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Hypergroups Obtained from a Latin Square

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"This paper is dedicated to Professor Ali Akbar Mohammadi Hassanabadi on his 80th birthday."

Abstract. In this study, we explore the construction of a sequence of hypergroupoids derived from a quasigroup (Latin square). We demonstrate that cyclic groups yield sequences of commutative hypergroups. Additionally, under specific conditions, we establish the formation of H_v-groups and hypergroups. Various examples are provided to illustrate and support the theoretical concepts presented in this paper, offering insights into their structure and applicability.

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Key Words and Phrases: Latin Square, Quasigroup, Quasihypergroup, hypergroup, H_v-group

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1. Introduction

The linkage between algebraic structures and combined design problems in combinatorial models has been a field of great interdisciplinary potential, both in abstract algebra theory and in practical applications ranging from coding theory, cryptography, and experimental design [1, 2]. One of the fundamental conductivities is the relation between quasigroups and Latin squares. In recent years, the topic has also been extended to hyperstructures, and here also to quasihypergroups and hypergroups, providing new avenues for algebraic generalization. The paper by Iranmanesh and Ashrafi [3] opens new fields of algebraic generalization involving the connections between quasihypergroups and Latin squares with their hyperstructural homologues.

The study of Latin squares dates back to ancient mathematical traditions, with early examples in Arabic manuscripts and medieval combinatorial puzzles. However, formalization started in the 18th century, beginning with Leonhard Euler's famous "36 officers problem" that launched a systematic investigation of their properties. A Latin square of order n is a $n \times n$ grid of points, whose values are described by [1, 2, 4]. This set of n independent symbols (which can all occur simultaneously in rows and columns) became

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a key object of design theory. By the 20^{th} century Latin squares had rigorous applications in statistical experimental design (through R. A. Fisher's work), error-correcting codes, and group theory where they became known as multiplication tables of quasigroups (non-associative algebraic structures satisfying the Latin square property)[1, 2, 4].

The desire to use the problem of Latin squares for applications outside of algebra enabled a number of generalizations, for example, reduced Latin squares, orthogonal arrays, and partial Latin squares. All three variants relaxed or extended classical constraints in some way. They proved helpful in solving problems in graph theory, finite geometry, and cryptography. At the same time, quasigroups (algebraic systems with multiplication tables of Latin squares) developed as the basis of non-associative algebras and mathematically relevant solutions of equations. Quasigroups are not associative but contain cancellativity in their behavior. They therefore offer the bridge between discrete mathematics and algebraic abstraction.

On the other hand, the mid-20th century also brought about the development of hyperstructures, a generalization of classical algebraic operations by changing binary operations (contrary to traditional algebraic operations) to set-valued operations. The fields were opened by the work of F. Marty (1934) in the invention of hypergroups–structures in which the product of two variables is a nonempty set—in the direction of work from Corsini, Davvaz and Vougiouklis [5–8]. hyperstructures and their variants—such as quasihypergroups and H_v-structures—have been used for modelling uncertainty theory, granular computing, and non-deterministic systems. Since hyperstructures are compatible with multivaluedness, they offer more powerful algebraic representations for complex relational systems.

The construction of a sequence of hypergroupoids derived from a quasigroup (Latin square) is investigated here. We prove that cyclic groups are capable of yielding sequences of commutative hypergroups. In addition, under some special conditions, we show that H_v-groups and hypergroups are formed. Experiments are given in this paper to show and support the theoretical notions presented, and to show how they are formulated and how they may be extended to practice.

2. Main Results

A quasigroup $(Q, \star)$ is a non-empty set Q with a binary operation $\star$, obeying the Latin square property[4, 9]. This states that, for each a and b in Q, there exist unique elements x and y in Q such that both

$$a \star x = b,$$

$$y \star a = b$$

hold. In other words, each element of the set occurs exactly once in each row and exactly once in each column of the quasigroup's multiplication table, or Cayley table. This property ensures that the Cayley table of a finite quasigroup, and, in particular, a finite group, is a Latin square.

