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On Analysis of Single Solution for a Class of BVP with Generalized Caputo-Katugampola Fractional Derivative

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Abstract. In this paper, we endeavor to simulate the existence of a single solution for a BVP for Caputo-Katugampola fractional derivative in the manner theorem of contraction of Banach. We extrapolate similar examples to interpret the conclusions reached.

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1. Introduction

Fractional differential equations are currently witnessing rapid development and an advanced pace of research creativity. This is due to the growth of physical, technological, biological, economic and various new scientific phenomena. Mathematical modeling plays an important role in the mathematical interpretation of these various scientific phenomena. It produced ordinary and fractional differential equations, as well as partial differential equations. Simulation is also of great importance in providing analysis, differentiated deduction, and numerical interpretation of these recent scientific phenomena, the simulation

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used in this document is a theoretical simulation applied with an ordinary differential equation to a fractional differential equation [1–12].

The following simulation is derived from one of the theorems declared in [13], which was also classified without proof in [14, Theorem 3.3] and it remained as a neglected issue without a solution [15, Problem (41.6)] for the researcher.

Theorem 1 ([13]). *Presume $\Xi : [\theta, \vartheta] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function is continuous and verifies a condition of uniform Lipschitz with reference to μ and μ'*

$$\left| \Xi\left(\tau, \mu, \mu'\right) - \Xi\left(\tau, \nu, \nu'\right) \right| \leq \zeta|\mu - \nu| + \eta|\mu' - \nu'|, \tag{1.1}$$

for (τ, μ, μ'), $(\tau, \nu, \nu') \in [\theta, \vartheta] \times \mathbb{R}^2$, where $\zeta \geq 0$ and $\eta > 0$ are reals. If

$$\zeta \frac{(\vartheta - \theta)^2}{8} + \eta \frac{(\vartheta - \theta)}{2} < 1, \tag{1.2}$$

so the BVP

$$\mu'' = -\Xi\left(\tau, \mu, \mu'\right), \quad \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \tag{1.3}$$

admits a unique solution.

In the content of this article, we intend to develop the aforementioned consequences through viewing of a fractional generalized Caputo derivative (we advise the researcher to see [16–23] in order to the fundamental concepts and basic consequences on calculus of fractional order with applications) instead of the ordinary operator μ'', that is to say, we achieve the existence of a singularity of solution to the BVP of Caputo-Katugampola fractional derivative order

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{\theta^+}\mu(\tau) = -\Xi\left(\tau, \mu(\tau), {}^{\varrho}_{c}D^{\varsigma}_{\theta^+}\mu(\tau)\right), & \theta < \tau < \vartheta, \\ \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \end{cases} \tag{1.4}$$

where $1 < \sigma \leq 2$, $0 < \varsigma \leq 1$. Previously, we searched the existence of a singularity of a solve to the boundary value problem for Caputo-Katugampola fractional derivative (see, [24]) and the references mentioned in its content). Under previous research, the simulation of our problem produces new results by applying Theorem 1.

2. Principal concepts

Starting, we review some basic properties of fractional calculus for investigating BVPs, lookup in [25–29].

Definition 1. *On the left-sided in the generalized fractional integral order ${}^{\varrho}I^{\sigma}_{\theta^+}\mu$ for $\sigma \in \mathbb{C}(Re(\sigma) > 0)$ is given by*

$$\left({}^{\varrho}I^{\sigma}_{\theta^+}\mu\right)(\tau) = \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)} \int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho} - r^{\varrho})^{\sigma-1}\mu(r)\,\mathrm{d}r, \tag{2.1}$$

where $\tau > 0$, $\varrho > 0$. According to the formula of the generalized fractional integrals (2.1)*, we define the Caputo-Katugampola fractional derivative for $\tau > 0$ by*

$$\begin{aligned}\left({}^{\varrho}D^{\sigma}_{\theta^+}\mu\right)(\tau) &= \left(\tau^{1-\varrho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n \left({}^{\varrho}I^{n-\sigma}_{\theta^+}\mu\right)(\tau) \\ &= \frac{\varrho^{\sigma-n+1}}{\Gamma(n-\sigma)}\left(\tau^{1-\varrho}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n \int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{n-1-\sigma}\mu(r)\,\mathrm{d}r. \end{aligned} \tag{2.2}$$

Definition 2. *By using the above Caputo-Katugampola fractional derivative* (2.2)*, the generalized Caputo non-classical derivative with the operator notation ${}^{\varrho}_{c}D^{\sigma}_{\theta^+}$ is displayed by*

$${}^{\varrho}_{c}D^{\sigma}_{\theta^+}\mu(\tau) = \left({}^{\varrho}D^{\sigma}_{\theta^+}\left[\mu(\tau) - \sum_{l=0}^{n-1}\frac{\mu^{(l)}(\theta)}{l!}(\tau-\theta)^l\right]\right)(\tau), \quad n = [Re(\sigma)]. \tag{2.3}$$

Lemma 1 ([25])**.** *Let $\sigma, \varrho > 0$ and $\mu \in \mathcal{C}(J,\mathbb{R}) \cap \mathcal{C}^1(J,\mathbb{R})$. Then*

1. *The Caputo-Katugampola fractional derivative differential equation*

$${}^{\varrho}_{c}D^{\sigma}_{\theta^+}\mu(\tau) = 0,$$

has a solution.

$$\mu(\tau) = p_0 + p_1\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right) + p_2\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^2 + \dots + p_{n-1}\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{n-1},$$

where $p_i \in \mathbb{R}$, $i = 0,1,2,...,n-1$ and $n = [\sigma]+1$.

