EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6131
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Uniqueness of Stationary Distribution in Markov Processes: A Quintuple Fixed Point and Coincidence Point Approach

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Abstract. This article introduces the concept of quintuple fixed points and coincidence points for matrix-related mappings in generalized metric spaces. Furthermore, the existence of quintuple coincidence points is established. This task is achieved by leveraging the structure of matrices. We derive several corollaries as special cases of our main results. These corollaries provide evidence for the authentication of the proven results. To validate the significance of our findings, we provide a selection of non-trivial examples. Eventually, we demonstrate the practical applicability of our established results by applying them to determine the stationary distribution of a Markov process.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Generalized metric space (GMs), Markov Process, Partially Ordered Metric Spaces (PoM), Tripled fixed point (TFp), Quintuple Fixed Point(QFP)

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1. Introduction

Functional analysis has far-reaching applications in various fields, including linear and nonlinear analysis, calculus of variations, approximation theory, numerical analysis, and differential and integral equations. In nonlinear analysis, metric fixed point theory is a fundamental tool. Currently, finding solutions to differential and integral equations is a

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6131

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crucial research area. This task can be achieved by converting the equation into a fixed point problem for a suitable mapping and an appropriate domain. In 1922, Stephan Banach [1] presented a vital result known as the Banach contraction principle (BCP), which erected a base for metric fixed-point theory. In 1964, Perov [2] extended the classical BCP on generalized metric spaces for contraction mappings. Filip and Petrusel [3] further generalized these results on vector-valued metric spaces for self-mappings. A deeper concept of Perov-type contractions can be perceived from [4]. Later on, authors proved further extensions of BCP on generalized structures of metric spaces [5–7].
Working on a new track, Bhaskar and Lakshmikantham [8] established the coupled fixed point for mixed-monotone mappings under partially ordered metric spaces (PoM). Furthermore, a few important partial-order metric space results were presented in [9]. In 2011, Berinde and Borcut [10] extended the idea of coupled fixed point to tripled fixed point (TFp) for self-mappings and set up some significant results in PoMs. One can refer to [11] and [12] for a detailed review of these ideas. Subsequently, generalizing the concept of TFp in 2012, Karapınar [13] opened a gateway for researchers in a new direction by proposing the theory of quadruple fixed point of mappings and established exciting consequences in this regard, see also [14] and [15].

Working on a similar track, we extend the idea of a quadruple fixed point and introduced the notion of a quintuple fixed point(QFP). Motivated by the work of Hammad [16] in 2022, we generalize crucial results for the existence of $QFPs$ in generalized metric spaces and provide supportive examples and applications related to Markov process stationary distribution analysis.

2. Preliminaries

Now, onward in this manuscript, $M_{n,n}(\mathbb{R}^+)$, $\mathcal{I}$, $\mathcal{O}$ are the symbol representations for the set of all $n \times n$ matrices over $\mathbb{R}^+$, identity and zero matrices respectively, and $\mathbb{W} = \{0, 1, 2, 3, \cdots\}$ is the set of inetgers. Suppose that $\tilde{\Gamma} \in M_{n,n}(\mathbb{R}^+)$, then $\tilde{\Gamma}$ is said a convergent matrix, i.e, converges to zero matrix $\mathcal{O}$ if and only if $\lim_{n\to\infty} \tilde{\Gamma}^n = \mathcal{O}$. A deeper concept can be built through [17].
Denote a set of all $n \times n$ matrices $\tilde{\Gamma} \in M_{n,n}(\mathbb{R}^+)$, with $\tilde{\Gamma}^n \to \mathcal{O}$, whenever $n \to \infty$ by ZM.

Example 1. *Let* $\tilde{\Gamma} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix}$ *be a matrix in* $M_{2,2}(\mathbb{R}^+)$ *with the condition that* $\xi_1 + \xi_2 < 1$, *for some* $\xi_1, \xi_2 \in \mathbb{R}^+$, *then* $\tilde{\Gamma} \in ZM$.

Example 2. *Suppose a matrix* $\tilde{\Upsilon} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} \in M_{2,2}(\mathbb{R}^+)$, *such that* $\xi_1 + \xi_2 \geq 1$, *for some* $\xi_1, \xi_2 \in \mathbb{R}^+$, *then* $\tilde{\Upsilon}$ *does not belong to ZM.*

For a k dimensional vector space $\mathbb{R}^k$, let $\mathbf{0}$, $\mathbf{1}$ be the zero vector and identity vector, respectively. Also, addition and multiplication in $\mathbb{R}^k$ are defined as under:

$$\xi + \xi^* = (\xi_1 + \xi_1^*, \xi_2 + \xi_2^*, \xi_3 + \xi_3^*, \cdots, \xi_k + \xi_k^*) \quad \text{and} \quad \xi.\xi^* = (\xi_1.\xi_1^*, \xi_2.\xi_2^*, \cdots, \xi_k.\xi_k^*),$$

for any $\xi, \xi^* \in \mathbb{R}^k$, where $\xi = (\xi_1, \xi_2, \xi_3, \cdots, \xi_k)$ and $\xi^* = (\xi_1^*, \xi_2^*, \xi_3^*, \cdots, \xi_k^*)$. One can have a detailed study from [3].
The proof of the subsequent lemma is discussed in matrix analysis in [3].

Lemma 1. *Suppose that $\tilde{\Gamma}$ is a square matrix with entries from $\mathbb{R}^+$, then the following statements are equivalent:*

(L1) $\tilde{\Gamma} \to \mathcal{O}$;

(L2) $\tilde{\Gamma}^n \to \mathcal{O}$ *as* $n \to \infty$;

(L3) for each $\mathrm{z} \in \mathbb{C}$, $|\mathrm{z}| < 1$ *with* $\det(\tilde{\Gamma} - \mathrm{z}\mathcal{I}) = 0$;

(L4) for a non-singular matrix $\mathcal{I} - \tilde{\Gamma}$

$$(\mathcal{I} - \tilde{\Gamma})^{-1} = \mathcal{I} + \tilde{\Gamma} + \cdots + \tilde{\Gamma}^n + \cdots;$$

(L5) two matrices $\tilde{\Gamma}^n w$ *and* $w\tilde{\Gamma}^n$ *tend to zero as* $n \to \infty$, *for some* $w \in \mathbb{R}^k$.

Definition 1. *A mapping* $\mathfrak{T} : \mathcal{G}^2 \to \mathbb{R}^k$, *where* $\mathcal{G} \neq \emptyset$, *is named as a vector-valued metric over* $\mathcal{G}$, *whenever the conditions below are fulfilled, that is, for each* $\xi_1, \xi_2, \xi_3 \in \mathcal{G}$,

($\mathcal{G}_1$) $\mathfrak{T}(\xi_1, \xi_2) \geq \mathbf{0}, \quad \mathfrak{T}(\xi_1, \xi_2) = \mathbf{0} \ \Leftrightarrow \ \xi_1 = \xi_2$,

($\mathcal{G}_2$) $\mathfrak{T}(\xi_1, \xi_2) = \mathfrak{T}(\xi_2, \xi_1)$,

($\mathcal{G}_3$) $\mathfrak{T}(\xi_1, \xi_2) \leq \mathfrak{T}(\xi_1, \xi_3) + \mathfrak{T}(\xi_3, \xi_2)$.

If $\xi_1, \xi_2 \in \mathbb{R}^k$, *where* $\xi_1 = (\xi_1^1, \xi_1^2, \cdots, \xi_1^k)$ *and* $\xi_2 = (\xi_2^1, \xi_2^2, \cdots, \xi_2^k)$, *then* $\xi_1 \leq \xi_2$ *if and only if* $\xi_1^i \leq \xi_2^i$, *for* $1 \leq i \leq k$. *Thus,* $(\mathcal{G}, \mathfrak{T})$ *is a generalized metric space. [3]*

Bhaskar and Lakshmikantham [8] established the following concepts.

Definition 2. *An element* $(\xi_1, \xi_2) \in \mathcal{G}^2$ *is named as a coupled fixed point of the mapping* $P : \mathcal{G}^2 \to \mathcal{G}$ *if* $P(\xi_1, \xi_2) = \xi_1$ *and* $P(\xi_2, \xi_1) = \xi_2$.

Definition 3. *Two mappings* $P : \mathcal{G}^2 \to \mathcal{G}$ *and* $p : \mathcal{G} \to \mathcal{G}$ *have a couple fixed point* $(\xi_1, \xi_2) \in \mathcal{G}^2$ *if* $P(\xi_1, \xi_2) = p(\xi_1)$ *and* $P(\xi_2, \xi_1) = p(\xi_2)$.

Definition 4. *A mapping* $P : \mathcal{G}^2 \to \mathcal{G}$ *on a partially ordered set* $(\mathcal{G}, \preceq)$ *possesses the mixed-monotone property (MMP) if* $P(\xi_1, \xi_2)$ *is non-decreasing in* ξ_1 *and non-increasing in* ξ_2, *i.e, for any* $\xi_1, \xi_2 \in \mathcal{G}$,

$$\begin{aligned} \xi_1^1, \xi_1^2 \in \mathcal{G}, \ \xi_1^1 \preceq \xi_1^2 \quad &\Rightarrow \quad P(\xi_1^1, \xi_2) \preceq P(\xi_1^2, \xi_2) \\ \xi_2^1, \xi_2^2 \in \mathcal{G}, \ \xi_2^1 \preceq \xi_2^2 \quad &\Rightarrow \quad P(\xi_1, \xi_2^1) \succeq P(\xi_1, \xi_2^2). \end{aligned}$$

Berinde and Borcut [10] constructed the idea of a tripled fixed point by generalizing the term of a coupled fixed point.

Definition 5. *An element* $(\xi_1, \xi_2, \xi_3) \in \mathcal{G}^3$ *is termed as a triple fixed point of the mapping* $P : \mathcal{G}^3 \to \mathcal{G}$ *if* $P(\xi_1, \xi_2, \xi_3) = \xi_1,\ P(\xi_2, \xi_3, \xi_1) = \xi_2, P(\xi_3, \xi_1, \xi_2) = \xi_3$.

The definition of a quadruple fixed point introduced by Karapınar [18] is stated as follows:

Definition 6. *[18] Let* $\mathcal{G} \neq \emptyset$. *An element* $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{G}^4$ *is stated as the quadruple fixed point of the mapping* $P : \mathcal{G}^4 \to \mathcal{G}$ *if*

$$P(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1,\ P(\xi_2, \xi_3, \xi_4, \xi_1) = \xi_2,$$
$$P(\xi_3, \xi_4, \xi_1, \xi_2) = \xi_3,\ P(\xi_4, \xi_1, \xi_2, \xi_3) = \xi_4.$$

Berinde extended the idea of mixed monotone property (MMP) [10] on $\mathcal{G}^3$, whereas Karapınar [18] introduced the concept of MMP on $\mathcal{G}^4$.

Definition 7. *A partially ordered set* $(\mathcal{G}, \preceq)$ *is termed as regular if the conditions below are satisfied:*

(A) for $m \geq 0,\ \xi_1^m \preceq \xi_1$ *if a non-decreasing sequence* $\xi_1^m \to \xi_1$,

(B) for $m \geq 0,\ \xi_2 \leq \xi_2^m$ *if a non-increasing sequence* $\xi_2^m \to \xi_2$. *[19]*

3. Main Results

Inspired by the results on couple, triple and quadruple fixed points, we initiate the term of quintuple fixed points to present some related fixed point results in the partially ordered complete generalized metric space (POCGMs) $(\mathcal{G}, \mathfrak{T}, \preceq)$.