We recall basic definitions from [8]. A hypergroupoid or hyperstructure is a non-empty set H with a hyperoperation $\circ$ defined on H, that is, a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is defined by $x \circ y$. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y | x \in A, y \in B\}$. The notation $a \circ A$ is used for $\{a\} \circ A$, and $A \circ a$ for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a.

The relational notation $A \approx B$ (read A meets B) is used to assert that the sets A and B have an element in common, that is, $A \cap B \neq \emptyset$.

A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all $x, y, z \in H$ we have $x \circ (y \circ z) = (x \circ y) \circ z$. Moreover, a hypergroupoid $(H, \circ)$ is called a H_v-semigroup if for all $x, y, z \in H$ we have $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$ or $x \circ (y \circ z) \approx (x \circ y) \circ z$.

A hypergroupoid $(H, \circ)$ is called a commutative hypergroupoid if $x \circ y = y \circ x$, for all $x, y \in H$. Moreover it is called weak commutative if $x \circ y \cap y \circ x \neq \emptyset$ or $x \circ y \approx y \circ x$, for all $x, y \in H$.

A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for all $x \in H$ we have $x \circ H = H \circ x = H$.

We defin two hypercompositions on $(H, \circ)$, right extension / and left extension \, each an inverse to $\cdot$, are defined by:

$$a/b = \{x \mid a \in x \circ b\} \quad \text{and} \quad b\backslash a = \{x \mid a \in b \circ x\}.$$

Hence, $x \approx a/b$ if and only if $a \approx x \circ b$, and $x \approx b\backslash a$ if and only if $a \approx b \circ x$.

Definition 1. *Let X be a n-set and let $A = [a_{ij}]$ be a $n \times n$ matrix with $a_{ij} \subseteq X$ for all of $1 \leq i, j \leq n$. A is called a generalized Latin square on n-set X if the following condition is satisfied:*

$$\bigcup_{i=1}^{n} a_{ij} = X = \bigcup_{j=1}^{n} a_{ij}.$$

Example 1. *Let $n = 4$, $X = \{a, b, c, d\}$ and $a_{ii} = X$ and $a_{ij} = \{a\}$, where $i \neq j$. Then $A = [a_{ij}]$ is a generalized Latin square. In fact*

$A =$

X	a	a	a
a	X	a	a
a	a	X	a
a	a	a	X

Definition 2. *Let X be a n-set and let $A = [a_{ij}]$ be a $n \times n$ matrix with $a_{ij} \subseteq X$ for all of $1 \leq i, j \leq n$. A is called a Latin k-hypersquare on n-set X if the following condition is satisfied:*

(1) $|a_{ij}| = k$.

(2) Equations $a \cdot x = c$ and $y \cdot b = d$ has exactly k solutions in X, Where the hyperoperation $\cdot$ on X is defined as $x \cdot y = a_{xy}$.

Example 2. *The generalized Latin square A in Example 1 is not a Latin k-hypersquare. But the generalized Latin square B is a is not a Latin 2-hypersquare, where*

$B =$

e, c	e, a	a, b	b, c
e, a	a, b	b, c	e, c
b, c	e, c	e, a	a, b
a, b	b, c	e, c	e, a

Let $(H, \circ)$ be a semihypergroup. The relation β^* is the transitive closure of the relation $\beta = \cup_{n\geq 1}\beta_n$, where β_1 is the diagonal relation and, for every integer $n > 1$, β_n is the relation defined as follows:

$$x\beta_n y \;\Leftrightarrow\; \exists(z_1, \ldots, z_n) \in H^n : \{x, y\} \subseteq \prod_{i=1}^{n} z_i.$$

β^* is the smallest strongly regular equivalence on H. Moreover, the canonical projection $\psi : H \to H/\beta^*$ is a homomorphism and if H is a hypergroup, the kernel of ψ is called heart of H and is denoted with ω_H.

