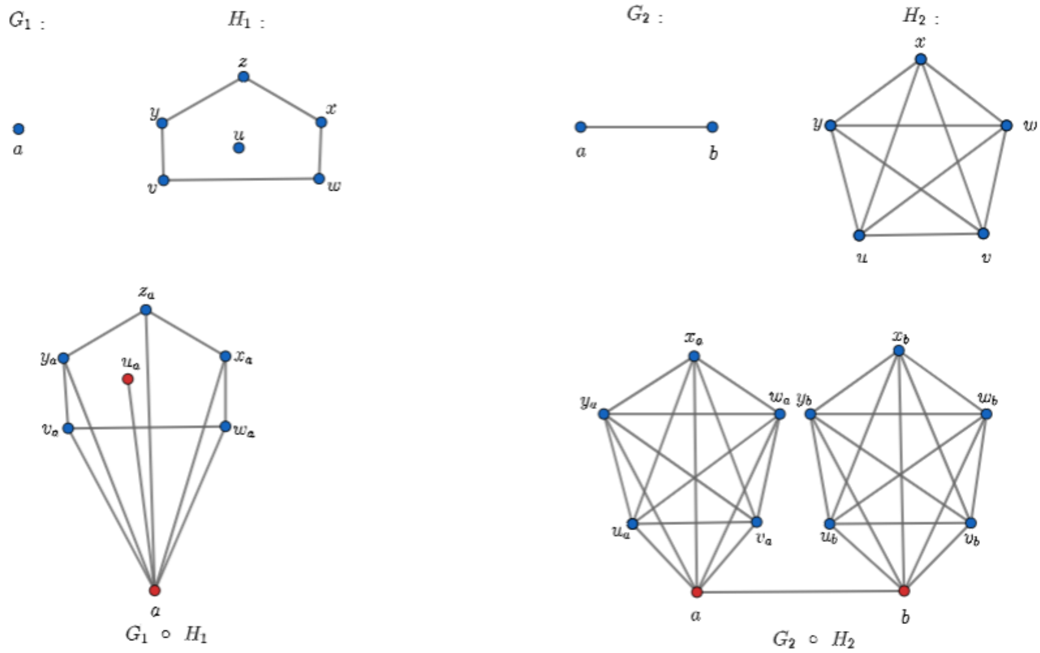


Figure 8: Graphs $G_1 \circ H_1$ and $G_2 \circ H_2$ with $\gamma_{te}(G_1 \circ H_1) = \gamma_{te}(G_2 \circ H_2) = 2$.

Corollary 11. *If $G \ncong P_1$ or $G \ncong P_2$, then for any graph H, $G \circ H$ is a non-γ_{te}-graph.*

Corollary 12. *Let G be a disconnected graph with k components such that each component is isomorphic to either P_1 or P_2 and H be any graph with an isolated vertex. Then*

$$\gamma_{te}(G \circ H) = 2k.$$

Example 5. *The following example verifies the result of Corollary 12.*

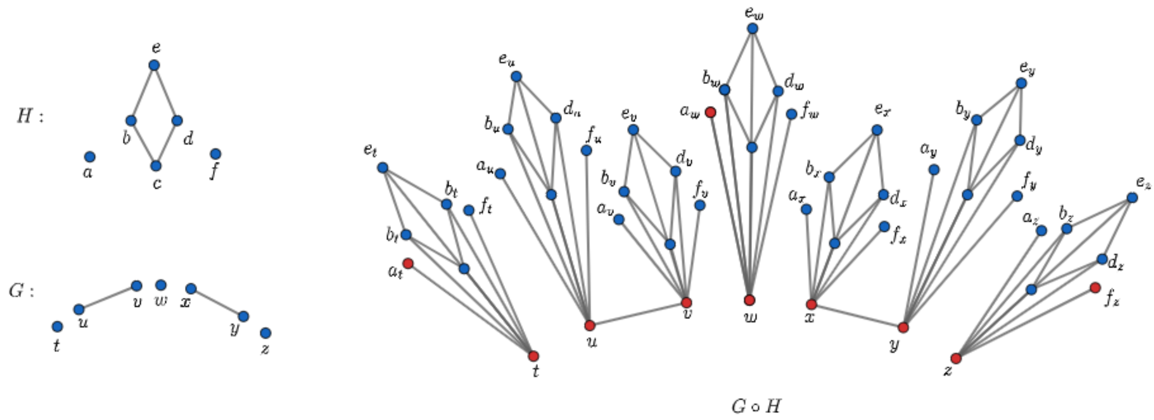


Figure 9: Graph $G \circ H$ with $\gamma_{te}(G \circ H) = 2(5) = 10$.