2. *If μ, ${}^{\varrho}_{c}D^{\sigma}_{\theta^+}\mu \in \mathcal{C}(J,\mathbb{R}) \cap \mathcal{C}^1(J,\mathbb{R})$. Then*

$${}^{\varrho}I^{\sigma}_{\theta^+}\,{}^{\varrho}_{c}D^{\sigma}_{\theta^+}\mu(\tau) = \mu(\tau) + p_0 + p_1\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right) + p_2\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^2 + \dots + p_{n-1}\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{n-1}, \tag{2.4}$$

where $p_i \in \mathbb{R}$, $i = 0,1,2,...,n-1$ and $n = [\sigma]+1$.

3. Main results

At the heart of this passage, we witness significant propositions and theorems on which all this work is based. We give the integral formula for the generalized fractional BVP (1.4) from the principle of the Green function.

Lemma 2. *Presume that Ξ is a function is continuous and either a function $\mu \in C[\theta,\vartheta]$ is a solution of* (1.4) *equivalent that μ check the integral equation*

$$\mu(\tau) = \left[(\lambda_2-\lambda_1)\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1\right] + \int_{\theta}^{\vartheta}\hbar(\tau,r)\;\Xi\left(r,\mu(r),{}^{\varrho}_{c}D^{\varsigma}_{\theta^+}\mu(r)\right)\mathrm{d}r, \tag{3.1}$$

where for $r,\tau \in [\theta,\vartheta]$,

$$\hbar(\tau,r) = \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\begin{cases}\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\;r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1} - r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}, & r \le \tau, \\ \frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\;r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}, & \tau \le r.\end{cases} \tag{3.2}$$

Proof. By the Lemma(1),we solve this problem

$$ {}^{\varrho}_{c}D^{\sigma}_{\theta^{+}}\mu(\tau) = -q(\tau). $$

According to (2.4), we obtain

$$ \begin{aligned} {}^{\varrho}I^{\sigma}_{\theta^{+}}\ {}^{\varrho}_{c}D^{\sigma}_{\theta^{+}}\mu(\tau) &= -\,{}^{\varrho}I^{\sigma}_{\theta^{+}}q(\tau) + p_0 + p_1\frac{(\tau^{\varrho}-\theta^{\varrho})}{\varrho} \\ \mu(\tau) &= -\,{}^{\varrho}I^{\sigma}_{\theta^{+}}q(\tau) + p_0 + p_1\frac{(\tau^{\varrho}-\theta^{\varrho})}{\varrho} \\ \mu(\tau) &= -\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}q(r)dr + p_0 + p_1\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{\varrho}, \end{aligned} $$

by using boundary conditions

$$ \begin{aligned} &\mu(\theta) = \lambda_1 \Longrightarrow p_0 = \lambda_1, \\ &\mu(\vartheta) = \lambda_2 \Longrightarrow p_1 = \tfrac{\varrho(\lambda_2-\lambda_1)}{(\vartheta^{\varrho}-\theta^{\varrho})} + \tfrac{\varrho^{2-\sigma}}{(\vartheta^{\varrho}-\theta^{\varrho})\Gamma(\sigma)}\int_{\theta}^{\vartheta} r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r. \end{aligned} $$

By replacing in $\mu(\tau)$, we get

$$ \begin{aligned} \mu(\tau) = &-\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r \\ &+\left[\varrho(\lambda_2-\lambda_1) + \tfrac{\varrho^{2-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\vartheta} r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)dr\right]\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1, \end{aligned} $$

and

$$ \begin{aligned} \mu(\tau) = &-\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r + \tfrac{\varrho^{1-\sigma}(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})\Gamma(\sigma)} \\ &\int_{\theta}^{\vartheta} r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r + (\lambda_2-\lambda_1)\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1. \end{aligned} $$

Therefore,

$$ \begin{aligned} \mu(\tau) = &\left[(\lambda_2-\lambda_1)\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1\right] \\ &+\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau}\left(\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1} - r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}\right)q(r)\,\mathrm{d}r \\ &+\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\tau}^{\vartheta}\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r, \end{aligned} $$

Thus the proof is complete.

Immediately, we will present the important salient rules that will make it easier for us to achieve our desired objectives.