Definition 8. *Let* $P : \mathcal{G}^5 \to \mathcal{G}$ *be a mapping. If* P *is monotonically non-increasing in* ξ_2, ξ_4 *and non-decreasing in* ξ_1, ξ_3, ξ_5, *then* P *is said to have MMP. In other words, for any* $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathcal{G}$,

$$\begin{aligned}
\xi_1^1, \xi_1^2 \in \mathcal{G},\ \xi_1^1 \preceq \xi_1^2 \ &\Rightarrow\ P(\xi_1^1, \xi_2, \xi_3, \xi_4, \xi_5) \preceq P(\xi_1^2, \xi_2, \xi_3, \xi_4, \xi_5),\\
\xi_2^1, \xi_2^2 \in \mathcal{G},\ \xi_2^1 \preceq \xi_2^2 \ &\Rightarrow\ P(\xi_1, \xi_2^1, \xi_3, \xi_4, \xi_5) \succeq P(\xi_1, \xi_2^2, \xi_3, \xi_4, \xi_5),\\
\xi_3^1, \xi_3^2 \in \mathcal{G},\ \xi_3^1 \preceq \xi_3^2 \ &\Rightarrow\ P(\xi_1, \xi_2, \xi_3^1, \xi_4, \xi_5) \preceq P(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5),\\
\xi_4^1, \xi_4^2 \in \mathcal{G},\ \xi_4^1 \preceq \xi_4^2 \ &\Rightarrow\ P(\xi_1, \xi_2, \xi_3, \xi_4^1, \xi_5) \succeq P(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5),\\
\xi_5^1, \xi_5^2 \in \mathcal{G},\ \xi_5^1 \preceq \xi_5^2 \ &\Rightarrow\ P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5^1) \preceq P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5^2).
\end{aligned}$$

The above definition can be generalized for two mappings as follows:

Definition 9. *Let* $P : \mathcal{G}^5 \to \mathcal{G}$ *and* $p : \mathcal{G} \to \mathcal{G}$ *be two mappings. Then,* P *exhibits mixed p-monotone property (MpMP) if, for any* $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathcal{G}$,

$$\begin{aligned}
\xi_1^1, \xi_1^2 \in \mathcal{G},\ p(\xi_1^1) \preceq p(\xi_1^2) \ &\Rightarrow\ P(\xi_1^1, \xi_2, \xi_3, \xi_4, \xi_5) \preceq P(\xi_1^2, \xi_2, \xi_3, \xi_4, \xi_5),\\
\xi_2^1, \xi_2^2 \in \mathcal{G},\ p(\xi_2^1) \preceq p(\xi_2^2) \ &\Rightarrow\ P(\xi_1, \xi_2^1, \xi_3, \xi_4, \xi_5) \succeq P(\xi_1, \xi_2^2, \xi_3, \xi_4, \xi_5),\\
\xi_3^1, \xi_3^2 \in \mathcal{G},\ p(\xi_3^1) \preceq p(\xi_3^2) \ &\Rightarrow\ P(\xi_1, \xi_2, \xi_3^1, \xi_4, \xi_5) \preceq P(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5),\\
\xi_4^1, \xi_4^2 \in \mathcal{G},\ p(\xi_4^1) \preceq p(\xi_4^2) \ &\Rightarrow\ P(\xi_1, \xi_2, \xi_3, \xi_4^1, \xi_5) \succeq P(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5),\\
\xi_5^1, \xi_5^2 \in \mathcal{G},\ p(\xi_5^1) \preceq p(\xi_5^2) \ &\Rightarrow\ P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5^1) \preceq P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5^2).
\end{aligned}$$

Definition 10. *An element* $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \in \mathcal{G}$ *is called a quintuple fixed point (QFP) of the mapping* $P : \mathcal{G}^5 \to \mathcal{G}$ *if*

$$P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \xi_1,\ P(\xi_2, \xi_3, \xi_4, \xi_5, \xi_1) = \xi_2,\ P(\xi_3, \xi_4, \xi_5, \xi_1, \xi_2) = \xi_3,$$
$$P(\xi_4, \xi_5, \xi_1, \xi_2, \xi_3) = \xi_4,\ P(\xi_5, \xi_1, \xi_2, \xi_3, \xi_4) = \xi_5.$$

For a generalized metric space $(\mathcal{G}, \mathfrak{T})$*, a function* $\bar{\mathfrak{T}} : \mathcal{G}^5 \times \mathcal{G}^5 \to \mathbb{R}^k$*, given by*

$$\bar{\mathfrak{T}}((\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)) = \mathfrak{T}(\xi_1, \mathfrak{q}_1)+\mathfrak{T}(\xi_2, \mathfrak{q}_2)+\mathfrak{T}(\xi_3, \mathfrak{q}_3)+\mathfrak{T}(\xi_4, \mathfrak{q}_4)+\mathfrak{T}(\xi_5, \mathfrak{q}_5),$$

is a GMs on $\mathcal{G}^5$*, that is,* $(\mathcal{G}^5, \bar{\mathfrak{T}})$ *is a GMs induced by* $\mathfrak{T}$.

Definition 11. *An element* $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \in \mathcal{G}^5$ *is called a quintuple coincidence point (QCP) or a common quintuple fixed point of the mappings* $P : \mathcal{G}^5 \to \mathcal{G}$ *and* $p : \mathcal{G} \to \mathcal{G}$ *if*

$$P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = p(\xi_1),\ P(\xi_2, \xi_3, \xi_4, \xi_5, \xi_1) = p(\xi_2),\ P(\xi_3, \xi_4, \xi_5, \xi_1, \xi_2) = p(\xi_3),$$
$$P(\xi_4, \xi_5, \xi_1, \xi_2, \xi_3) = p(\xi_4),\ P(\xi_5, \xi_1, \xi_2, \xi_3, \xi_4) = p(\xi_5).$$

Moreover, for $p : \mathcal{G} \to \mathcal{G}$ being the identity map, Definition 9 and Definition 11 reduce into Definition 8 and Definition 10, respectively.

Definition 12. *Let* $P : \mathcal{G}^5 \to \mathcal{G}$ *and* $p : \mathcal{G} \to \mathcal{G}$ *be two mappings. Then,* P *and* p *are said to be commutable if*

$$p(P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)) = P(p(\xi_1), p(\xi_2), p(\xi_3), p(\xi_4), p(\xi_5)),$$

for all $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathcal{G}$.

Definition 13. *The mappings* $P : \mathcal{G}^5 \to \mathcal{G}$ *and* $p : \mathcal{G} \to \mathcal{G}$ *over a metric space* $(\mathcal{G}, \mathfrak{T})$ *are compatible if the following conditions hold:*

$$\lim_{n\to+\infty} \mathfrak{T}(p(U^1), P(V^1)) = 0,$$
$$\text{where} \quad U^1 = P(\xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n) \quad \text{and} \quad V^1 = (p(\xi_1^n), p(\xi_2^n), p(\xi_3^n), p(\xi_4^n), p(\xi_5^n)),$$
$$\lim_{n\to+\infty} \mathfrak{T}(p(U^2), P(V^2)) = 0,$$
$$\text{where} \quad U^2 = P(\xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n, \xi_1^n) \quad \text{and} \quad V^2 = (p(\xi_2^n), p(\xi_3^n), p(\xi_4^n), p(\xi_5^n), p(\xi_1^n)),$$
$$\lim_{n\to+\infty} \mathfrak{T}(p(U^3), P(V^3)) = 0,$$
$$\text{where} \quad U^3 = P(\xi_3^n, \xi_4^n, \xi_5^n, \xi_1^n, \xi_2^n) \quad \text{and} \quad V^3 = (p(\xi_3^n), p(\xi_4^n), p(\xi_5^n), p(\xi_1^n), p(\xi_2^n)),$$
$$\lim_{n\to+\infty} \mathfrak{T}(p(U^4), P(V^4)) = 0,$$
$$\text{where} \quad U^4 = P(\xi_4^n, \xi_5^n, \xi_1^n, \xi_2^n, \xi_3^n) \quad \text{and} \quad V^4 = (p(\xi_4^n), p(\xi_5^n), p(\xi_1^n), p(\xi_2^n), p(\xi_3^n)),$$
$$\lim_{n\to+\infty} \mathfrak{T}(p(U^5), P(V^5)) = 0,$$
$$\text{where} \quad U^5 = P(\xi_5^n, \xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n) \quad \text{and} \quad V^5 = (p(\xi_5^n), p(\xi_1^n), p(\xi_2^n), p(\xi_3^n), p(\xi_4^n)),$$

whenever $\{\xi_1^n\}, \{\xi_2^n\}, \{\xi_3^n\}, \{\xi_4^n\}$ *and* $\{\xi_5^n\}$ *are sequences in* $\mathcal{G}$ *in such a way that*

$$\lim_{n\to+\infty} U^1 = \lim_{n\to+\infty} p(\xi_1^n) = \xi_1, \quad \lim_{n\to+\infty} U^2 = \lim_{n\to+\infty} p(\xi_2^n) = \xi_2, \quad \lim_{n\to+\infty} U^3 = \lim_{n\to+\infty} p(\xi_3^n) = \xi_3,$$
$$\lim_{n\to+\infty} U^4 = \lim_{n\to+\infty} p(\xi_4^n) = \xi_4, \quad \lim_{n\to+\infty} U^5 = \lim_{n\to+\infty} p(\xi_5^n) = \xi_5,$$

for some $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathcal{G}$.

Definition 14. *The mappings* $P : \mathcal{G}^5 \to \mathcal{G}$ *and* $p : \mathcal{G} \to \mathcal{G}$ *are termed as reciprocally continuous if for some* $\xi_i \in \mathcal{G}$, *where* $1 \leq i \leq 5$, *we have*

$$\begin{aligned}
\lim_{n\to+\infty} p(U^1) = p(\xi_1) \quad &\textit{and} \quad \lim_{n\to+\infty} P(V^1) = P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5),\\
\lim_{n\to+\infty} p(U^2) = p(\xi_2) \quad &\textit{and} \quad \lim_{n\to+\infty} P(V^2) = P(\xi_2, \xi_3, \xi_4, \xi_5, \xi_1),\\
\lim_{n\to+\infty} p(U^3) = p(\xi_3) \quad &\textit{and} \quad \lim_{n\to+\infty} P(V^3) = P(\xi_3, \xi_4, \xi_5, \xi_1, \xi_2),\\
\lim_{n\to+\infty} p(U^4) = p(\xi_4) \quad &\textit{and} \quad \lim_{n\to+\infty} P(V^4) = P(\xi_4, \xi_5, \xi_1, \xi_2, \xi_3),\\
\lim_{n\to+\infty} p(U^5) = p(\xi_5) \quad &\textit{and} \quad \lim_{n\to+\infty} P(V^5) = P(\xi_5, \xi_1, \xi_2, \xi_3, \xi_4),
\end{aligned}$$

whenever $\{\xi_1^n\}, \{\xi_2^n\}, \{\xi_3^n\}, \{\xi_4^n\}$ *and* $\{\xi_5^n\}$ *are sequences in* $\mathcal{G}$ *such that*

$$\lim_{n\to+\infty} U^1 = \lim_{n\to+\infty} p(\xi_1^n) = \xi_1, \quad \lim_{n\to+\infty} U^2 = \lim_{n\to+\infty} p(\xi_2^n) = \xi_2,, \quad \lim_{n\to+\infty} U^3 = \lim_{n\to+\infty} p(\xi_3^n) = \xi_3,$$
$$\lim_{n\to+\infty} U^4 = \lim_{n\to+\infty} p(\xi_4^n) = \xi_4, \quad \lim_{n\to+\infty} U^5 = \lim_{n\to+\infty} p(\xi_5^n) = \xi_5,$$

for some $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathcal{G}$.