Corollary 13. *Let G be a disconnected graph with k components such that each component is isomorphic to P_2 and H be any graph. Then*

$$\gamma_{te}(G \circ H) = 2k.$$

4.3. Total Exact Dominating Set in the Lexicographic Product of Graphs

This section contains results when the lexicographic product $G[H]$ has either a $\gamma_{te} - set$ or has no $\gamma_{te} - set$ and its total exact domination number.

Theorem 15. *Let G and H be graphs such that at least one of them is connected, and at most one is an empty graph. Then $G[H]$ is a non-γ_{te}-graph if and only if one of the following is satisfied:*

(i) both G and H are nontrivial connected graphs,

(ii) G is an empty graph and H is a connected non- γ_{te}-graph, or

(iii) H is an empty graph and G is a connected non- γ_{te}-graph.

Proof. Suppose that G is an empty graph and H is a connected graph, and let T be a γ_{te}-set of H. Since G is an empty graph, $G[H]$ is a disjoint union of $|V(G)|$ copies of H. Clearly, $|C| = |V(G)||T|$ and C is a γ_{te}-set of $G[H]$. Hence, $G[H]$ is not a non-γ_{te}-graph.

Suppose that H is an empty graph and G is a connected graph, and S is a γ_{te}-set of G. Let $C = \bigcup_{x \in S}(\{x\} \times y)$ where $y \in V(H)$. Since H is an empty graph, all vertices of the $|V(H)|$ copies of G is adjacent to the exactly one vertex in C. Thus, $|N(u,v) \cap C| = 1$ for all vertices $(u,v) \in V(G[H])$. Therefore, C is a γ_{te}-set of $G[H]$. Hence, $G[H]$ is not a non-γ_{te}-graph.

Suppose that (i) holds, that is, both G and H are nontrivial connected graphs. Let C be a γ_{te}-set of $G[H]$. Note that for any two adjacent vertices $u, v \in V(G)$ and any two adjacent vertices $p, q \in V(H)$, the vertices formed in $G[H]$ are $(u,p), (v,p), (u,q)$, and (v,q). These vertices are all adjacent to each other, forming a complete subgraph K_4. This holds for every pair of adjacent vertices in $V(G)$ and $V(H)$ when constructing the vertices of $G[H]$. Moreover, note that two vertices in C must be adjacent. Thus, there exists a vertex in $V(G[H]) \setminus C$ such that $|N(s,t) \cap C| = 2$. This contradicts the definition of C. Hence, $G[H]$ is non-γ_{te}-graph.

Suppose that (ii) holds, that is, G is an empty graph and H is a connected non- γ_{te}-graph. Let C be a γ_{te}-set of $G[H]$. Since G is an empty graph, $G[H]$ is a disjoint union of $|V(G)|$ copies of H and so, the set C must be the union of γ_{te}-sets from each copy of H. However, since H is a non-γ_{te}-graph, no such γ_{te}-set can exist in any copy of H, making it impossible to form C. Hence, $G[H]$ is non-γ_{te}-graph.

Suppose that (iii) holds, that is, H is an empty graph and G is a connected non- γ_{te}-graph. Let C be a γ_{te}-set of $G[H]$. Since H is an empty graph and G is a connected graph, $G[H]$ is a connected graph such that the neighbors of $(u,p) \in V(G[H])$ come only from adjacent copies of H in $G[H]$. That is, $N(u,p)$ contains all vertices from independent sets corresponding to neighbors of u in G. Since the independent sets are fully connected through the edges of G and G is a non- γ_{te}-graph, it is impossible to pick a set C such that every vertex has exactly one neighbor in C. Thus, $G[H]$ is non-γ_{te}-graph. □

Theorem 16. *Let G be a connected graph, and H be any graph of order m. Then*

$$C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$$

is a γ_{te}-set of $G[H]$ if and only if either one of the following is satisfied:

(i) S is a γ_{te}-set of G and H is an empty graph or

(ii) T_x is a γ_{te}-set of H for every $x \in V(G)$ and G is an empty graph.

Proof. Let C be a γ_{te}-set of $G[H]$. By Theorem 15, either (i) H is an empty graph and G is a connected graph and not a non-γ_{te}-graph or (ii) G is an empty graph and H is a connected graph and not a non-γ_{te}-graph. Consider the following cases:

(i) Suppose that H is an empty graph and G is a connected graph and not a non-γ_{te}-graph. Let $u \in V(G)$. Pick any $v \in V(H)$. Since C is a γ_{te}-set , there exists $(y,z) \in C$ such that

$$N_{G[H]}(u,v) \cap C = \{(y,z)\}.$$

This implies that $N_G(u) \cap S = \{y\}$. Hence, every $u \in V(G)$ is dominated by exactly one vertex in S, and so, S is a γ_{te}-set of G.