Proposition 1. *Depending on the Green function $\hbar$ is given in Lemma 2. Therefore*

$$\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r \le \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\Big[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\Big], \tag{3.3}$$

and

$$\int_{\theta}^{\vartheta} \frac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r \le \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Big[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}\Big]. \tag{3.4}$$

Proof. We determine

$$\int_{\vartheta}^{\theta} |\hbar(\tau,r)|\,\mathrm{d}r, \qquad \hbar(\tau,r)\ge 0. \quad \forall\, \theta\le\tau,\ r\le\vartheta. \tag{3.5}$$

Therefore

$$\begin{aligned}\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r = \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\Bigg[\int_{\theta}^{\tau}\Big(\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}-r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}\Big)\,\mathrm{d}r\\ +\int_{\tau}^{\vartheta}\Big(\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}\Big)\,\mathrm{d}r\Bigg].\end{aligned}$$

we calculate the primitives by integration by a change of variable

$$\begin{aligned}\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r &= \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\Bigg[\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\frac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\Big[\Big(1-\frac{\theta^{\varrho}}{\vartheta^{\varrho}}\Big)^{\sigma}-\Big(1-\frac{\tau^{\varrho}}{\vartheta^{\varrho}}\Big)^{\sigma}\Big]\\ &\quad -\frac{\tau^{\varrho\sigma}}{\varrho}\Big(\frac{1}{\sigma}\Big(1-\frac{\theta^{\varrho}}{\tau^{\varrho}}\Big)^{\sigma}\Big)+\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\frac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\Big(1-\frac{\tau^{\varrho}}{\vartheta^{\varrho}}\Big)^{\sigma}\Bigg]\\ &= \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\Bigg[\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\frac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\Big[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}-\frac{(\vartheta^{\varrho}-\tau^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}\Big]\\ &\quad -\frac{\tau^{\varrho\sigma}}{\varrho}\Big(\frac{1}{\sigma}\frac{(\tau^{\varrho}-\theta^{\varrho})^{\sigma}}{\tau^{\varrho\sigma}}\Big)+\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\frac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\frac{(\vartheta^{\varrho}-\tau^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}\Bigg]\\ &= \frac{\varrho^{1-\sigma}}{\varrho\sigma\Gamma(\sigma)}\Big[\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\Big].\end{aligned}$$

Then,

$$\int_{\theta}^{\vartheta} |\hbar(\tau,r)|\,\mathrm{d}r = \frac{1}{\varrho^{\sigma}\sigma\Gamma(\sigma)}\Big[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\Big].$$

This implies that

$$\int_{\theta}^{\vartheta} \frac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r = \frac{1}{\varrho^{\sigma-1}\sigma\Gamma(\sigma)}\Big[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}\Big], \tag{3.6}$$

which ends the proof.

Corollary 1. *We can define the continuous functions ξ and ξ' by*

$$\xi(\tau) = (\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}, \qquad \tau\in[\theta,\vartheta], \tag{3.7}$$

and

$$\xi'(\tau) = (\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}, \quad \tau\in[\theta,\vartheta]. \tag{3.8}$$

Proposition 2. *By* (3.3) *and* (3.4), *suppose that* $\theta = 0$, $\theta < \vartheta$ *then*

$$\int_0^{\vartheta} |\hbar(\tau, r)|\, \mathrm{d}r \leq \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{1/\sigma-1}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/\sigma-1}}\right], \tag{3.9}$$

and

$$\int_0^{\vartheta} \frac{\partial|\hbar(\tau,r)|}{\partial\tau}\, \mathrm{d}r \leq \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\vartheta^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)} - \sigma(\vartheta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right]. \tag{3.10}$$

Proof. According by Corollary 1, differentiating the functions ξ, ξ' and directly deduce that the maximum was reached at the points

$$\tau^* = \frac{\vartheta}{\sigma^{1/\varrho(\sigma-1)}}, \qquad \tau_1^* = \vartheta\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{1/\varrho(\sigma-1)}.$$

Moreover,

$$\begin{aligned} \xi(\tau^*) &= \frac{1}{\varrho^{\sigma}}\left(\frac{\vartheta^{\varrho\sigma}}{\sigma^{1/\sigma-1}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/\sigma-1}}\right), \\ \xi'(\tau_1^*) &= \frac{1}{\varrho^{\sigma-1}}\left(\vartheta^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)} - \sigma(\vartheta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right), \end{aligned}$$

that finishes the proof.

Theorem 2. *Assume* $\Xi : [0,\vartheta] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on* $[0,\vartheta]\times\mathbb{R}^2$ *with Lipschitz real* ζ, *thus,*

$$\left|\Xi\left(\tau,\mu,\mu'\right) - \Xi(\tau,\nu,\nu')\right| \leq \zeta|\mu-\nu| + \eta|\mu'-\nu'|, \tag{3.11}$$

for (τ,μ,μ'), $(\tau,\nu,\nu') \in [0,\vartheta]\times\mathbb{R}^2$, *where* $\eta \geq 0$, $\zeta > 0$ *are constants. If*

$$\begin{aligned} \frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{1/\sigma-1}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/\sigma-1}}\right] + \frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Bigg[\vartheta^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)} \\ -\sigma(\vartheta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\Bigg] < 1. \end{aligned} \tag{3.12}$$

Then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0^+}\mu(\tau) = -\Xi\left(\tau, \mu(\tau), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(\tau)\right), & 0 < \tau < \vartheta, \\ \mu(0) = \lambda_1, \qquad \mu(\vartheta) = \lambda_2, \end{cases} \tag{3.13}$$

admits a single solution.