Definition 15. *Two mappings* $P : \mathcal{G}^5 \to \mathcal{G}$ *and* $p : \mathcal{G} \to \mathcal{G}$ *are known as weakly reciprocally continuous if for some* $\xi_i \in \mathcal{G}$, *where* $1 \leq i \leq 5$, *we have*

$$\begin{aligned}
\lim_{n\to+\infty} p(U^1) = p(\xi_1) \quad &\textit{or} \quad \lim_{n\to+\infty} P(V^1) = P(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5),\\
\lim_{n\to+\infty} p(U^2) = p(\xi_2) \quad &\textit{or} \quad \lim_{n\to+\infty} P(V^2) = P(\xi_2, \xi_3, \xi_4, \xi_5, \xi_1),\\
\lim_{n\to+\infty} p(U^3) = p(\xi_3) \quad &\textit{or} \quad \lim_{n\to+\infty} P(V^3) = P(\xi_3, \xi_4, \xi_5, \xi_1, \xi_2),\\
\lim_{n\to+\infty} p(U^4) = p(\xi_4) \quad &\textit{or} \quad \lim_{n\to+\infty} P(V^4) = P(\xi_4, \xi_5, \xi_1, \xi_2, \xi_3),\\
\lim_{n\to+\infty} p(U^5) = p(\xi_5) \quad &\textit{or} \quad \lim_{n\to+\infty} P(V^5) = P(\xi_5, \xi_1, \xi_2, \xi_3, \xi_4),
\end{aligned}$$

whenever $\{\xi_1^n\}, \{\xi_2^n\}, \{\xi_3^n\}, \{\xi_4^n\}$ *and* $\{\xi_5^n\}$ *are few sequences within the set* $\mathcal{G}$ *in such a way that*

$$\lim_{n\to+\infty} U^1 = \lim_{n\to+\infty} p(\xi_1^n) = \xi_1, \quad \lim_{n\to+\infty} U^2 = \lim_{n\to+\infty} p(\xi_2^n) = \xi_2,, \quad \lim_{n\to+\infty} U^3 = \lim_{n\to+\infty} p(\xi_3^n) = \xi_3,$$
$$\lim_{n\to+\infty} U^4 = \lim_{n\to+\infty} p(\xi_4^n) = \xi_4, \quad \lim_{n\to+\infty} U^5 = \lim_{n\to+\infty} p(\xi_5^n) = \xi_5,$$

for some $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathcal{G}$.

Example 3. *Let $\mathcal{G} = [0,1]$. Let $\mathfrak{T}(\xi_1,\xi_2) = |\xi_1 - \xi_2|$ and "$\preceq$" be the partial order on $\mathcal{G}$ defined for all $\xi_1, \xi_2 \in \mathcal{G}$, $\xi_1 \preceq \xi_2 \ \Leftrightarrow \xi_1 \leq \xi_2$. Let $P : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings defined as*

$$P(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = \frac{\xi_1\xi_2 - \xi_3\xi_4 + \xi_5}{5} \quad \textit{and} \quad p(\xi_1) = \xi_1, \quad \forall\ \xi_1,\xi_2,\xi_3,\xi_4,\xi_5 \in \mathcal{G}.$$

Consider the sequences $\{\xi_1^n\}$, $\{\xi_2^n\}$, $\{\xi_3^n\}$, $\{\xi_4^n\}$ and $\{\xi_5^n\}$ defined by

$$\xi_1^n = \frac{1}{n},\ \xi_2^n = \frac{1}{n^3+1},\ \xi_3^n = \frac{1}{\sqrt{n^3+1}},\ \xi_4^n = \frac{1}{n^3}, \quad \textit{and} \quad \xi_5^n = \frac{1}{n^3+2} \quad \forall\ n \in \mathbb{N}.$$

Clearly, $(\mathcal{G},\mathfrak{T})$ is a partially ordered metric space and $\mathcal{G}$ is complete. In addition,

$$\begin{aligned}
&\lim_{n\to+\infty} U^1 = \lim_{n\to+\infty} p(\xi_1^n) = 0, \quad \lim_{n\to+\infty} U^2 = \lim_{n\to+\infty} p(\xi_2^n) = 0,\\
&\lim_{n\to+\infty} U^3 = \lim_{n\to+\infty} p(\xi_3^n) = 0, \quad \lim_{n\to+\infty} U^4 = \lim_{n\to+\infty} p(\xi_4^n) = 0,\\
&\lim_{n\to+\infty} U^5 = \lim_{n\to+\infty} p(\xi_5^n) = 0.
\end{aligned}$$

Moreover, defined sequences, functions and metrics satisfy compatibility conditions, reciprocal continuity and weakly reciprocal continuity for P and p. In light of this, both P and p exhibit compatibility, reciprocal continuity and weak reciprocal continuity.

Definition 16. *For two mappings $\zeta^n : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ defined on the metric space $(\mathcal{G},\mathfrak{T})$, the sequence $\{\zeta^n\}_{n\in\mathbb{W}}$ and p are known as compatible if*

$$\begin{aligned}
&\lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^1}),\zeta^n(\bar{V^1})) = 0,\\
&\textit{where} \quad \bar{U^1} = \zeta^n(\xi_1^n,\xi_2^n,\xi_3^n,\xi_4^n,\xi_5^n) \quad \textit{and} \quad \bar{V^1} = (p(\xi_1^n),p(\xi_2^n),p(\xi_3^n),p(\xi_4^n),p(\xi_5^n)),\\
&\lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^2}),\zeta^n(\bar{V^2})) = 0,\\
&\textit{where} \quad \bar{U^2} = \zeta^n(\xi_2^n,\xi_3^n,\xi_4^n,\xi_5^n,\xi_1^n) \quad \textit{and} \quad \bar{V^2} = (p(\xi_2^n),p(\xi_3^n),p(\xi_4^n),p(\xi_5^n),p(\xi_1^n)),\\
&\lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^3}),\zeta^n(\bar{V^3})) = 0,\\
&\textit{where} \quad \bar{U^3} = \zeta^n(\xi_3^n,\xi_4^n,\xi_5^n,\xi_1^n,\xi_2^n) \quad \textit{and} \quad \bar{V^3} = (p(\xi_3^n),p(\xi_4^n),p(\xi_5^n),p(\xi_1^n),p(\xi_2^n)),\\
&\lim_{n\to\infty} \mathfrak{T}(p(\bar{U^4}),\zeta^n(\bar{V^4})) = 0,\\
&\textit{where} \quad \bar{U^4} = \zeta^n(\xi_4^n,\xi_5^n,\xi_1^n,\xi_2^n,\xi_3^n) \quad \textit{and} \quad \bar{V^4} = (p(\xi_4^n),p(\xi_5^n),p(\xi_1^n),p(\xi_2^n),p(\xi_3^n)),\\
&\lim_{n\to\infty} \mathfrak{T}(p(\bar{U^5}),\zeta^n(\bar{V^5})) = 0,\\
&\textit{where} \quad \bar{U^5} = \zeta^n(\xi_5^n,\xi_1^n,\xi_2^n,\xi_3^n,\xi_4^n) \quad \textit{and} \quad \bar{V^5} = (p(\xi_5^n),p(\xi_1^n),p(\xi_2^n),p(\xi_3^n),p(\xi_4^n)),
\end{aligned}$$

whenever $\{\xi_1^n\},\{\xi_2^n\},\{\xi_3^n\},\{\xi_4^n\}$ and $\{\xi_5^n\}$ are sequences in $\mathcal{G}$ such that

$$\lim_{n\to+\infty} \bar{U^1} = \lim_{n\to\infty} p(\xi_1^{n+1}) = \xi_1, \quad \lim_{n\to+\infty} \bar{U^2} = \lim_{n\to\infty} p(\xi_2^{n+1}) = \xi_2, \quad \lim_{n\to+\infty} \bar{U^3} = \lim_{n\to\infty} p(\xi_3^{n+1}) = \xi_3$$
$$\lim_{n\to+\infty} \bar{U^4} = \lim_{n\to\infty} p(\xi_4^{n+1}) = \xi_4, \quad \lim_{n\to+\infty} \bar{U^5} = \lim_{n\to\infty} p(\xi_5^{n+1}) = \xi_5,$$

for some $\xi_1,\xi_2,\xi_3,\xi_4,\xi_5 \in \mathcal{G}$.

Definition 17. *Let $\zeta^n : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings on a metric space $(\mathcal{G}, \mathfrak{T})$, then the following conditions describe weak reciprocal continuity of $\{\zeta^n\}_{n\in\mathbb{N}}$ and p.*

$$\lim_{n\to+\infty} p(\bar{U^1}) = p(\xi_1), \quad \lim_{n\to+\infty} p(\bar{U^2}) = p(\xi_2), \quad \lim_{n\to+\infty} p(\bar{U^3}) = p(\xi_3)$$
$$\lim_{n\to+\infty} p(\bar{U^4}) = p(\xi_4), \quad \lim_{n\to+\infty} p(\bar{U^5}) = p(\xi_5),$$

whenever $\{\xi_1^n\}, \{\xi_2^n\}, \{\xi_3^n\}, \{\xi_4^n\}$ and $\{\xi_5^n\}$ are sequences in $\mathcal{G}$ such that

$$\lim_{n\to+\infty} \bar{U^1} = \lim_{n\to\infty} p(\xi_1^{n+1}) = \xi_1, \quad \lim_{n\to+\infty} \bar{U^2} = \lim_{n\to\infty} p(\xi_2^{n+1}) = \xi_2, \quad \lim_{n\to+\infty} \bar{U^3} = \lim_{n\to\infty} p(\xi_3^{n+1}) = \xi_3$$
$$\lim_{n\to+\infty} \bar{U^4} = \lim_{n\to\infty} p(\xi_4^{n+1}) = \xi_4, \quad \lim_{n\to+\infty} \bar{U^5} = \lim_{n\to\infty} p(\xi_5^{n+1}) = \xi_5,$$

for some ξ_i in $\mathcal{G}$, where $1 \le i \le 5$.