Let $(H, \circ)$ be a hypergroupoid. Let U denote the set of all finite products of elements of H. Then relation β can be defined on H as follows:

$$x\beta y \Leftrightarrow \exists u \in U \text{such that} \{x, y\} \subseteq u.$$

If $(H, \circ)$ is a hypergroupoid then The relation β^* is the transitive closure of the relation β.

Theorem 1. *(Theorem 81 in [5]) If H is a hypergroup then $\beta = \beta^*$.*

As a consequence of Theorem 1, in every hypergroup the relation β is transitive. But in semihypergroups this not true and in H_v-groups this is an open problem.

Definition 3. *Let $(H, \circ)$ be a hypergroupoid. The hyperoperation $\circ$ is called total associative, if*

$$x \circ (y \circ z) = H = (x \circ y) \circ z, \;\forall x, y, z \in H.$$

Example 3. *Let $(H, \circ_T)$ be a total hypergroup, i. e., for all $x, y \in H$, $x \circ_T y = H$. Then $\circ$ is a total associative hyperoperation.*

Example 4. *Let $H \neq \emptyset$ and $|H| > 2$. For every $x, y \in H$, define* $x \circ y = \begin{cases} H - \{y\}, x \neq y \\ H, \; x = y. \end{cases}$ *Then for every $x, y, z \in H$, we have $y \circ z = H - \{z\}$ and so $|H - \{z\} = 2$. So*

$$x \circ (y \circ z) = x \circ (H - \{z\}) = \cup_{w \in H - \{z\}} x \circ w = H.$$

In the similar way we have $H = (x \circ y) \circ z$. Therefore the hyperoperation $\circ$ is total associative

Now, we construct some hyperoperations from a quasigroup as follows:

Definition 4. *Let $(H,\star)$ be a finite quasigroup. Let $H=\{a_0,a_1,\ldots,a_{n-1}\}$ and $a_i\star a_j=a_{ij}$, for every $i,j=0,1,\ldots,n-1$. For $k\in\{0,1,\ldots,n-1\}$ set $a_i\star_k^C a_j=a_{ih}$, when $j-k\equiv h \mod n$. It easily to see that $(H,\star_k)$ is a quasigroup. Now for every $x,y\in H$ define the hyperoperation $\circ_k^C$ as follows*

$$x\circ_k^C y=\{x\star_m^C y|m=0,1,\ldots,k-1\}.$$

Definition 5. *Let $(H,\star)$ be a finite quasigroup. Let $H=\{a_0,a_1,\ldots,a_{n-1}\}$ and $a_i\star a_j=a_{ij}$, for every $i,j=0,1,\ldots,n-1$. For $k\in\{0,1,\ldots,n-1\}$ set $a_i\star_k^R a_j=a_{hj}$, when $j-k\equiv h$ mod n. It easily to see that $(H,\star_k^R)$ is a quasihypergroup. Now for every $x,y\in H$ define the hyperoperation $\circ_k^R$ as follows*

$$x\circ_k^R y=\{x\star_m^R y|m=0,1,\ldots,k-1\}.$$

Proposition 1. *Let $(H,\star)$ be a finite quasigroup and $\circ_k^R$ and $\circ_k^C$ be the hyperoperations in Definition 4 and 5. Then $(H,\circ_{n-1}^C)$ and $(H,\circ_{n-1}^R)$ are total hypergroup.*

Proof. Let $H=\{a_0,a_1,\ldots,a_{n-1}\}$. Since $(H,\star)$ is a quasigroup then for every $x,y\in H$, $|x\circ_{n-1}^C y|=n$ and $x\circ_{n-1}^R y\subseteq H$. So for every $x,y\in H$, $x\circ_{n-1}^R y=H$.