Proof. Suppose Π is a space of Banach fitted with continuous applications defined on $[0, \vartheta]$ with the norm

$$\|\mu\| = \max_{\tau\in[0,\vartheta]} \left\{\zeta\left|\mu(\tau)\right| + \eta\left|\mu'(\tau)\right|\right\}.$$

According to Lemma 2, we have $\mu \in C[0, \vartheta]$ is a solution of (3.13) equivalent that this is the same as the solving an equation in integral form

$$\mu(\tau) = \left[(\lambda_2 - \lambda_1)\left(\tfrac{\tau}{\vartheta}\right)^{\varrho} + \lambda_1\right] + \int_0^{\vartheta} \hbar(\tau, r)\ \Xi\left(r, \mu(r), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(r)\right) \mathrm{d}r. \tag{3.14}$$

Define the operator $\Sigma : \Pi \to \Pi$ by

$$\Sigma\mu(\tau) = \left[(\lambda_2 - \lambda_1)\left(\tfrac{\tau}{\vartheta}\right)^{\varrho} + \lambda_1\right] + \int_0^{\vartheta} \hbar(\tau, r)\ \Xi\left(r, \mu(r), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(r)\right) \mathrm{d}r \tag{3.15}$$

for $\tau \in [0, \vartheta]$. We should interpret that the application Σ admits a single fixed point. Assume $\mu, \nu \in \Pi$. Therefore

$$\begin{aligned}\zeta\left|\Sigma\mu(\tau) - \Sigma\nu(\tau)\right| &\le \int_0^{\vartheta} |\hbar(\tau, r)| \left|\Xi\left(r, \mu(r), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(r)\right) - \Xi\left(r, \nu(r), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\nu(r)\right)\right| \mathrm{d}r \\ &\le \int_0^{\vartheta} |\hbar(\tau, r)| \left(\zeta|\mu(\tau) - \nu(\tau)| + \eta|\mu'(\tau) - \nu'(\tau)|\right) \mathrm{d}r \\ &\le \zeta \int_0^{\vartheta} |\hbar(\tau, r)|\, \mathrm{d}r \|\mu - \nu\| \le \frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{1/\sigma-1}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/\sigma-1}}\right] \|\mu - \nu\|,\end{aligned}$$

for $\tau \in [0, \vartheta]$, and similarly,

$$\begin{aligned}\eta\left|(\Sigma\mu)'(\tau) - (\Sigma\nu)'(\tau)\right| &\le \int_0^{1} \frac{|\hbar(\tau,r)|}{\partial\tau} \left|\Xi\left(r, \mu(r), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(r)\right) - \Xi\left(r, \nu(r), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\nu(r)\right)\right| \mathrm{d}r \\ &\le \int_0^{\vartheta} \frac{|\hbar(\tau,r)|}{\partial\tau} \left(\zeta|\mu(\tau) - \nu(\tau)| + \eta|\mu'(\tau) - \nu'(\tau)|\right) \mathrm{d}r \\ &\le \eta \int_0^{\vartheta} \frac{|\hbar(\tau,r)|}{\partial\tau}\, \mathrm{d}r \|\mu - \nu\| \le \frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \Bigg[\vartheta^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)} \\ &\quad -\sigma(\vartheta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\Bigg] \|\mu - \nu\|, \qquad \tau \in [0, \vartheta].\end{aligned}$$

Hence, $\|\Sigma\mu - \Sigma\nu\| \le \varpi\|\mu - \nu\|$, where

$$\begin{aligned}\varpi = {}& \frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{1/\varrho(\sigma-1)}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/\sigma-1}}\right] + \frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \Bigg[\vartheta^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)} \\ & -\sigma(\vartheta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\Bigg] < 1.\end{aligned}$$

Whither, we have martyred Proposition 2. According to (3.12), we extrapolate that Σ is a contracting operator on Π, so, by the theorem of contraction mapping of Banach we culminate in the possible outcome. This means that, we conclude that Σ accepts a single fixed point in $C[0, \vartheta]$, this requires that the BVP (3.13) admits a single solution.

Remark 1. *We analyze this when taking $\sigma = 2$, $\varsigma = 1$, $\theta = 0$ and $\varrho = 1$ in Theorem 2, through condition* (3.12)*, we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{1/\sigma-1}}-\frac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/\sigma-1}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\vartheta^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)}-\sigma(\vartheta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right]$$

$$=\zeta\frac{\vartheta^{2}}{4\Gamma(3)}+\eta\frac{\vartheta}{\Gamma(3)}<1.$$

Proposition 3 ([17])**.** *By* (3.3) *and* (3.4)*, suppose that $\theta = 0$, $\vartheta = 1$ then*

$$\int_0^1 |\hbar(\tau,r)|\,\mathrm{d}r \leq \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{1/\varrho(\sigma-1)}}-\frac{1}{\sigma^{\sigma/(\sigma-1)}}\right], \tag{3.16}$$

and

$$\int_0^1 \frac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r \leq \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho-1)/\varrho(\sigma-1)}-\sigma\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right]. \tag{3.17}$$

Theorem 3 ([17])**.** *Assume $\Xi : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on $[0,1]\times\mathbb{R}^2$ with Lipschitz real ζ, thus,*

$$\left|\Xi\left(\tau,\mu,\mu'\right)-\Xi\left(\tau,\nu,\nu'\right)\right| \leq \zeta|\mu-\nu|+\eta|\mu'-\nu'|, \tag{3.18}$$

for (τ,μ,μ'), $(\tau,\nu,\nu') \in [0,1]\times\mathbb{R}^2$, where $\eta \geq 0$, $\zeta > 0$ are constants. If

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{1/\sigma-1}}-\frac{1}{\sigma^{\sigma/\sigma-1}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)}-\sigma\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right]<1, \tag{3.19}$$

then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0^+}\mu(\tau) = -\Xi\left(\tau,\mu(\tau),{}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(\tau)\right), & 0<\tau<1,\\ \mu(0)=\lambda_1, \quad \mu(1)=\lambda_2, \end{cases} \tag{3.20}$$

has a unique solution.