Example 4. *Let $\mathcal{G} = [0,1]$ be endowed with the metric $\mathfrak{T}(\xi_1, \xi_2) = |\xi_1 - \xi_2|$. Let $\zeta^n : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings defined as*

$$\zeta^n(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \frac{1}{5^n} - \frac{\xi_1\xi_2\xi_3\xi_4\xi_5}{5} \quad \textit{and} \quad p(\xi_1) = \xi_1, \quad \forall\ \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in \mathcal{G}.$$

Consider the five sequences $\{\xi_1^n\}, \{\xi_2^n\}, \{\xi_3^n\}, \{\xi_4^n\}$ and $\{\xi_5^n\} \in \mathcal{G}$ defined as

$$\xi_1^n = \frac{1}{n^2+1},\ \xi_2^n = \frac{1}{\sqrt{n^2+1}},\ \xi_3^n = \frac{1}{n+1},\ \xi_4^n = \frac{1}{\sqrt{n+1}} \ \textit{and}\ \xi_5^n = \frac{1}{n^3+1}, \quad \forall\ n \in\ \mathbb{N}.$$

Then,

$$\lim_{n\to+\infty} \bar{U^1} = \lim_{n\to\infty} p(\xi_1^{n+1}) = 0, \quad \lim_{n\to+\infty} \bar{U^2} = \lim_{n\to\infty} p(\xi_2^{n+1}) = 0,$$
$$\lim_{n\to+\infty} \bar{U^3} = \lim_{n\to\infty} p(\xi_3^{n+1}) = 0, \lim_{n\to+\infty} \bar{U^4} = \lim_{n\to\infty} p(\xi_4^{n+1}) = 0,$$
$$\lim_{n\to+\infty} \bar{U^5} = \lim_{n\to\infty} p(\xi_5^{n+1}) = 0.$$

Also,

$$\lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^1}), \zeta^n(\bar{V^1})) = 0, \ \lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^2}), \zeta^n(\bar{V^2})) = 0,$$
$$\lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^3}), \zeta^n(\bar{V^3})) = 0, \ \lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^4}), \zeta^n(\bar{V^4})) = 0,$$
$$\lim_{n\to+\infty} \mathfrak{T}(p(\bar{U^5}), \zeta^n(\bar{V^5})) = 0,$$

and

$$\lim_{n\to+\infty} p(\bar{U^1}) = p(\xi_1) = 0, \quad \lim_{n\to+\infty} p(\bar{U^2}) = p(\xi_2) = 0, \quad \lim_{n\to+\infty} p(\bar{U^3}) = p(\xi_3) = 0$$
$$\lim_{n\to+\infty} p(\bar{U^4}) = p(\xi_4) = 0, \quad \lim_{n\to+\infty} p(\bar{U^5}) = p(\xi_5) = 0,$$

which proves the compatibility and weak reciprocal continuity of $\{\zeta^n\}_{n\in\mathbb{N}}$ and p.

Definition 18. *Let $(\mathcal{G}, \preceq)$ be a partially ordered set (PoS) and $\zeta^n : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings on $\mathcal{G}$, then $\{\zeta^n\}_{n\in\mathbb{W}}$ is said to have MpMP if for any $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5 \in \mathcal{G}$,*

$$\begin{aligned}
p(\xi_1) \preceq p(\mathfrak{q}_1) \quad &\Rightarrow \quad \zeta^n(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \preceq \zeta^{n+1}(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5),\\
p(\xi_2) \succeq p(\mathfrak{q}_2) \quad &\Rightarrow \quad \zeta^n(\xi_2, \xi_3, \xi_4, \xi_5, \xi_1) \succeq \zeta^{n+1}(\mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_1),\\
p(\xi_3) \preceq p(\mathfrak{q}_3) \quad &\Rightarrow \quad \zeta^n(\xi_3, \xi_4, \xi_5, \xi_1, \xi_2) \preceq \zeta^{n+1}(\mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_1, \mathfrak{q}_2),\\
p(\xi_4) \succeq p(\mathfrak{q}_4) \quad &\Rightarrow \quad \zeta^n(\xi_4, \xi_5, \xi_1, \xi_2, \xi_3) \succeq \zeta^{n+1}(\mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3),\\
p(\xi_5) \preceq p(\mathfrak{q}_5) \quad &\Rightarrow \quad \zeta^n(\xi_5, \xi_1, \xi_2, \xi_3, \xi_4) \preceq \zeta^{n+1}(\mathfrak{q}_5, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4).
\end{aligned}$$

Definition 19. *Let $\zeta^i : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings. Then, $\{\zeta^i\}_{i\in\mathbb{W}}$ and p are said to satisfy the (C) condition if*

$$\begin{aligned}
\mathfrak{T}(\zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)) \leq \tilde{\Gamma}[&\mathfrak{T}(p(\xi_1), \zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5))\\
&+ \mathfrak{T}(p(\mathfrak{q}_1), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5))] \qquad (1)\\
&+ \tilde{\Upsilon}[\mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1))],
\end{aligned}$$

for some $\xi_i, \mathfrak{q}_i \in \mathcal{G}$, where $1 \leq i \leq 5$, provided that $p(\xi_i) \preceq p(\mathfrak{q}_i)$, for $1 \leq i \leq 5$, or $p(\xi_i) \succeq p(\mathfrak{q}_i)$, for $1 \leq i \leq 5$. In addition, $\mathcal{I} \neq \tilde{\Gamma} = (\tilde{\Gamma}_{ij})$ and $\mathcal{I} \neq \tilde{\Upsilon} = (\tilde{\Upsilon}_{ij}) \in ZM$ satisfy the condition that $(\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1} \in ZM$.

Example 5. *Let $\mathcal{G} = [0, 1]$ be equipped with metric $\mathfrak{T}(\xi_1, \xi_2) = |\xi_1 - \xi_2|$, for all $\xi_1, \xi_2 \in \mathcal{G}$.*

(1) Let $\tilde{\Gamma} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$ and $\tilde{\Upsilon} = \begin{pmatrix} 0 & \frac{1}{5} \\ \frac{1}{5} & 0 \end{pmatrix} \in ZM$. Then, $(\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1} \in; ZM$.

(2) Let $\tilde{\Gamma} = \xi\mathcal{I}$, and $\tilde{\Upsilon} = ((1-\xi)^3 - \xi)\mathcal{I} \in ZM$ such that $\xi = \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}$, then $(\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1} \in ZM$.

Definition 20. *Let $\zeta^o : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings, then ζ^o and p are said to have mixed quintuple transcendence point (MQTP) if there exists some $\xi_1^o, \xi_2^o, \xi_3^o, \xi_4^o, \xi_5^o \in \mathcal{G}$ such that*

$$\begin{aligned}
&\zeta^o(\xi_1^o, \xi_2^o, \xi_3^o, \xi_4^o, \xi_5^o) \succeq p(\xi_1^o), \quad \zeta^o(\xi_2^o, \xi_3^o, \xi_4^o, \xi_5^o, \xi_1^o) \preceq p(\xi_2^o),\\
&\zeta^o(\xi_3^o, \xi_4^o, \xi_5^o, \xi_1^o, \xi_2^o) \succeq p(\xi_3^o), \quad \zeta^o(\xi_4^o, \xi_5^o, \xi_1^o, \xi_2^o, \xi_3^o) \preceq p(\xi_4^o), \qquad (2)\\
&\zeta^o(\xi_5^o, \xi_1^o, \xi_2^o, \xi_3^o, \xi_4^o) \succeq p(\xi_5^o),
\end{aligned}$$

given that ζ^o and p have non-decreasing transcendent point in $\xi_1^o, \xi_3^o, \xi_5^o$ and a non-increasing transcendence point in ξ_2^o, ξ_4^o.

Lemma 2. *Let $\zeta^i : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings in the setting of a partially ordered complete generalized metric space (POCGMs) $(\mathcal{G}, \mathfrak{T}, \preceq)$. Suppose that $\{\zeta^i\}_{i\in\mathbb{W}}$ have MpMP such that $\zeta^i(\mathcal{G}^5) \subseteq p(\mathcal{G})$. If ζ^o and p have MQTp, then* *(a):* $\exists$ *sequences* $\{\xi_1^n\}, \{\xi_2^n\}, \{\xi_3^n\}, \{\xi_4^n\}$ *and* $\{\xi_5^n\} \in \mathcal{G}$ *such that*

$$p(\xi_1^n) = \zeta^{n-1}(\xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1}), \quad p(\xi_2^n) = \zeta^{n-1}(\xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1}, \xi_1^{n-1}),$$
$$p(\xi_3^n) = \zeta^{n-1}(\xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1}, \xi_1^{n-1}, \xi_2^{n-1}), \quad p(\xi_4^n) = \zeta^{n-1}(\xi_4^{n-1}, \xi_5^{n-1}, \xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}),$$
$$p(\xi_5^n) = \zeta^{n-1}(\xi_5^{n-1}, \xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}).$$

(b): $\{p(\xi_1^n)\}, \{p(\xi_3^n)\}, \{p(\xi_5^n)\}$ *are non-decreasing sequences and* $\{p(\xi_2^n)\}, \{p(\xi_4^n)\}$ *are non-increasing sequences.*

Proof. (a): Suppose that condition in (2) is fulfilled for some $\xi_1^o, \xi_2^o, \xi_3^o, \xi_4^o, \xi_5^o \in \mathcal{G}$. Since $\zeta^o(\mathcal{G}^5) \subseteq p(\mathcal{G})$, then some elements $\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1, \xi_5^1 \in \mathcal{G}$

$$\begin{aligned} p(\xi_1^1) =& \zeta^o(\xi_1^o, \xi_2^o, \xi_3^o, \xi_4^o, \xi_5^o),\ p(\xi_2^1) = \zeta^o(\xi_2^o, \xi_3^o, \xi_4^o, \xi_5^o, \xi_1^o), \\ p(\xi_3^1) =& \zeta^o(\xi_3^o, \xi_4^o, \xi_5^o, \xi_1^o, \xi_2^o),\ p(\xi_4^1) = \zeta^o(\xi_4^o, \xi_5^o, \xi_1^o, \xi_2^o, \xi_3^o), \\ p(\xi_5^1) =& \zeta^o(\xi_5^o, \xi_1^o, \xi_2^o, \xi_3^o, \xi_4^o). \end{aligned} \tag{3}$$

Since $\zeta^o(\mathcal{G}^5) \subseteq p(\mathcal{G})$, then some elements can be set as $\xi_1^2, \xi_2^2, \xi_3^2, \xi_4^2, \xi_5^2 \in \mathcal{G}$ such that

$$\begin{aligned} p(\xi_1^2) =& \zeta^1(\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1, \xi_5^1),\ p(\xi_2^2) = \zeta^1(\xi_2^1, \xi_3^1, \xi_4^1, \xi_5^1, \xi_1^1), \\ p(\xi_3^2) =& \zeta^1(\xi_3^1, \xi_4^1, \xi_5^1, \xi_1^1, \xi_2^1),\ p(\xi_4^2) = \zeta^1(\xi_4^1, \xi_5^1, \xi_1^1, \xi_2^1, \xi_3^1), \\ p(\xi_5^2) =& \zeta^1(\xi_5^1, \xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1). \end{aligned}$$

Proceeding similarly, we obtain

$$\begin{aligned} p(\xi_1^n) &= \zeta^{n-1}(\xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1}), \\ p(\xi_2^n) &= \zeta^{n-1}(\xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1}, \xi_1^{n-1}), \\ p(\xi_3^n) &= \zeta^{n-1}(\xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1}, \xi_1^{n-1}, \xi_2^{n-1}), \\ p(\xi_4^n) &= \zeta^{n-1}(\xi_4^{n-1}, \xi_5^{n-1}, \xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}), \\ p(\xi_5^n) &= \zeta^{n-1}(\xi_5^{n-1}, \xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}). \end{aligned} \tag{4}$$

(b): Now, from (2) and (3),

$$p(\xi_1^o) \preceq p(\xi_1^1), \quad p(\xi_3^o) \preceq p(\xi_3^1), \quad p(\xi_5^o) \preceq p(\xi_5^1), \quad p(\xi_2^o) \succeq s(\xi_2^1) \quad \text{and} \quad s(\xi_4^o) \succeq s(\xi_4^1).$$

Then, $\forall\ n \geq 0$, by mathematical induction, it is obtained that

$$\begin{aligned} &p(\xi_1^n) \preceq p(\xi_1^{n+1}), \quad p(\xi_3^n) \preceq p(\xi_3^{n+1}), \quad p(\xi_5^n) \preceq p(\xi_5^{n+1}), \\ &p(\xi_2^n) \succeq p(\xi_2^{n+1}), \quad \text{and} \quad p(\xi_4^n) \succeq p(\xi_4^{n+1}). \end{aligned} \tag{5}$$

Hence, (4) and (5) complete the required result.