Proposition 2. *Let $(H,\star)$ be a finite quasigroup and $\circ_k^R$ and $\circ_k^C$ be the hyperoperations in Definition 4 and 5. Then for $k\geq\frac{|H|}{2}$, $(H,\circ_k^R)$ and $(H,\circ_k^C)$ are weak commutative hypergroupoids.*

Proof. For every $x,y\in H$, $|x\circ_k^C y|=k=|y\circ_k^C x|$. If $x\circ_k^C y\cap y\circ_k^C x=\emptyset$ then $|x\circ_k^C y\cup y\circ_k^C x|=k+k+2>|H|$ and this is a contradiction. Therefore $x\circ_k^C y\cap y\circ_k^C x\neq\emptyset$.

In Theorem 2, the given boundary for k is the best boundary. See the next Example.

Example 5. *Let $(H,\star)$ be a quasigroup.*

$\star_0$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	c	e	a	b
c	b	c	e	a

$\star_0$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	c	e	a	b
c	b	c	e	a

$\star_1$	e	a	b	c
e	c	e	a	b
a	e	a	b	c
b	b	c	e	a
c	a	b	c	e

$\star_2$	e	a	b	c
e	b	c	e	a
a	c	e	a	b
b	a	b	c	e
c	e	a	b	c

$\star_3$	e	a	b	c
e	a	b	c	e
a	b	c	e	a
b	e	a	b	c
c	c	e	a	b

$\circ_0^C$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	c	e	a	b
c	b	c	e	a

$\circ_1^C$	e	a	b	c
e	e, c	e, a	a, b	b, c
a	e, a	a, b	b, c	e, c
b	b, c	e, c	e, a	a, b
c	a, b	b, c	e, c	e, a

$\circ_2^C$	e	a	b	c
e	e, b, c	e, a, c	e, a, b	a, b, c
a	e, a, c	e, a, b	a, b, c	e, b, c
b	a, b, c	e, b, c	e, a, c	e, a, b
c	e, a, b	a, b, c	e, b, c	e, a, c

$(H, \circ_1^C)$ *is not weak commutative.*

Proposition 3. *Let $(H, \star)$ be a finite quasigroup and $\circ_k^C$ be the hyperoperation in Definition 4. If $\star$ is commutative then $\circ_k^C$ is weak commutative, for all k.*

Proof. For every $x, y \in H$, $x \star y = y \star x$. We have $x \star y \in x \circ_k^C y$ and $y \star x \in y \circ_k^C x$. Therefore $x \circ_k^C y \cap y \circ_k^C x \neq \emptyset$ and proof is complete.

Proposition 4. *Let $(H, \star)$ be a finite quasigroup and $\circ_k^C$ be the hyperoperation in Definition 4. If $(H, \star)$ is a cyclic group, then for every k, $\circ_k^C$ is commutative;*

Proof. Let $(H, \star)$ is a cyclic group then $(H, \star) \cong (\mathbb{Z}, +)$ or $(H, \star) \cong (\mathbb{Z}_n, +)$. If $(H, \star) \cong (\mathbb{Z}_n, +)$ then for every $x, y \in \mathbb{Z}_n$,

$$x \circ_k^C y = \{x + y, x + y - 1, \dots, x + y - k\} = \{y + x, y + x - 1, \dots, y + x - k\} = x \circ_k^C y.$$

Therefore $(H, \circ_k^C)$ is a commutative hypergroupoid.

Lemma 1. $(H, \circ_k^C)$ *and* $(H, \circ_k^R)$ *are quasihypergroups.*

Proof. For every $a, b \in H$, $a \star x = b$ and $y \star a = b$ have solutions in H. Since for every k, $a \star x \subseteq a \circ_k^C x$ and $y \star a \subseteq y \circ_k^C a$ therefore $b \in a \circ_k^C x$ and $b \in y \circ_k^C a$ have solutions in H.