Proof. Using the same method to prove Proposition 3 and Theorem 3 which are used in Theorem 2 and Proposition 2.

Remark 2. *The same remark 1, we notice that when $\sigma = 2$, $\varsigma = 1$, $\theta = 0$, $\vartheta = 1$ and $\varrho = 1$ in Theorem 3, through condition* (3.19)*, we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{1/\sigma-1}}-\frac{1}{\sigma^{\sigma/\sigma-1}}\right]+\frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)}-\sigma\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right]$$

$$=\zeta\frac{1}{4\Gamma(3)}+\eta\frac{1}{\Gamma(3)}<1.$$

Proposition 4 ([17]). *By* (3.3) *and* (3.4)*, suppose that* $\theta < \vartheta$ *then*

$$\int_0^{\vartheta} |\hbar(\tau, r)| \, \mathrm{d}r \leq \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{1/\sigma-1}} - \frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\sigma/\sigma-1}} \right], \tag{3.21}$$

and

$$\int_0^{\vartheta} \frac{\partial|\hbar(\tau,r)|}{\partial\tau} \, \mathrm{d}r \leq \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left[(\vartheta - \theta)^{(\varrho\sigma-1)} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)} \right)^{\varrho-1/\varrho(\sigma-1)} - \sigma(\vartheta - \theta)^{\varrho\sigma-1} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)} \right)^{\varrho\sigma-1/\varrho(\sigma-1)} \right]. \tag{3.22}$$

Theorem 4 ([17]). *Assume* $\Xi : [\theta, \vartheta] \times \mathbb{R}^2 \to \mathbb{R}$ *is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on* $[\theta, \vartheta] \times \mathbb{R}^2$ *with Lipschitz real* ζ*, thus,*

$$\left| \Xi\left(\tau, \mu, \mu'\right) - \Xi\left(\tau, \nu, \nu'\right) \right| \leq \zeta|\mu - \nu| + \eta|\mu' - \nu'|, \tag{3.23}$$

for (τ, μ, μ')*,* $(\tau, \nu, \nu') \in [\theta, \vartheta] \times \mathbb{R}^2$*, where* $\eta \geq 0$*,* $\zeta > 0$ *are constants. If*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{1/\sigma-1}} - \frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\sigma/\sigma-1}} \right] + \frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left[(\vartheta - \theta)^{(\varrho\sigma-1)} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)} \right)^{\varrho-1/\varrho(\sigma-1)} \right.$$

$$\left. -\sigma(\vartheta - \theta)^{\varrho\sigma-1} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)} \right)^{\varrho\sigma-1/\varrho(\sigma-1)} \right] < 1, \tag{3.24}$$

then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0^+}\mu(\tau) = -\Xi\left(\tau, \mu(\tau), {}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(\tau)\right), \quad 0 < \tau < 1, \\ \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \end{cases} \tag{3.25}$$

has a unique solution.

Proof. Using the same method to prove Proposition 4 and Theorem 4 which are used in Proposition 2 and also applies to Theorem 2.

Remark 3. *Same previous notes. We notice them in the general case. We apply them when* $\sigma = 2$*,* $\theta < \vartheta$ *and* $\varrho = 1$ *on Theorem 4, through condition* (3.24)*, we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{1/\sigma-1}} - \frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\sigma/\sigma-1}} \right] + \frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left[(\vartheta - \theta)^{(\varrho\sigma-1)} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)} \right)^{\varrho-1/\varrho(\sigma-1)} \right.$$

$$\left. -\sigma(\vartheta - \theta)^{\varrho\sigma-1} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)} \right)^{\varrho\sigma-1/\varrho(\sigma-1)} \right] = \zeta \frac{(\vartheta-\theta)^2}{4\Gamma(3)} + \eta \frac{(\vartheta-\theta)}{\Gamma(3)} < 1.$$

Proposition 5 ([17]). *By* (3.3) *and* (3.4)*, suppose that* $\theta < \vartheta = 1$ *then*

$$\int_0^{\vartheta} |\hbar(\tau, r)| \, \mathrm{d}r \leq \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{1/\sigma-1}} - \frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\sigma/\sigma-1}} \right], \tag{3.26}$$

and

$$\int_0^{\vartheta} \frac{\partial |\hbar(\tau,r)|}{\partial \tau}\, \mathrm{d}r \le \frac{1}{\varrho^{\sigma-1}\Gamma(\sigma+1)} \left[(1-\theta)^{(\varrho\sigma-1)} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)} - \sigma(1-\theta)^{\varrho\sigma-1} \left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)} \right]. \tag{3.27}$$