Our next theorem is the core part of this section.

Theorem 1. *Suppose that all suppositions of Lemma 2 hold, let* $\{\zeta^i\}_{i\in\mathbb{W}}$ *and* p *be two monotonically decreasing mappings such that they satisfy (C) condition. In addition, suppose that both mappings are compatible and weakly reciprocally continuous and* p *is continuous. If* $p(\mathcal{G}) \subseteq \mathcal{G}$ *is complete and regular, then there exists a quintuple coincidence point (QCP) of* $\{\zeta^i\}_{i\in\mathbb{W}}$ *and* p *provided that* $\tilde{\Gamma}, \tilde{\Upsilon} \neq \mathcal{O}$ *belongs to ZM.*

Proof. Let $\{\xi_1^n\}, \{\xi_2^n\}, \{\xi_3^n\}, \{\xi_4^n\}$ and $\{\xi_5^n\}$ be the sequences in $\mathcal{G}$ constructed by Lemma 2, then from (2) it follows that

$$\begin{aligned}
\mathfrak{T}(p(\xi_1^n), (\xi_1^{n+1})) &= \mathfrak{T}(\zeta^{n-1}(\xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1}), \zeta^n(\xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n)) \\
&\leq \tilde{\Gamma}[\mathfrak{T}(p(\xi_1^{n-1}), \zeta^{n-1}(\xi_1^{n-1}, \xi_2^{n-1}, \xi_3^{n-1}, \xi_4^{n-1}, \xi_5^{n-1})) \\
&\quad + \mathfrak{T}(p(\xi_1^n), \zeta^n(\xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n))] + \tilde{\Upsilon}(\mathfrak{T}(p(\xi_1^{n-1}), p(\xi_1^n))) \\
&= (\tilde{\Gamma} + \tilde{\Upsilon})\mathfrak{T}(p(\xi_1^{n-1}), p(\xi_1^n)) + \tilde{\Gamma}\mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+1})).
\end{aligned}$$

It results in

$$\mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+1})) \leq (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}\mathfrak{T}(p(\xi_1^{n-1}), p(\xi_1^n)). \tag{6}$$

Similar operations generate

$$\begin{aligned}
\mathfrak{T}(p(\xi_2^n), p(\xi_2^{n+1})) &\leq (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}\mathfrak{T}(p(\xi_2^{n-1}), p(\xi_2^n)), \\
\mathfrak{T}(p(\xi_3^n), p(\xi_3^{n+1})) &\leq (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}\mathfrak{T}(p(\xi_3^{n-1}), p(\xi_3^n)), \\
\mathfrak{T}(p(\xi_4^n), p(\xi_4^{n+1})) &\leq (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}\mathfrak{T}(p(\xi_4^{n-1}), p(\xi_4^n)),
\end{aligned} \tag{7}$$

and

$$\mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+1})) \leq (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}\mathfrak{T}(p(\xi_5^{n-1}), p(\xi_5^n)). \tag{8}$$

Adding (3.6) - (3.8), one writes

$$\begin{aligned}
\mu^n &= \mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+1})) + \mathfrak{T}(p(\xi_2^n), p(\xi_2^{n+1})) + \mathfrak{T}(p(\xi_3^n), p(\xi_3^{n+1})) + \mathfrak{T}(p(\xi_4^n), p(\xi_4^{n+1})) \\
&\quad + \mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+1})) \\
&\leq (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}[\mathfrak{T}(p(\xi_1^{n-1}), p(\xi_1^n)) + \mathfrak{T}(p(\xi_2^{n-1}), p(\xi_2^n)) + \mathfrak{T}(p(\xi_3^{n-1}), p(\xi_3^n)) \\
&\quad + \mathfrak{T}(p(\xi_4^{n-1}), P(\xi_4^n)) + \mathfrak{T}(p(\xi_5^{n-1}), P(\xi_5^n))] \\
&= (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}\mu^{n-1}.
\end{aligned}$$

Take $(\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1} = Q$, hence for $n \in \mathbb{N}$,

$$\mathcal{O} \leq \mu^n \leq Q\mu^{n-1} \leq Q^2\mu^{n-2} \leq \cdots \leq Q^n\mu^o.$$

In light of triangular inequality, for $m > 0$,

$$\begin{aligned}
&\mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+m})) + \mathfrak{T}(p(\xi_2^n), p(\xi_2^{n+m})) + \mathfrak{T}(p(\xi_3^n), p(\xi_3^{n+m})) + \mathfrak{T}(p(\xi_4^n), p(\xi_4^{n+m})) \\
&\quad + \mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+m}))
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+1})) + \mathfrak{T}(p(\xi_2^n), (\xi_2^{n+1})) + \mathfrak{T}(p(\xi_3^n), p(\xi_3^{n+1})) + \mathfrak{T}(p(\xi_4^n), p(\xi_4^{n+1}))\\
&\quad + \mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+1})) + \mathfrak{T}(p(\xi_1^{n+1}), p(\xi_1^{n+2})) + \mathfrak{T}(p(\xi_2^{n+1}), p(\xi_2^{n+2})) + \mathfrak{T}(p(\xi_3^{n+1}), p(\xi_3^{n+2}))\\
&\quad + \mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+2})) + \mathfrak{T}(p(\xi_4^{n+1}), p(a_4^{\xi+2})) + \cdots + \mathfrak{T}(p(\xi_1^{n+m-1}), p(\xi_1^{n+m}))\\
&\quad + \mathfrak{T}(p(\xi_2^{n+m-1}), p(\xi_2^{n+m}))\\
&\quad + \mathfrak{T}(p(\xi_3^{n+m-1}), p(\xi_3^{n+m})) + \mathfrak{T}(p(\xi_4^{n+m-1}), p(\xi_4^{n+m})) + \mathfrak{T}(p(\xi_5^{n+m-1}), p(\xi_5^{n+m}))\\
&= \mu^n + \mu^{n+1} + \cdots + \mu^{n+m-1}\\
&\leq (Q^n + Q^{n+1} + \cdots + Q^{n+m-1})\mu^o\\
&= Q^n(\mathcal{I} + Q + \cdots + Q^{m-1} + \cdots)\mu^o\\
&= \ Q^n(\mathcal{I} - Q)^{-1}\mu^o.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\lim_{n\to+\infty}&[\mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+m})) + \mathfrak{T}(p(\xi_2^n), p(\xi_2^{n+m})) + \mathfrak{T}(p(\xi_3^n), p(\xi_3^{n+m})) + \mathfrak{T}(p(\xi_4^n), p(\xi_4^{n+m}))\\
&\quad + \mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+m}))]\\
&\leq [(\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}]^n[\mathcal{I} - (\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}]^{-1}\mu^o\\
&= [(\tilde{\Gamma} + \tilde{\Upsilon})(\mathcal{I} - \tilde{\Gamma})^{-1}]^n[\mathcal{I} - (\tilde{\Gamma} + \tilde{\Upsilon})(I - \tilde{\Gamma})^{-1}]^{-1}\mu^o.
\end{aligned}
$$

Applying $\lim\limits_{n\to+\infty}$ on both sides yields that

$$
\begin{aligned}
\lim_{n\to+\infty}&[\mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+m})) + \mathfrak{T}(p(\xi_2^n), p(\xi_2^{n+m})) + \mathfrak{T}(p(\xi_3^n), p(\xi_3^{n+m})) + \mathfrak{T}(p(\xi_4^n), p(\xi_4^{n+m}))\\
&\quad + \mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+m}))] = 0,
\end{aligned}
$$

or

$$
\begin{aligned}
\lim_{n\to+\infty}&\mathfrak{T}(p(\xi_1^n), p(\xi_1^{n+m})) = \lim_{n\to+\infty}\mathfrak{T}(p(\xi_2^n), p(\xi_2^{n+m})) = \lim_{n\to+\infty}\mathfrak{T}(p(\xi_3^n), p(\xi_3^{n+m})) =\\
&= \lim_{n\to+\infty}\mathfrak{T}(p(\xi_4^n), p(\xi_4^{n+m}))] = \lim_{n\to+\infty}\mathfrak{T}(p(\xi_5^n), p(\xi_5^{n+m}))] = 0.
\end{aligned}
$$

Hence, $\{p(\xi_1^n)\}, \{p(\xi_2^n)\}, \{p(\xi_3^n)\}, \{p(\xi_4^n)\}$ and $\{p(\xi_5^n)\}$ are Cauchy sequences within the set $\mathcal{G}$. Moreover, by completeness of $p(\mathcal{G})$, there must exist $(\xi_1^*, \xi_2^*, \xi_3^*, \xi_4^*, \xi_5^*) \in \mathcal{G}^5$ such that

$$
\begin{aligned}
&\lim_{n\to+\infty}p(\xi_1^n) = p(\xi_1^*) = \xi_1, \ \lim_{n\to+\infty}p(\xi_2^n) = p(\xi_2^*) = \xi_2,\\
&\lim_{n\to+\infty}p(\xi_3^n) = p(\xi_3^*) = \xi_3, \lim_{n\to+\infty}p(\xi_4^n) = p(\xi_4^*) = \xi_4,\\
&\lim_{n\to+\infty}p(\xi_5^n) = p(\xi_5^*) = \xi_5,
\end{aligned}
$$

which results in

$$
\begin{aligned}
&\lim_{n\to+\infty}p(\xi_1^{n+1}) = \lim_{n\to+\infty}\zeta^n(\xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n), \ \lim_{n\to+\infty}p(\xi_2^{n+1}) = \lim_{n\to+\infty}\zeta^n(\xi_2^n, \xi_3^n, \xi_4^n, \xi_5^n, \xi_1^n),\\
&\lim_{n\to+\infty}p(\xi_3^{n+1}) = \lim_{n\to+\infty}\zeta^n(\xi_3^n, \xi_4^n, \xi_5^n, \xi_1^n, \xi_2^n), \ \lim_{n\to+\infty}p(\xi_4^{n+1}) = \lim_{n\to+\infty}\zeta^n(\xi_4^n, \xi_5^n, \xi_1^n, \xi_2^n, \xi_3^n),\\
&\lim_{n\to+\infty}p(\xi_5^{n+1}) = \lim_{n\to+\infty}\zeta^n(\xi_5^n, \xi_1^n, \xi_2^n, \xi_3^n, \xi_4^n).
\end{aligned}
$$