Lemma 2. *We have*

$$\star = \circ_0^C \subset \circ_1^C \subset \ldots \subset \circ_{n-2}^C \subset \circ_{n-1}^C = \circ_T,$$

and

$$\star = \circ_0^R \subset \circ_1^R \subset \ldots \subset \circ_{n-2}^R \subset \circ_{n-1}^R = \circ_T.$$

When $\star$ is the operation in Definition 5 and $\circ_T$ is the hyperoperation in Example 3.

Proof. It is straightforward.

Example 6. *Let $(H, \star)$ be a cycle group of order 3 by the following Cayley table*

$\star$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

then we obtain

$\star_0$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\star_1$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

$\star_3$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

and therefore

$\circ_0^C$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\circ_1^C$	a	b	c
a	a, c	a, b	b, c
b	a, b	b, c	a, c
c	b, c	a, c, a	a, b

$\circ_2^C$	a	b	c
a	H	H	H
b	H	H	H
c	H	H	H

.

$(H, \circ_k^C)$ are commutative hypergroups.

Example 7. *Let $(H, \star)$ be a finite quasigroup by the following Cayley table*

$\star$	a	b	c
a	b	c	a
b	a	b	c
c	c	b	a

then we obtain

$\star_0$	a	b	c
a	b	c	a
b	a	b	c
c	c	a	b

$\star_1$	a	b	c
a	a	b	c
b	c	a	b
c	b	c	a

$\star_2$	a	b	c
a	c	a	b
b	b	c	a
c	a	b	c

and therefore

$\circ_0^C$	a	b	c
a	b	c	a
b	a	b	c
c	c	a	b

$\circ_1^C$	a	b	c
a	a, b	b, c	a, c
b	a, c	a, b	b, c
c	b, c	c, a	b, a

$\circ_2^C$	a	b	c
a	H	H	H
b	H	H	H
c	H	H	H

.

$(H, \circ_0^C)$ is neither commutative nor weak commutative (because $a \circ_0^C c \not\approx c \circ_0^C a$).
$(H, \circ_1^C)$ is not commutative but it is weak commutative (because $x \circ_1^C y \approx y \circ_1^C x$ for all $x, y \in H$).

Theorem 2. *For every $x, y \in H$,*

(1) $|x \circ_k^C y| = k + 1$;

(2) $|\{u | x \in u \circ_k^C y\}| = k + 1$;

(3) $|\{u | x \in y \circ_k^C u\}| = k + 1$.

Proof.
It obtains from definition $\circ_k^C$.

Corollary 1. *for all $k = 1, 2, \ldots, n$, the quasihypergroups $(H, \circ_{k-1}^C)$ are Latin k-hypersquare.*

In the next Theorem we show that for all $k = \frac{|H|}{2}, \frac{|H|}{2} + 1, \ldots, n$, the quasihypergroups $(H, \circ_k^C)$ are hypergroups,

Theorem 3. *If $k \geq \frac{|H|}{2}$ then $\circ_k^C$ is a total associative hyperoperation.*

Proof. Let $H = \{a_1, a_2, \ldots, a_n\}$ and $k \geq \frac{n}{2}$. Suppose that $x, y, z \in H$. Then by part (1) of Theorem 2, $|y \circ_k^C z| = k + 1 > \frac{n}{2}$. So we have $y \circ_k^C z = \{a_{i_0}, a_{i_2}, \ldots, a_{i_k}\}$. Now if $h \in H$ then by part (3) of Theorem 2, there exists $j = 0, 2, \ldots, k$ such that $u \in x \circ_k^C a_{i_j}$. Hence $x \circ_k^C (y \circ_k^C z) = H$. In the similar way we obtain $(x \circ_k^C y) \circ_k^C z = H$. Therefore $\circ_k^C$ is a total associative hyperoperation.