Theorem 5 ([17]). *Assume $\Xi : [\theta,1]\times\mathbb{R}^2 \to \mathbb{R}$ is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on $[\theta,1]\times\mathbb{R}^2$ with Lipschitz real ζ, thus,*

$$\left|\Xi\left(\tau,\mu,\mu'\right) - \Xi\left(\tau,\nu,\nu'\right)\right| \le \zeta|\mu-\nu| + \eta|\mu'-\nu'|, \tag{3.28}$$

for (τ,μ,μ'), $(\tau,\nu,\nu') \in [\theta,1]\times\mathbb{R}^2$, where $\eta \ge 0$, $\zeta > 0$ are constants. If

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{1/\sigma-1}} - \frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\sigma/\sigma-1}}\right] + \frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[(1-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)}\right.$$

$$\left. -\sigma(1-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right] < 1, \tag{3.29}$$

then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0^+}\mu(\tau) = -\Xi\left(\tau,\mu(\tau),{}^{\varrho}_{c}D^{\varsigma}_{0^+}\mu(\tau)\right), & 0<\tau<1, \\ \mu(\theta) = \lambda_1, \quad \mu(1) = \lambda_2, \end{cases} \tag{3.30}$$

has a unique solution.

Proof. Using the same method to prove Proposition 5 and Theorem 5 which are used in Proposition 2 and also applies to Theorem 2.

Remark 4. *Same previous notes. We notice them in the general case. We apply them when $\sigma = 2$, $\theta < \vartheta = 1$ and $\varrho = 1$ on Theorem 5, through condition (3.29), we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{1/\sigma-1}} - \frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\sigma/\sigma-1}}\right] + \frac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[(1-\theta)^{(\varrho\sigma-1)}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho-1/\varrho(\sigma-1)}\right.$$

$$\left. -\sigma(1-\theta)^{\varrho\sigma-1}\left(\frac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{\varrho\sigma-1/\varrho(\sigma-1)}\right] = \zeta\frac{(1-\theta)^2}{4\Gamma(3)} + \eta\frac{(1-\theta)}{\Gamma(3)} < 1.$$

4. Applications

To prove the desired results above, we take some applications.

Example 1. *Extrapolate the following application of BVP*

$$\begin{cases} {}^{1}_{c}D^{\sigma}_{0^+}\mu(\tau) = 7 - \tau^5 - \sin(\mu(\tau)) - {}^{1}_{c}D^{1}_{0^+}\cos(\mu(\tau)), & 0<\tau<\vartheta, \\ \mu(0) = 2, \quad \mu(\vartheta) = 3. \end{cases} \tag{4.1}$$

Set, $\varrho = 1$, $\sigma = \{1.5, 1.65, 1.8, 1.95\}$, $\varsigma = 1$, $\theta = 0$ *and*

$$\Xi\left(\tau, \mu(\tau), {}^{\varrho}_{c}D^{\varsigma}_{\theta^+}\mu(\tau)\right) = \tau^5 - 7 + \sin(\mu(\tau)) + {}^{1}_{c}D^{1}_{0^+}\cos(\mu(\tau)).$$

Here,

$$\begin{aligned}
&\left|\Xi\left(\tau, \mu, \mu'\right) - \Xi\left(\tau, \nu, \nu'\right)\right| \\
&\quad = \left|\tau^5 - 7 + \sin(\mu) + \cos(\mu') - \left(\tau^5 - 7 + \sin(\nu) + \cos(\nu')\right)\right| \\
&\quad \leq \left|\sin(\mu) - \sin(\nu)\right| + \left|\cos(\mu') - \cos(\nu')\right| \\
&\quad \leq |\mu - \nu| + \left|-2\sin\tfrac{\mu'+\nu'}{2}\sin\tfrac{\mu'-\nu'}{2}\right| \\
&\quad \leq \zeta|\mu - \nu| + \eta|\mu' - \nu'|,
\end{aligned}$$

for (τ, μ, μ'), $(\tau, \nu, \nu') \in [0, \vartheta] \times \mathbb{R}^2$, *where* $\eta = 1 \geq 0$, $\zeta = 2 > 0$. *Moreover, we have*

$$\begin{aligned}
\varpi = {}& \tfrac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\tfrac{\vartheta^{\varrho\sigma}}{\sigma^{1/(\sigma-1)}} - \tfrac{\vartheta^{\varrho\sigma}}{\sigma^{\sigma/(\sigma-1)}}\right] + \tfrac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\Bigg[\vartheta^{(\varrho\sigma-1)}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho-1)/\varrho(\sigma-1)} \\
& -\sigma(\vartheta)^{(\varrho\sigma-1)}\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho\sigma-1)/\varrho(\sigma-1)}\Bigg] \approx \left\{\begin{array}{ll} 0.9751, & \sigma = 1.50, \\ 0.9188, & \sigma = 1.65, \\ 0.8508, & \sigma = 1.80, \\ 0.7758, & \sigma = 1.95, \end{array}\right\} < 1.
\end{aligned}$$

The curves drawn in Figure 1 show how the ϖ *changes for different derivative orders* σ. *The important point is that all of them are less than the line* $y = 1$ *in the interval* $[0, \vartheta]$, *and as the order of the derivative approaches the number one, the parameter* ϖ *decreases, but they are still less than one. These results are shown in Table 1. By the applications of Theorem 2, and the condition* (3.12) *is agreed. Then the BVP* (4.1) *accepts an unique solution.*

Table 1: Numerical results ϖ in Example 1 for four values of σ.