Also, from the weak reciprocal continuity and compatibility of $\{\zeta^i\}_{i\in\mathbb{W}}$ and p, it is derived that

$$\begin{aligned}
&\lim_{n\to+\infty}\zeta^n(p(\xi_1^n),p(\xi_2^n),p(\xi_3^n),p(\xi_4^n),p(\xi_5^n))=p(\xi_1),\\
&\lim_{n\to+\infty}\zeta^n(p(\xi_2^n),p(\xi_3^n),p(\xi_4^n),p(\xi_5^n),p(\xi_1^n))=p(\xi_2),\\
&\lim_{n\to+\infty}\zeta^n(p(\xi_3^n),p(\xi_4^n),p(\xi_5^n),p(\xi_1^n),p(\xi_2^n))=p(\xi_3),\\
&\lim_{n\to+\infty}\zeta^n(p(\xi_4^n),p(\xi_5^n),p(\xi_1^n),p(\xi_2^n),p(\xi_3^n))=p(\xi_4),\\
&\lim_{n\to+\infty}\zeta^n(p(\xi_5^n),p(\xi_1^n),p(\xi_2^n),s(\xi_3^n),p(\xi_4^n))=p(\xi_5).
\end{aligned}$$

Since $\{p(\xi_1^n)\}$, $\{p(\xi_3^n)\}$, $\{p(\xi_5^n)\}$ are non-decreasing sequences and $\{p(\xi_2^n)\}$, $\{p(\xi_4^n)\}$ are non-increasing sequences, from the regularity of $\mathcal{G}$, for all $n\geq 0$, it is obtained that $p(\xi_1^n)\preceq\xi_1,\xi_2\preceq p(\xi_2^n),p(\xi_3^n)\preceq\xi_3,\xi_4\preceq p(\xi_4^n),p(\xi_5^n)\preceq\xi_5$. Then, from (1), it is computed as

$$\begin{aligned}
&\mathfrak{T}(\zeta^i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5),\zeta^n(p(\xi_1^n),p(\xi_2^n),p(\xi_3^n),p(\xi_4^n),p(\xi_5^n))\\
&\leq\tilde{\Gamma}[\mathfrak{T}(p(\xi_1),\zeta^i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5))\\
&\quad+\mathfrak{T}(p(p(\xi_1^n)),\zeta^n(p(\xi_1^n),p(\xi_2^n),p(\xi_3^n),p(\xi_4^n),p(\xi_5^n))]\\
&\quad+\tilde{\Upsilon}\mathfrak{T}(p(\xi_1),p(p(\xi_1^n))).
\end{aligned}$$

Hence, taking $n\to+\infty$, one gets

$$\mathfrak{T}(\zeta^i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5),p(\xi_1))\leq\tilde{\Gamma}\mathfrak{T}(p(\xi_1),\zeta^i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)),$$

which only holds if

$$\mathfrak{T}(\zeta^i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5),p(\xi_1))=0\ \Rightarrow\ \zeta^i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)=p(\xi_1).$$

Similar operation generates

$$\begin{aligned}
&\zeta^i(\xi_2,\xi_3,\xi_4,\xi_5,\xi_1)=p(\xi_2),\ \zeta^i(\xi_3,\xi_4,\xi_5,\xi_1,\xi_2)=p(\xi_3),\\
&\zeta^i(\xi_4,\xi_5,\xi_1,\xi_2,\xi_3)=p(\xi_4)\text{ and }\zeta^i(\xi_5,\xi_1,\xi_2,\xi_3,\xi_4)=p(\xi_5).
\end{aligned}$$

Hence, $(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)$ is a QCP of $\{\zeta^i\}_{i\in\mathbb{W}}$ and p.

The next result is an extension of Theorem 1 by introducing $s=\mathcal{I}_d$ as an identity map.

Corollary 1. *Let $\{\zeta^i\}_{i\in\mathbb{W}}:\mathcal{G}^5\to\mathcal{G}$ be a mixed-monotone sequence over a POCGMs $(\mathcal{G},\mathfrak{T},\preceq)$ such that $\{\zeta^i\}_{i\in\mathbb{W}}$ and $\mathcal{I}_d:\mathcal{G}\to\mathcal{G}$ satisfy (C) condition and $\mathcal{I}_d(\aleph)$ is regular. If $\mathcal{I}_d$ and ζ^o have MQTP, then $\exists$ $(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\in\mathcal{G}^5$ such that*

$$\begin{aligned}
&\zeta^i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)=\xi_1,\quad \zeta^i(\xi_2,\xi_3,\xi_4,\xi_5,\xi_1)=\xi_2,\\
&\zeta^i(\xi_3,\xi_4,\xi_5,\xi_1,\xi_2)=\xi_3,\quad \zeta^i(\xi_4,\xi_5,\xi_1,\xi_2,\xi_3)=\xi_4,\\
\textit{and}\quad &\zeta^i(\xi_5,\xi_1,\xi_2,\xi_3,\xi_4)=\xi_5.
\end{aligned}$$

for $i\in\mathbb{W}$.

By excluding some of the conditions from Corollary 1, taking $\tilde{\Gamma}$ as a zero matrix and expanding the distance $\mathfrak{T}(\xi_1, \xi_2)$, we conclude an important outcome.

Corollary 2. *Let $\mathcal{F} : \mathcal{G}^5 \to \mathcal{G}$ be a mixed-monotone mapping in the setting of a POCGMs $(\mathcal{G}, \mathfrak{T}, \preceq)$ such that $\mathcal{F}$ has a MQTP and $\mathcal{F}$ satisfy the condition*

$$\mathfrak{T}(\mathcal{F}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \mathcal{F}(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)) \leq \tilde{\Upsilon}(\mathfrak{T}((\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5))).$$

Then, their exists a QFP of $\mathcal{F}$ in $\mathcal{G}$.

Definition 21. *Two points $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ and $(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5) \in \mathcal{G}$ are called quintuple comparable (QC) if and only if*

$$\begin{array}{lll}
\xi_1 \preceq \mathfrak{q}_1,\ \xi_2 \succeq \mathfrak{q}_2,\ \xi_3 \preceq \mathfrak{q}_3,\ \xi_4 \succeq \mathfrak{q}_4,\ \xi_5 \preceq \mathfrak{q}_5 & \textit{or} & \xi_1 \succeq \mathfrak{q}_1,\ \xi_2 \preceq \mathfrak{q}_2,\ \xi_3 \succeq \mathfrak{q}_3,\ \xi_4 \preceq \mathfrak{q}_4,\ \xi_5 \succeq \mathfrak{q}_5 \quad \textit{or} \\
\xi_1 \preceq \mathfrak{q}_2,\ \xi_2 \succeq \mathfrak{q}_3,\ \xi_3 \preceq \mathfrak{q}_4,\ \xi_4 \succeq \mathfrak{q}_5,\ \xi_5 \preceq \mathfrak{q}_1 & \textit{or} & \xi_1 \succeq \mathfrak{q}_2,\ \xi_2 \preceq \mathfrak{q}_3,\ \xi_3 \succeq \mathfrak{q}_4,\ \xi_4 \preceq \mathfrak{q}_5,\ \xi_5 \succeq \mathfrak{q}_1 \quad \textit{or} \\
\xi_1 \preceq \mathfrak{q}_3,\ \xi_2 \succeq \mathfrak{q}_4,\ \xi_3 \preceq \mathfrak{q}_5,\ \xi_4 \succeq \mathfrak{q}_1,\ \xi_5 \preceq \mathfrak{q}_2 & \textit{or} & \xi_1 \succeq \mathfrak{q}_3,\ \xi_2 \preceq \mathfrak{q}_4,\ \xi_3 \succeq \mathfrak{q}_5,\ \xi_4 \preceq \mathfrak{q}_1,\ \xi_5 \succeq \mathfrak{q}_2 \quad \textit{or} \\
\xi_1 \preceq \mathfrak{q}_4,\ \xi_2 \succeq \mathfrak{q}_5,\ \xi_3 \preceq \mathfrak{q}_1,\ \xi_4 \succeq \mathfrak{q}_2,\ \xi_5 \preceq \mathfrak{q}_3 & \textit{or} & \xi_1 \succeq \mathfrak{q}_4,\ \xi_2 \preceq \mathfrak{q}_5,\ \xi_3 \succeq \mathfrak{q}_1,\ \xi_5 \preceq \mathfrak{q}_2,\ \xi_5 \succeq \mathfrak{q}_3 \quad \textit{or} \\
\xi_1 \preceq \mathfrak{q}_5,\ \xi_2 \succeq \mathfrak{q}_1,\ \xi_3 \preceq \mathfrak{q}_2,\ \xi_4 \succeq \mathfrak{q}_3,\ \xi_5 \preceq \mathfrak{q}_4 & \textit{or} & \xi_1 \succeq \mathfrak{q}_5,\ \xi_2 \preceq \mathfrak{q}_1,\ \xi_3 \succeq \mathfrak{q}_2,\ \xi_4 \preceq \mathfrak{q}_3,\ \xi_5 \succeq \mathfrak{q}_4.
\end{array}$$

If we replace $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ and $(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)$ with $(p(\xi_1), p(\xi_2), p(\xi_3), p(\xi_4), p(\xi_5))$ and $(p(\mathfrak{q}_1),$
$p(\mathfrak{q}_2), p(\mathfrak{q}_3), p(\mathfrak{q}_4), p(\mathfrak{q}_5))$ in above Definition, then we say that $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ a QC with $(\mathfrak{q}, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4,$
$\mathfrak{q}_5)$ with respect to (w.r.t) p.

Theorem 2. *Let $\{\zeta^i\}_{i\in\mathbb{W}} : \mathcal{G}^5 \to \mathcal{G}$ and $p : \mathcal{G} \to \mathcal{G}$ be two mappings over a POCGMs $(\mathcal{G}, \mathfrak{T}, \preceq)$ such that $\{\zeta^i\}_{i\in\mathbb{W}}$ and p have quintuple coincidence point with quintuple comparable (w.r.t) p and satisfy (C) condition. Then, there is a unique quintuple coincidence point $\{\zeta^i\}_{i\in\mathbb{W}}$ and p.*

Proof. Theorem 1 shows that the existence of a QCPs of mappings is ensured. Let $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ and $((\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)$ be QCPs, that is, if

$$\begin{aligned}
&p(\xi_1) = \zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5),\ p(\xi_2) = \zeta^i(\xi_2, \xi_3, \xi_4, \xi_5, \xi_1),\ p(\xi_3) = \zeta^i(\xi_3, \xi_4, \xi_5, \xi_1, \xi_2), \\
&p(\xi_4) = \zeta^i(\xi_4, \xi_5, \xi_1, \xi_2, \xi_3),\ p(\xi_5) = \zeta^i(\xi_5, \xi_1, \xi_2, \xi_3, \xi_4), \\
&p(\mathfrak{q}_1) = \zeta^i((\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5),\ p(\mathfrak{q}_2) = \zeta^i(\mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_1),\ p(\mathfrak{q}_3) = \zeta^i(\mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_1, \mathfrak{q}_2), \\
&p(\mathfrak{q}_4) = \zeta^i(\mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3),\ p(\mathfrak{q}_5) = \zeta^i(\mathfrak{q}_5, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4),
\end{aligned}$$

then, $p(\xi_1) = p(\mathfrak{q}_1),\ p(\xi_2) = p(\mathfrak{q}_2),\ p(\xi_3) = p(\mathfrak{q}_3),\ p(\xi_4) = p(\mathfrak{q}_4)$ and $p(\xi_5) = p(\mathfrak{q}_5)$. Since, QCPs are also QC, then from (1), it is obtained that

$$\begin{aligned}
\mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1)) &= \mathfrak{T}(\zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)) \\
&\leq \tilde{\Gamma}[\mathfrak{T}(p(\xi_1), \zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)) + \mathfrak{T}(p(\mathfrak{q}_1), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5))] \\
&\quad + \tilde{\Upsilon}[\mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1))], \\
\Rightarrow \quad \mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1)) &\leq \tilde{\Upsilon}[\mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1))].
\end{aligned}$$