(ii) Suppose that G is an empty graph and H is a connected graph and not a non-γ_{te}-graph. Suppose that T_x is not a γ_{te}-set of H for every $x \in V(G)$. Then, since G is an empty graph, $G[H]$ is a disjoint union of $|V(G)|$ copies of H. Thus,

$$C = \bigcup_{x \in S} (\{x\} \times T_x)$$

is not a γ_{te}-set of $G[H]$ since T_x is not a γ_{te}-set of H for every $x \in V(G)$. This is a contradiction. Therefore, T_x is a γ_{te}-set of H for every $x \in V(G)$.

The converse is easy. □

Corollary 14. *Let G be a connected graph such that G has a total exact dominating set and H be an empty graph. Then*

$$\gamma_{te}(G[H]) = \gamma_{te}(G).$$

Proof. Since H is an empty graph, by Theorem 16 (i), $\gamma_{te}(G[H]) = \gamma_{te}(G)$. □

Corollary 15. *Let H be a connected graph such that H has a total exact dominating set and G be an empty graph of order n. Then*

$$\gamma_{te}(G[H]) = n\gamma_{te}(H).$$

Proof. Since G is an empty graph of order n and H has a total exact dominating set, by Theorem 16 (ii), $\gamma_{te}(G[H]) = n\gamma_{te}(H)$. □

Example 6. *The following examples verify the results of Corollaries 14 and 15.*

Figure 10: Graph $G_1[H_1]$ with $\gamma_{te}(G_1[H_1]) = \gamma_{te}(G_1) = 4$.

Figure 11: Graph $G_2[H_2]$ with $\gamma_{te}(G_2[H_2]) = 3(\gamma_{te}(H_2)) = 3(4) = 12$.

The next results follow directly from Corollaries 14 and 15.

Corollary 16. *Let m and n be positive integers where $m, n \geq 2$. Then*

$$\gamma_{te}(\overline{K}_m[P_n]) = \begin{cases} \frac{mn}{2}, & \text{if } n \equiv 0 \pmod 4 \\ m\left(\frac{n+2}{2}\right), & \text{if } n \equiv 2 \pmod 4 \\ m\left(\frac{n+1}{2}\right), & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

and

$$\gamma_{te}(P_n[\overline{K}_m]) = \gamma_{te}(P_n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+2}{2}, & \text{if } n \equiv 2 \pmod 4 \\ \frac{n+1}{2}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Corollary 17. *The total exact domination number of the Lexicographic product of $\overline{K}_m$ and C_n is given by*

$$\gamma_{te}(\overline{K}_m[C_n]) = \frac{mn}{2}, \quad \forall\, n \equiv 0 \pmod 4$$

and

$$\gamma_{te}(C_n[\overline{K}_m]) = \frac{n}{2}, \quad \forall\, n \equiv 0 \pmod 4.$$

The next result follows directly from Theorems 6, 8 and 15.

Corollary 18. *The following lexicographic products are non-γ_{te}-graphs:*

i. $P_n[\overline{K}_m]$ $\forall\, n \equiv 1 \pmod 4$,

ii. $\overline{K}_m[P_n]$ $\forall\, n \equiv 1 \pmod 4$,

iii. $C_n[\overline{K}_m]$ $\forall\, n \not\equiv 0 \pmod 4$,

iv. $\overline{K}_m[C_n]$ $\forall\, n \not\equiv 0 \pmod 4$.

5. Conclusion

In this study, we explored and introduced the concept of total exact domination in graphs. We computed $\gamma_{te}(G)$ for several families of special graphs such as paths, cycles, stars, and complete bipartite graphs, as well as for graphs resulting from some binary operations including the join, corona, and lexicographic product. Graphs that do not have total exact dominating sets were identified as non-γ_{te}-graphs, examples of which include complete graphs, fan graphs, and wheel graphs with more than two vertices. In addition, we considered some disconnected graphs in the binary operations, enriching the scope of the investigation.

The total exact domination has applications in communication and sensor networks, where nodes must be fully covered (total domination) with minimal overlapping influence (exact domination), similar to applications discussed in network design scenarios [15]. The total exact domination model ensures efficient placement of resources or control centers with distinct coverage, reducing redundancy.

For future research, we recommend studying the total exact domination number under other binary operations not mentioned in this study, such as the strong product, Cartesian product, and tensor product of graphs. Additionally, extending the study to weighted graphs, directed graphs, or random graphs could broaden its computational relevance and applicability.

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