τ	ϖ			
	$\sigma = 1.5$	$\sigma = 1.65$	$\sigma = 1.80$	$\sigma = 1.95$
0.00	0.0000	0.0000	0.0000	0.0000
0.05	0.1707	0.0978	0.0555	0.0311
0.10	0.2449	0.1562	0.0986	0.0616
0.15	0.3043	0.2069	0.1391	0.0926
0.20	0.3564	0.2538	0.1786	0.1244
0.25	0.4040	0.2984	0.2177	0.1571
0.30	0.4487	0.3415	0.2568	0.1909
0.35	0.4912	0.3837	0.2960	0.2256
0.40	0.5322	0.4253	0.3355	0.2614
0.45	0.5719	0.4664	0.3753	0.2983
0.50	0.6107	0.5073	0.4156	0.3362
0.55	0.6488	0.5481	0.4564	0.3753
0.60	0.6863	0.5888	0.4978	0.4154
0.65	0.7233	0.6295	0.5397	0.4566
0.70	0.7599	0.6703	0.5822	0.4989
0.75	0.7962	0.7112	0.6254	0.5423
0.80	0.8323	0.7523	0.6691	0.5868
0.85	0.8682	0.7936	0.7136	0.6324
0.90	0.9040	0.8351	0.7586	0.6791
0.95	0.9396	0.8768	0.8044	0.7269
1.00	0.9751	0.9188	0.8508	0.7758

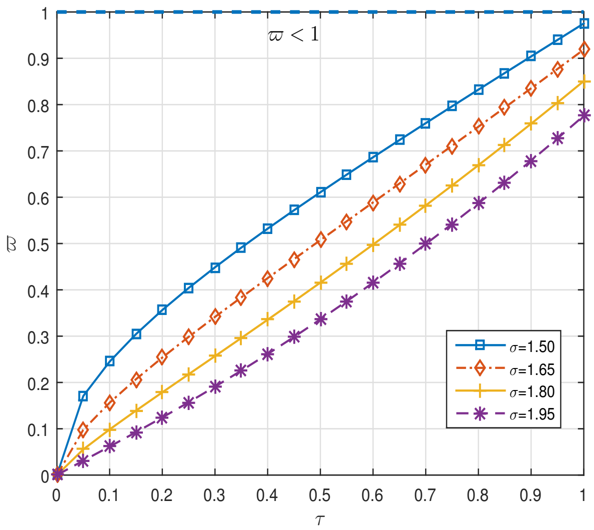

Figure 1: Representation of ϖ for BVP (4.1) in Example 1 for four case σ.

Example 2. *Extrapolate the following application of BVP*

$$\begin{cases} {}^1_cD^{\sigma}_{0^+}\mu(\tau) = 2 - \tau^3 + \frac{1}{2}\cos(\mu(\tau)) - {}^1_cD^1_{0^+}\sin(\mu(\tau)), \quad 0 < \tau < \vartheta \\ \mu(0) = 5, \quad \mu(1) = 6. \end{cases} \tag{4.2}$$

Set, $\varrho = 1$, $\sigma \in \{\frac{4}{3}, \frac{3}{2}, \frac{7}{4}\}$, $\varsigma = 1$, $\theta = 0$, $\vartheta = 1$, *and*

$$\Xi\left(\tau, \mu(\tau), {}^{\varrho}_{c}D^{\varsigma}_{\theta^+}\mu(\tau)\right) = \tau^3 - 2 - \tfrac{1}{2}\cos(\mu(\tau)) + {}^{1}_{c}D^{1}_{0^+}\sin\left(\mu(\tau)\right).$$

Here,

$$\begin{aligned}
&\left|\Xi(\tau,\mu,\mu') - \Xi(\tau,\nu,\nu')\right| \\
&\quad = \left|\tau^3 - 2 - \tfrac{1}{2}\cos(\mu) + \sin\left(\mu'\right) - \left(\tau^3 - 2 - \tfrac{1}{2}\cos(\nu) + \sin\left(\nu'\right)\right)\right| \\
&\quad \le \tfrac{1}{2}\left|\cos(\nu) - \cos(\mu)\right| + \left|\sin\left(\mu'\right) - \sin\left(\nu'\right)\right| \\
&\quad \le \tfrac{1}{2}\left| -2\sin\tfrac{\nu+\mu}{2}\sin\tfrac{\nu-\mu}{2}\right| + \left|\mu' - \nu'\right| \\
&\quad \le \left|\sin\tfrac{\nu-\mu}{2}\right| + \left|\mu' - \nu'\right| \le \zeta|\mu - \nu| + \eta|\mu' - \nu'|,
\end{aligned}$$