Since $\mathcal{I} \neq \tilde{\Upsilon} \in$ ZM, then $\mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1)) = \mathbf{0}$, or $p(\xi_1) = p(\mathfrak{q}_1)$. Similarly, it is obtained that $p(\xi_2) = p(\mathfrak{q}_2)$, $p(\xi_3) = p(\mathfrak{q}_3)$, $p(\xi_4) = p(\mathfrak{q}_4)$ and $p(\xi_5) = p(\mathfrak{q}_5)$. Hence, $p(\xi_1) = p(\xi_2) = p(\xi_3) = p(\xi_4) = p(\xi_5) = p(\mathfrak{q}_1) = p(\mathfrak{q}_2) = p(\mathfrak{q}_3) = p(\mathfrak{q}_4) = p(\mathfrak{q}_5)$. Which shows that $(p(\xi_1), p(\xi_2), p(\xi_3), p(\xi_4), p(\xi_5))$ is a unique quintuple coincidence point of $\{\zeta^i\}_{i\in\mathbb{W}}$ and p. Moreover, $\{\zeta^i\}_{i\in\mathbb{W}}$ and p being compatible are also commutable, which proves the uniqueness of the QFP $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ of $\{\zeta^i\}_{i\in\mathbb{W}}$ and p.

Example 6. *Let $\mathcal{G} = [0,1]$ be the non-empty set equipped with the metric $\mathfrak{T}(\xi_1, \xi_2) = \begin{pmatrix} |\xi_1 - \xi_2| \\ |\xi_1 - \xi_2| \end{pmatrix}$ and $\tilde{\Gamma} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$, $\tilde{\Upsilon} = \begin{pmatrix} 0 & \frac{1}{25} \\ \frac{1}{25} & 0 \end{pmatrix} \in$ ZM. Hence, $(\mathcal{G}, \mathfrak{T}, \leq)$ is clearly a POCGMs. Let $\zeta^i : \mathcal{G}^5 \to \mathcal{G}$ and $s : \mathcal{G} \to \mathcal{G}$ be two mappings defined as $\zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \frac{\xi_1}{5^i}$ and $p(\xi_1) = 5\xi_1$ respectively. Then,*

$$
\begin{aligned}
\mathfrak{T}(\zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)) &= \begin{pmatrix} |\frac{\xi_1}{5^i} - \frac{\mathfrak{q}_1}{5^j}| \\ |\frac{\xi_1}{5^i} - \frac{\mathfrak{q}_1}{5^j}| \end{pmatrix} \\
&= \frac{1}{5}\begin{pmatrix} |(\frac{6\xi_1}{5^i} - \frac{\xi_1}{5^i}) + (\frac{\mathfrak{q}_1}{4^j} - \frac{6\mathfrak{q}_1}{5^j})| \\ |(\frac{6\xi_1}{5^i} - \frac{\xi_1}{5^i}) + (\frac{\mathfrak{q}_1}{4^j} - \frac{6\mathfrak{q}_1}{5^j})| \end{pmatrix} \\
&\leq \frac{1}{5}\begin{pmatrix} |(5\xi_1 - \frac{\xi_1}{5^i}) + (\frac{\mathfrak{q}_1}{5^j} - 5\mathfrak{q}_1) + (\xi_1 - \mathfrak{q}_1)| \\ |(5\xi_1 - \frac{\xi_1}{5^i}) + (\frac{\mathfrak{q}_1}{4^j} - 4\mathfrak{q}_1) + (\xi_1 - \mathfrak{q}_1)| \end{pmatrix} \\
&\leq \frac{1}{5}\left(\begin{pmatrix} |5\xi_1 - \frac{\xi_1}{5^i}| \\ |5\xi_1 - \frac{\xi_1}{5^i}| \end{pmatrix} + \begin{pmatrix} |5\mathfrak{q}_1 - \frac{\mathfrak{q}_1}{5^j}| \\ |\mathfrak{q}_1 - \frac{\mathfrak{q}_1}{5^j}| \end{pmatrix} \right) \\
&\quad + \frac{1}{25}\begin{pmatrix} |5\xi_1 - 5\mathfrak{q}_1| \\ |5\xi_1 - 5\mathfrak{q}_1| \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \big(\mathfrak{T}(p(\xi_1), \zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)) \\
&\quad + \mathfrak{T}(p(\mathfrak{q}_1), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5))\big) \\
&\quad + \begin{pmatrix} 0 & \frac{1}{25} \\ \frac{1}{25} & 0 \end{pmatrix} \mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1)).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mathfrak{T}(\zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)) \leq \tilde{\Gamma}[&\mathfrak{T}(p(\xi_1), \zeta^i(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)) \\
&+ \mathfrak{T}(p(\mathfrak{q}_1), \zeta^j(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5))] \\
&+ \tilde{\Upsilon}\mathfrak{T}(p(\xi_1), p(\mathfrak{q}_1)).
\end{aligned}
$$

Hence, the (C) condition is fulfilled. All the conditions of Theorem 1 are accomplished. Moreover, $(0, 0, 0, 0, 0)$ being a QCP of $\{\zeta^i\}_{i\in\mathbb{W}}$ and p is also a unique quintuple coincidence point of both mappings according to Theorem 2.

4. Application

Suppose that $\mathbb{R}^n_+ = \{\xi_1 = (\xi_1^1, \xi_1^2, \xi_1^3, \cdots, \xi_1^n) : \xi_i \geq, i \geq 1\}$ and Θ^5_{n-1} being $5(n-1)$ dimensional unit simplex defined as

$$\Theta^5_{n-1} = \{\theta = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \sum_{i=1}^n \theta_i = \sum_{i=1}^n (\xi_1^i + \xi_2^i + \xi_3^i + \xi_4^i + \xi_5^i) = 1\}.$$

Let $\theta \in \Theta^5_{n-1}$ be the probability over $5n$ prospective states. As a stochastic process, the Markov process asserts that $5n$ states are achieved in each period $\pi = 1, 2, 3, \cdots$ with the probability events over the currently attained states. For each $\pi = 1, 2, 3, \cdots$, e_{ij} shows the probability event achieved by state i in the next period starting from state j. Then, the preceding probability vector θ^π and the succeeding probability vector $\theta^{\pi+1}$ in the period π and $\pi + 1$ respectively, written as $\theta_i^{\pi+1} = \sum_j e_{ij} \theta_j^\pi$, for each $j \geq 1$. Let θ^π be a column vector, then to obtain matrix form, consider the mapping $\theta^{\pi+1} = \mathcal{F}\theta^\pi$. In addition with that for all $e_{ij} \geq 0, \sum_{i=1}^n e_{ij} = 1$, required for conditional probability. Finding the stationary distribution for the Markov process is the same as finding the fixed point of $\mathcal{F}$, i.e., there exists some $\theta \in \Theta^5_{n-1}$ such that $\mathcal{F}\theta^\pi = \theta^\pi$, whenever $\theta^{\pi+1} = \theta^\pi$. The period θ^π is called the stationary distribution of the Markov process. Suppose that $\phi_i = \min_j e_{ij}$, for each i and $\phi = \sum_{i=1}^n \phi_i$.
Now, the major part of this section is stated here.

Theorem 3. *By supposition $e_{ij} \geq 0$, the Markov process has a unique stationary distribution.*

Proof. Let $\mathfrak{T} : \Theta^5_{n-1} \times \Theta^5_{n-1} \to \mathbb{R}^2$ be a mapping defined as

$$\begin{aligned}\mathfrak{T}(M, N) =& \mathfrak{T}((\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)) \\ =& \Bigg(\sum_{i=1}^n (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|), \\ & \sum_{i=1}^n (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|) \Bigg),\end{aligned}$$

where $M = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ and $N = (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5)$ belongs to Θ^5_{n-1}.
Since, $\mathfrak{T}(M, N) \geq (0, 0)$ for all M and N in Θ^5_{n-1}. Also, if $\mathfrak{T}(M, N) = (0, 0)$, then this implies that

$$\Bigg(\sum_{i=1}^n (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|), \\ \sum_{i=1}^n (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|) \Bigg) = (0, 0),$$

or

$$\begin{aligned} & |\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i| = 0, \\ \Rightarrow \quad & |\xi_1^i - \mathfrak{q}_1^i| = |\xi_2^i - \mathfrak{q}_2^i| = |\xi_3^i - \mathfrak{q}_3^i| = |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i| = 0, \\ \Rightarrow \quad & \xi_1^i = \mathfrak{q}_1^i,\ \xi_2^i = \mathfrak{q}_2^i,\ \xi_3^i = \mathfrak{q}_3^i,\ \xi_4^i = \mathfrak{q}_4^i,\ \xi_5^i = \mathfrak{q}_5^i. \end{aligned}$$

Hence, $M = N$.
Conversely, let $M = N$, then $\xi_1^i = \mathfrak{q}_1^i,\ \xi_2^i = \mathfrak{q}_2^i,\ \xi_3^i = \mathfrak{q}_3^i,\ \xi_4^i = \mathfrak{q}_4^i,\ \xi_5^i = \mathfrak{q}_5^i$,

$$\Rightarrow \quad |\xi_1^i - \mathfrak{q}_1^i| = |\xi_2^i - \mathfrak{q}_2^i| = |\xi_3^i - \mathfrak{q}_3^i| = |\xi_4^i - \mathfrak{q}_4^i| = |\xi_5^i - \mathfrak{q}_5^i| = 0.$$

Hence,

$$\begin{aligned} & \Bigg(\sum_{i=1}^{n} (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi 5^i - \mathfrak{q}_5^i|), \\ & \sum_{i=1}^{n} (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|) \Bigg) = (0,0) \\ \Rightarrow \quad & \mathfrak{T}(M,N) = (0,0). \end{aligned}$$

Moreover,

$$\begin{aligned} \mathfrak{T}(M,N) =& \Bigg(\sum_{i=1}^{n} (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|), \\ & \sum_{i=1}^{n} (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|) \Bigg) \\ =& \Bigg(\sum_{i=1}^{n} (|\mathfrak{q}_1^i - \xi_1^i| + |\mathfrak{q}_2^i - \xi_2^i| + |\mathfrak{q}_3^i - \xi_3^i| + |\mathfrak{q}_4^i - \xi_4^i| + |\mathfrak{q}_5^i - \xi_5^i|), \\ & \sum_{i=1}^{n} (|\mathfrak{q}_1^i - \xi_1^i| + |\mathfrak{q}_2^i - \xi_2^i| + |\mathfrak{q}_3^i - \xi_3^i| + |\mathfrak{q}_4^i - \xi_4^i| + |\mathfrak{q}_5^i - \xi_5^i|) \Bigg) \\ =& \mathfrak{T}(N,M). \end{aligned}$$