Theorem 4. *If $k \geq \frac{|H|}{2}$ then $(H, \circ_k^C)$ is a hypergroup.*

Proof. By Theorem 3 $(H, \circ_k^C)$ is a semihypergroup and by Lemma 1 $(H, \circ_k^C)$ is a quasihypergroup. Therefore $(H, \circ_k^C)$ is a hypergroup

Example 8. *In Theorem 4, the given boundary for k is the best boundary. For example if $(H, \star)$ is the quasigroup in Example 7 then $(H, \circ_1^C)$ is not hypergroup.*

Theorem 5. *Let $(H, \star)$ be a cyclic group of order n, i. e., $(H, \star) \cong (\mathbb{Z}_n, +)$. Then $(H, \circ_k^C)$ is a commutative hypergroup, for all $k = 0, 1, \ldots, n - 1$.*

Example 9. *Let $(H, \star) \cong (Z_2 \times \mathbb{Z}_2, +)$.*

$\star$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

then we obtain

$\star_0$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\star_1$	e	a	b	c
e	c	e	a	b
a	b	a	e	c
b	a	b	c	e
c	e	c	b	a

$\star_2$	e	a	b	c
e	b	c	e	a
a	c	b	a	e
b	e	a	b	c
c	a	e	c	b

$\star_3$	e	a	b	c
e	a	b	c	e
a	e	c	b	a
b	c	e	a	b
c	b	a	e	c

and therefore

$\circ_0^C$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\circ_1^C$	e	a	b	c
e	e, c	e, a	a, b	b, c
a	a, b	e, a	e, c	b, c
b	a, b	b, c	e, c	e, a
c	e, c	c, b	a, b	e, a

$\circ_2^C$	e	a	b	c
e	e, b, c	e, a, c	e, a, b	a, b, c
a	e, a, b	e, a, c	e, b, c	a, b, c
b	e, a, b	a, b, c	e, b, c	e, a, c
c	e, a, c	e, b, c	a, b, c	e, a, b

We have $(H,\star)$ is commutative but $(H,\circ_1^C)$ and $(H,\circ_2^C)$ are weak commutative(are not commutative)

Theorem 6. *Let $(H,\star)$ be a group of order n. Then $(H,\circ_k^C)$ is an H_v-group, for all $k=0,1,\ldots,n-1$.*

Proof. For every $x,y,z \in X$, $x\star(y\star z)=(x\star y)\star z$, $x\star(y\star z)\in x\circ_k^C(y\circ_k^C z)$ and $(x\star y)\star z)\in(x\circ_k^C y)\circ_k^C z$. Therefore $x\circ_k^C(y\circ_k^C z)\cap(x\circ_k^C y)\circ_k^C z\neq\emptyset$.

Theorem 7. *Let $(H,\star)$ be a quasigroup.*

(1) If $k=0$, then $\beta^*=\{(x,x)|x\in H\}$;

(2) If $k>0$, then $\beta^*=H\times H$.

Proof. If $k=0$ then $\prod_{i=1}^n z_i$ is singleton and so $\beta=\{(x,x)|x\in H\}$. Now, Let $k>0$ and $x,y\in H$. Consider $(a_1\star_0 a_1,\ldots,a_1\star_0 a_n)=(b_1,\ldots,b_n)$ so $(a_1\star_1 a_1,\ldots,a_1\star_1 a_n)=(b_2,\ldots,b_n,b_1)$. Therefore $\{b_i,b_{i+1}\}\subseteq a_1\circ_2^C a_i\subseteq a_1\circ_k^C a_i$. Since $H=\{b_1,\ldots,b_n\}$ then for every $x,y\in H$ there exist $1\leq i,j\leq n$ such that $i<j$ and $b_i=x$ and $b_j=y$. $x=b_i\beta b_{i+1}\beta\ldots\beta b_j=y$. and therefore $x\beta^*y$.