for (τ,μ,μ'), $(\tau,\nu,\nu') \in [0,1]\times\mathbb{R}^2$, *where* $\eta = 1 \ge 0$, $\zeta = 1 > 0$. *Moreover, we have*

$$\begin{aligned}
\varpi &= \tfrac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\tfrac{1}{\sigma^{1/(\sigma-1)}} - \tfrac{1}{\sigma^{\sigma/(\sigma-1)}}\right] + \tfrac{\eta}{\varrho^{\sigma-1}\Gamma(\sigma+1)}\left[\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho-1)/\varrho(\sigma-1)}\right. \\
&\quad \left. -\sigma\left(\tfrac{\varrho-1}{\sigma(\varrho\sigma-1)}\right)^{(\varrho\sigma-1)/\varrho(\sigma-1)}\right] \approx \left\{\begin{array}{ll} 0.9285, & \sigma = \frac{4}{3}, \\ 0.3989, & \sigma = \frac{3}{2}, \\ 0.7481, & \sigma = \frac{7}{4}, \end{array}\right\} < 1.
\end{aligned}$$

In the last row of data in Table 2, the values of parameter ϖ*, at point* ϑ*, for three different values of derivative order* σ *are shown. The curves of all three cases are presented in Figure 2, which are decreasing as the order of the derivative increases and in all cases are less than the* $y = 1$ *line. By the applications of Theorem 3, and the condition* (3.19) *is agreed. Then the BVP* (4.2) *accepts an unique solution.*

Table 2: Numerical results ϖ in Example 2 for three values of σ.

τ	ϖ		
	$\sigma = \frac{4}{3}$	$\sigma = \frac{5}{2}$	$\sigma = \frac{7}{4}$
0.00	0.0000	0.0000	0.0000
0.05	0.3110	0.1695	0.0664
0.10	0.3940	0.2414	0.1128
0.15	0.4533	0.2978	0.1544
0.20	0.5015	0.3464	0.1935
0.25	0.5430	0.3901	0.2310
0.30	0.5800	0.4303	0.2674
0.35	0.6137	0.4681	0.3030
0.40	0.6449	0.5040	0.3381
0.45	0.6742	0.5383	0.3728
0.50	0.7018	0.5713	0.4073
0.55	0.7281	0.6033	0.4415
0.60	0.7532	0.6345	0.4756
0.65	0.7774	0.6649	0.5095
0.70	0.8008	0.6946	0.5435
0.75	0.8234	0.7239	0.5775
0.80	0.8455	0.7526	0.6114
0.85	0.8669	0.7809	0.6455
0.90	0.8879	0.8088	0.6796
0.95	0.9084	0.8364	0.7138
1.00	0.9285	0.8637	0.7481

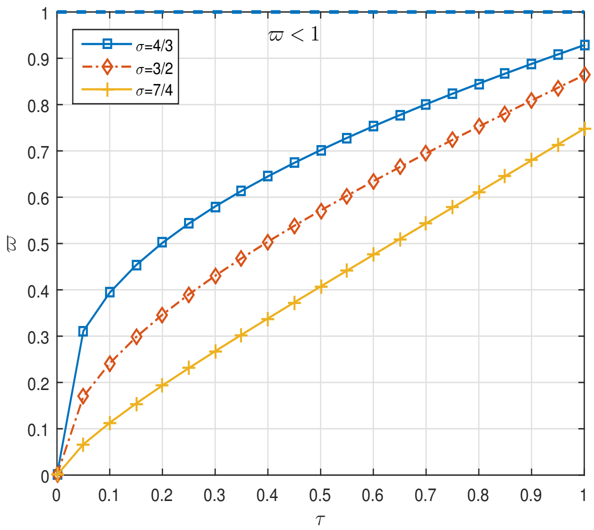

Figure 2: Representation of ϖ for BVP (4.2) in Example 2 for three case σ.

5. Conclusions

Through this project, we tried to simulate the Banach contraction theorem on the Caputo-Katugampola fractional derivative boundary value problem to achieve the existence of a single solution, the core of this work in the third chapter. We relied on the positivity of the Green function and its integral and obtained the two functions ξ and ξ' As shown in Corollary 1. By conclusion, we can determine the maximum of two derived functions in the general case of two boundary conditions "$\mu(\theta) = \lambda_1$, $\mu(\vartheta) = \lambda_2$". We studied two cases when "$\mu(0) = \lambda_1$, $\mu(\vartheta) = \lambda_2$" and "$\mu(0) = \lambda_1$, $\mu(1) = \lambda_2$" in Theorems 2 and 3, we obtain detailed results in this section. We also studied two cases when, the previous general case "$\mu(\theta) = \lambda_1$, $\mu(\vartheta) = \lambda_2$" and the case of "$\mu(\theta) = \lambda_1$, $\mu(1) = \lambda_2$", we can also add two examples with their simulation in these two cases of Theorems 4 and 5 in the Examples part. We also believe there are numerical methods to achieve Banach's theorem of contraction of a single solution to the problem (1.4) with the general boundary conditions. In the future, we can apply the Banach contraction to the Caputo-Fabrizio fractional BVP.

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Declarations

Authors' Contributions

Z.B., M.E.S., V.S.E, S.H. and N.M. wrote the main manuscript. All authors reviewed the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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