Now,

$$\begin{aligned} \mathfrak{T}(M,N) = & \Bigg(\sum_{i-1}^{n} (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|), \\ & \sum_{i=1}^{n} (|\xi_1^i - \mathfrak{q}_1^i| + |\xi_2^i - \mathfrak{q}_2^i| + |\xi_3^i - \mathfrak{q}_3^i| + |\xi_4^i - \mathfrak{q}_4^i| + |\xi_5^i - \mathfrak{q}_5^i|) \Bigg) \\ & = \left(\sum_{i=1}^{n} \begin{pmatrix} |(\xi_1^i - \kappa_1^i) + (\kappa_1^i - \mathfrak{q}_1^i)| + |(\xi_2^i - \kappa_2^i) + (\kappa_2^i - \mathfrak{q}_2^i)| + |(\xi_3^i - \kappa_3^i) \\ +(\kappa_3^i - \mathfrak{q}_3^i)| + |(\xi_4^i - \kappa_4^i) + (\kappa_4^i - \mathfrak{q}_4^i)| + |(\xi_5^i - \kappa_5^i) + (\kappa_5^i - \mathfrak{q}_5^i)| \end{pmatrix} \right., \end{aligned}$$

$$\sum_{i=1}^{n}\left(\begin{array}{c}|(\xi_1^i-\kappa_1^i)+(\kappa_1^i-\mathfrak{q}_1^i)|+|(\xi_2^i-\kappa_2^i)+(\kappa_2^i-\mathfrak{q}_2^i)|+|(\xi_3^i-\kappa_3^i)\\+(\kappa_3^i-\mathfrak{q}_3^i)|+|(\xi_4^i-\kappa_4^i)+(\kappa_4^i-\mathfrak{q}_4^i)|+|(\xi_5^i-\kappa_5^i)+(\kappa_5^i-\mathfrak{q}_5^i)|\end{array}\right)\Bigg)$$

$$\leq\left(\sum_{i-1}^{n}\left(\begin{array}{c}|\xi_1^i-\kappa_1^i|+|\kappa_1^i-\mathfrak{q}_1^i|+|\xi_2^i-\kappa_2^i|+|\kappa_2^i-\mathfrak{q}_2^i|+|\xi_3^i-\kappa_3^i|\\+|\kappa_3^i-\mathfrak{q}_3^i|+|\xi_4^i-\kappa_4^i|+|\kappa_4^i-\mathfrak{q}_4^i|+|\xi_5^i-\kappa_5^i|+|\kappa_5^i-\mathfrak{q}_5^i|\end{array}\right)\right.,$$

$$\left.\sum_{i=1}^{n}\left(\begin{array}{c}|\xi_1^i-\kappa_1^i|+|\kappa_1^i-\mathfrak{q}_1^i|+|\xi_2^i-\kappa_2^i|+|\kappa_2^i-\mathfrak{q}_2^i|+|\xi_3^i-\kappa_3^i|\\+|\kappa_3^i-\mathfrak{q}_3^i|+|\xi_4^i-\kappa_4^i|+|\kappa_4^i-\mathfrak{q}_4^i|+|\xi_5^i-\kappa_5^i|+|\kappa_5^i-\mathfrak{q}_5^i|\end{array}\right)\right)$$

$$=\{\left(\sum_{i=1}^{n}(|\xi_1^i-\kappa_1^i|^+|\xi_2^i-\kappa_2^i|+|\xi_3^i-\kappa_3^i|+|\xi_4^i-\kappa_4^i|+|\xi_5^i-\kappa_5^i|),\right.$$

$$\left.\sum_{i=1}^{n}(|\xi_1^i-\kappa_1^i|^+|\xi_2^i-\kappa_2^i|+|\xi_3^i-\kappa_3^i|+|\xi_4^i-\kappa_4^i|+|\xi_5^i-\kappa_5^i|)\right)$$

$$+\left(\sum_{i=1}^{n}(|\kappa_1^i-\mathfrak{q}_1^i|+|\kappa_2^i-\mathfrak{q}_2^i|+|\kappa_3^i-\mathfrak{q}_3^i|+|\kappa_4^i-\mathfrak{q}_4^i|+|\kappa_5^i-\mathfrak{q}_5^i|),\right.$$

$$\left.\sum_{i=1}^{n}(|\kappa_1^i-\mathfrak{q}_1^i|+|\kappa_2^i-\mathfrak{q}_2^i|+|\kappa_3^i-\mathfrak{q}_3^i|+|\kappa_4^i-\mathfrak{q}_4^i|+|\kappa_5^i-\mathfrak{q}_5^i|)\right)\}$$

$$=\mathfrak{T}(M,L)+\mathfrak{T}(L,N),$$

where, $L=(\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5)\in\Theta_{n-1}^5$. Hence, $(\Theta_{n-1}^5,\ \mathfrak{T})$ is a generalized metric space. Completeness of Θ_{n-1}^5 can be easily proved. Moreover, define the partial order on Θ_{n-1}^5 as for all $(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5),\ (\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{q}_4,\mathfrak{q}_5)\in\Theta_{n-1}^5$,

$$(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\preceq(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{q}_4,\mathfrak{q}_5)\iff\ \xi_1\preceq\mathfrak{q}_1,\xi_2\succeq\mathfrak{q}_2,\xi_3\preceq\mathfrak{q}_3,\ \xi_4\succeq\mathfrak{q}_4,\ \xi_5\preceq\mathfrak{q}_5.$$

Hence, $(\Theta_{n-1}^5,\ \mathfrak{T},\ \preceq)$ is a POCGMs.
Let $\mathcal{F}:\Theta_{n-1}^5\to\Theta_{n-1}^5$ be a mapping defined as for all $\theta\in\Theta_{n-1}^5,\ \mathcal{F}\theta=\alpha_j$ such that for each $j,\ \alpha_j=\sum_{i=1}^{n}e_{ij}\theta_j$. As,

$$\sum_{j=1}^{n}\alpha_j=\sum_{j=1}^{n}\sum_{i=1}^{n}e_{ij}\theta_j=\sum_{i=1}^{n}e_{ij}\sum_{j=1}^{n}(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j,\xi_5^j)$$
$$=\sum_{j=1}^{n}(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j,\xi_5^j)=1,$$

therefore, $\alpha_j\in\Theta_{n-1}^5$, so mapping is defined. Now, we have to show that $\mathcal{F}$ satisfies the contraction condition. For this, let α_i be the i^{th} row of α. Then, for all $(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5),\ (\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{q}_4,\mathfrak{q}_5)$ in Θ_{n-1}^5, we have

$$\mathfrak{T}(\mathcal{F}(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5),\mathcal{F}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{q}_4,\mathfrak{q}_5))$$
$$=\left(\sum_{i=1}^{n}\left(\left|\sum_{j=1}^{n}(e_{ij}(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-e_{ij}(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\right)\right|\right),$$

$$\sum_{i=1}^{n}\left(\left|\sum_{j=1}^{n}\Big(e_{ij}(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-e_{ij}(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\Big)\right|\right)\Bigg)$$

$$=\Bigg(\sum_{i=1}^{n}\Big(|\sum_{j=1}^{n}(e_{ij}-\phi_i)\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}$$
$$+\phi_i\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j,\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}|\Big),$$
$$\sum_{i=1}^{n}\Big(|\sum_{j=1}^{n}(e_{ij}-\phi_i)\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}$$
$$+\phi_i\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}|\Big)\Bigg)$$

$$\leq\Bigg(\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}|(e_{ij}-\phi_i)\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}|$$
$$+\sum_{i=1}^{n}\left|\phi_i\sum_{j=1}^{n}\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}\right|\Big),$$
$$\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(e_{ij}-\phi_i)\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}\right|$$
$$+\sum_{i=1}^{n}\left|\phi_i\sum_{j=1}^{n}\{(\xi_1^j+\xi_2^j+\xi_3^j+\xi_4^j+\xi_5^j)-(\mathfrak{q}_1^j+\mathfrak{q}_2^j+\mathfrak{q}_3^j+\mathfrak{q}_4^j+\mathfrak{q}_5^j)\}\right|\Big)\Bigg)$$

$$\leq\begin{pmatrix}\sum_{i=1}^{n}\sum_{j=1}^{n}(|\xi_1^j-\mathfrak{q}_1^j|+|\xi_2^j-\mathfrak{q}_2^j|+|\xi_3^j-\mathfrak{q}_3^j|+|\xi_4^j-\mathfrak{q}_4^j|+|\xi_5^j-\mathfrak{q}_5^j|)\times|e_{ij}-\phi_i|),\\ \sum_{i=1}^{n}\sum_{j=1}^{n}(|\xi_1^j-\mathfrak{q}_1^j|+|\xi_2^j-\mathfrak{q}_2^j|+|\xi_3^j-\mathfrak{q}_3^j|+|\xi_4^j-\mathfrak{q}_4^j|+|\xi_5^j-\mathfrak{q}_5^j|\times|e_{ij}-\phi_i|)\end{pmatrix}$$
$$=(\mathcal{I}-\phi)\begin{pmatrix}\sum_{j=1}^{n}(|\xi_1^j-\mathfrak{q}_1^j|+|\xi_2^j-\mathfrak{q}_2^j|+|\xi_3^j-\mathfrak{q}_3^j|+|\xi_4^j-\mathfrak{q}_4^j|+|\xi_5^j-\mathfrak{q}_5^j|),\\ \sum_{j=1}^{n}(|\xi_1^j-\mathfrak{q}_1^j|+|\xi_2^j-\mathfrak{q}_2^j|+|\xi_3^j-\mathfrak{q}_3^j|^2+|\xi_4^j-\mathfrak{q}_4^j|+|\xi_5^j-\mathfrak{q}_5^j|)\end{pmatrix}$$
$$=\tilde{\Upsilon}\mathfrak{T}((\xi_1,\xi_2,\xi_3,\xi_4,\xi_5),(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{q}_4,\mathfrak{q}_5)),$$

where $(\mathcal{I}-\phi)=\tilde{\Upsilon}\in$ ZM, therefore all conditions of Corollary 2 have been met. Then, there is a unique quintuple fixed point for $\mathcal{F}$ or, in other words, a unique stationary distribution of the Markov process. Moreover, the sequence $\{\mathcal{F}^n\theta^k\}$ converges to a unique stationary distribution for any $\theta^k\in\Theta_{n-1}^5$.

5. Conclusion

In this work, we found some Quadruple fixed point theorems for solving integral equations involved with matrices and the Markov process in generalized metric spaces. In this study, the notions introduced in [16] are structured with a mapping defined on quintuples.

The research study investigated the existence of quintuple fixed points ($QFPs$) for mappings in generalized metric spaces, utilizing matrix-based methods. Several definitions of ($QFPs$) are formulated, and new fixed point theorems is established. To illustrate the findings, an example is provided. Additionally, an application was developed to verify the results by determining the stationary distribution of a Markov process. Future research on quintuple fixed points could explore alternative approaches, such as:

(a): By working on different structures, e.g., by generalizing the obtained results in the setting of " b-metric spaces".

(b): Different contraction conditions can be adopted by involving new parameters and introducing more properties of contraction mappings.

Acknowledgements

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 25UQU4331214GSSR04.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Funding

This Research work was funded by Umm Al-Qura University, Saudi Arabia under grant number :
25UQU4331214GSSR04.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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