Corollary 2. *For quasihypergroups $(H,\circ_k^C)$, $k=0,1,\ldots,n-1$, we have*

(1) If $k=0$, then the fundamental quasigroup $\frac{H}{\beta^*}$ is isomorphic to H;

(2) If $k>0$, then the fundamental quasigroup $\frac{H}{\beta^*}$ is trivial group.

Definition 6. *A hypergroupoid $(H,\circ)$ is called a transposition hypergroup if it satisfies the axiom,*

$$(\textit{Transposition}) \quad b\setminus a\approx c/d \implies a\circ d\approx b\circ c \quad \textit{for all} \quad a,b,c,d\in H.$$

If the hypergroupoid $(H,\circ)$ is commutative then hyperoperation $\setminus = /$ and a commutative transposition hypergroup is called a join space.

Theorem 8. *If $k=\frac{|H|}{2},\ldots,n$ then the quasihypergroup $(H,\circ_k^C)$ is a transposition hypergroupoid.*

Proof. Let $a,b,c,d\in H$ and $b\setminus a\approx c/d$. Since $|a\circ_k^C d|=k+1=|b\circ_k^C c|$ then $a\circ_k^C d\cap b\circ_k^C c\neq\emptyset$.

Corollary 3. *Let $(H,\star)$ be a cyclic group of order n. Then $(H,\circ_k^C)$ is a join space.*

Proof. It obtains from Theorems 5 and 8.

Remark 1. *For every result true for $\circ_k^C$ then it also holds for $\circ_k^R$.*

3. conclusion

In this paper, we initiated the construction of sequences of hypergroupoids derived from quasigroups (Latin squares). Our findings demonstrate that cyclic groups can generate sequences of commutative hypergroups, providing a foundational framework for further exploration. Furthermore, under specific conditions, the formation of H_v-groups and hypergroups was established, enriching the theoretical understanding of these structures. To bridge theory with application, we conducted experiments that not only corroborate the presented theoretical concepts but also illustrate their formulation and potential practical extensions. These contributions pave the way for further research in hyperstructure theory and its real-world applicability.

Future work may delve into extending the current constructions from Γ-structures to Γ-hyperstructures, further enriching the theoretical framework. Moreover, this method can be applied to explore fuzzy algebraic structures, expanding its applicability. For example, recent advancements in the fuzzification of n-Lie algebras [10] and the structural aspects of Gamma rings [11] provide promising directions for future research.

References

[1] J Denes and A D Keedwell. *Latin Squares and Their Applications.* Academic Press Inc, 1974.

[2] V Shcherbacov. *Elements of Quasigroup Theory and Applications.* Chapman and Hall/CRC, 2017.

[3] A Iranmanesh and A R Ashrafi. Generalized latin square. *J. Appl. Math. & Computing*, 22(1-2):285–293, 2006.

[4] H O Pflugfelder O Chein and (eds) J D H Smith. *Quasigroups and Loops: Theory and Applications.* Heldermann, Berlin, 1990.

[5] P Corsini. *Prolegomena of hypergroup theory.* Aviani Editore, Aviani Editore, 1993.

[6] P Corsini and V Leoreanu. *Applications of hyperstructure theory.* Kluwer Academic Publishers, Advances in Mathematics, 2003.

[7] B Davvaz. *Semihypergroup theory.* Elsevier, 2016.

[8] B Davvaz and T Vougiouklis. *A walk through weak hyperstructures; H_v-structure.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ., 2019.

[9] R H Bruck. *A Survey of Binary Systems.* University of Michigan Press, Springer-Verlag, 1971.

[10] S Shaqaqha and M Y Al-Deiakeh. On lie homomorphisms of complex intuitionistic fuzzy lie algebras. *European Journal of Pure and Applied Mathematics*, 17(4):3291–3303, 2024.

[11] S Shaqaqha and A Dagher. Grading and filtrations of gamma rings. *Italian Journal of Pure and Applied Mathematics*, 47:958–970, 2022.