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Survival Weighted Pareto Distribution: Order Statistics and Its Applications

M. I. Khan[1,*], Abdelfattah Mustafa[1,2]

[1] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, 42351, Kingdom of Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

Abstract. Nagy et al. [1] proposed an extended Pareto distribution called a survival-weighted Pareto distribution (SWPD). The proposed model proves to be the best fit for several extensions of the Pareto distribution. The current study provides quantitative features of SWPD through order statistics. The SWPD is characterized through the hazard rate function (HRF). The appropriateness of the characterization result is verified empirically by using a real-life data set. Finally, the measures of inequality are taken into consideration due to the heavy tail properties of the model.

2020 Mathematics Subject Classifications: 60E05, 60E15, 62E10, 62G30

Key Words and Phrases: SWPD, Order statistics, Moments, Characterization, Inequality measures

1. Introduction

Arnold [2] discussed a profound historical survey on Pareto distribution and explained its wide use in the context of economic phenomena. The PD provides an extensive application in economics, actuarial, finance, social, natural sciences and various fields.
In the statistical arena, dealing with lifetime data is a main concern in several fields, including engineering, biomedical sciences, finance, and economics. Many continuous distributions have been introduced for dealing with such data because they can present a better fit than the based distribution.
The formation of new probability model devoted to serve for a vast range of real-world problem along with admissible statistical methodologies. In the last thirty years, there has been growingly interest in the formulation of extended flexible models of distribution in several areas of significance, where the original distribution fails for complex data.
Several authors developed the weighted model in the literature when a sample is not noted with equal probability. For reference, [3–11]. The weighted model was first formulated

*Corresponding author.
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Email addresses: izhar.stats@gmail.com (M. I. Khan), amelsayed@mans.edu.eg (A. Mustafa)

by Fisher [12] and showed its usefulness in ecological sciences, biostatistics, medicine, pharmacy, and environmental sciences.
However, there are still many situations where classical and weighted models are unsuitable for real-world data. To address this issue, we formulate the concept of the survival-weighted model to get the new distribution that enhances the flexibility of the existing new model technique for better adaption to complex data. The outcomes drawn from them appear quite sound, genuine and optimal.

2. Preliminaries

The survival-weighted model used in this article possesses the advantages of easy implementation, better fitting performance and solid theoretical foundation.
A random variable X is said to follow a Pareto distribution (PD) if it possesses the following probability density function (PDF)and the cumulative distribution function (CDF) respectively

$$f(x,\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}, \quad \alpha,\beta>0,\ x\geq\beta, \tag{1}$$

$$F(x,\alpha,\beta) = 1-\left(\frac{\beta}{x}\right)^{\alpha}, \quad \alpha,\beta>0,\ x\geq\beta. \tag{2}$$

A random variable X has the PDF $f(x)$. Then, the survival weighted distribution (SWD) of X is given as

$$f_{\mathrm{SWD}}(x) = \frac{S(x)f(x)}{E(S(x)} \tag{3}$$

where $S(x)$ and $f(x)$ are the SF and PDF of the base distribution.
The PDF of the SWPD distribution is generated from Equation (3) as follows

$$f_{\mathrm{SWPD}}(x) = \frac{2\alpha\beta^{2\alpha}}{x^{2\alpha+1}}, \quad \alpha,\beta>0,\ x\geq\beta, \tag{4}$$

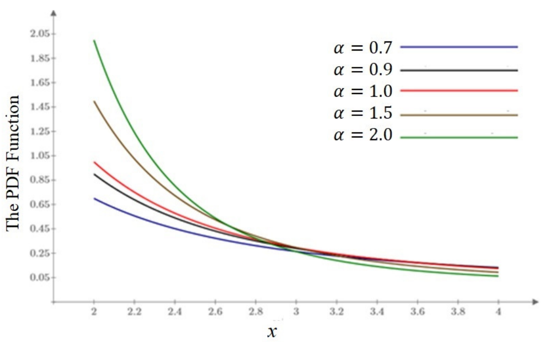

Figure 1: The plots of $f_{\mathrm{SWPD}}(x)$ at $\beta = 0.7$.

The plots of PDFs are decreasing as shown in Figure 1 at $\beta = 0.7$.

The CDF of SWPD is stated by

$$F_{\text{SWPD}}(x) = 1 - \left(\frac{\beta}{x}\right)^{2\alpha}, \quad \alpha, \beta > 0,\ x \geq \beta \tag{5}$$

The SF and hazard rate function (HRF)

$$S_{\text{SWPD}}(x) = \left(\frac{\beta}{x}\right)^{2\alpha}, \quad \alpha, \beta > 0,\ x \geq \beta, \tag{6}$$

$$h_{\text{SWPD}}(t) = \frac{f(t)}{\overline{F}(t)} = \frac{2\alpha}{t}, \tag{7}$$

where α and β represent the shape and scale parameters, respectively.

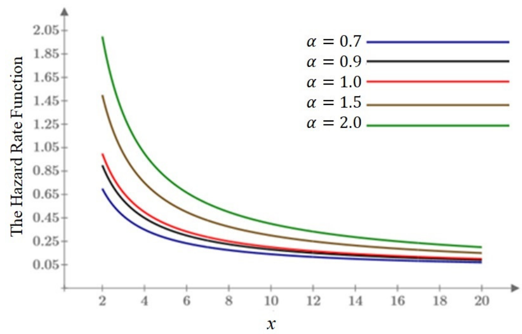

Figure 2: The plots of $h(t)$ at $\beta = 0.7$.

The $h(t)$ of SWPD distribution decreasing depending on the shape parameter.

The rest of manuscript outlined as follows. Section 3 provides order statistics from SWPD and its measures, whereas Section 4 contains recurrence relations. Section 5 is devoted to characterizing the distribution through HRF and supported with an application as well, while Section 6 introduces measures of inequality. Section 7 discusses some concluding remarks.

3. Order Statistics

The order statistics (O. S.) plays a key role in several areas of statistics and probability. For example, Meteorology, quality control, sports hydrology, several characterizations of probability distributions, detection of outliers, and so on. The voluminous theory and application of order statistics are presented by David [13] and Arnold et al. [14].

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from (4). Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the O.S. Then PDF of r-th O. S. is given by David and Nagaraja [15] for $1 \leq r \leq n$

$$f_{r:n}(x) = C(r,n)[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x), \quad 0 \leq x < \infty, \tag{8}$$

where

$$C(r,n) = \frac{n!}{(r-1)!(n-r)!}.$$

The PDF of r^{th} r-th O.S. of the SWPD is given by

$$f_{r:n}(x;\alpha,\beta) = C(r,n)\left[1-\left(\frac{\beta}{x}\right)^{2\alpha}\right]^{r-1}\left(\frac{\beta}{x}\right)^{2(n-r+1)\alpha}\frac{2\alpha}{x}. \tag{9}$$

Using binomially expansion of $\left[1-\left(\frac{\beta}{x}\right)^{2\alpha}\right]^{r-1}$, we have

$$f_{r:n}(x;\alpha,\beta) = C(r,n)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left(\frac{\beta}{x}\right)^{2(n-r+i+1)\alpha}\frac{2\alpha}{x}. \tag{10}$$

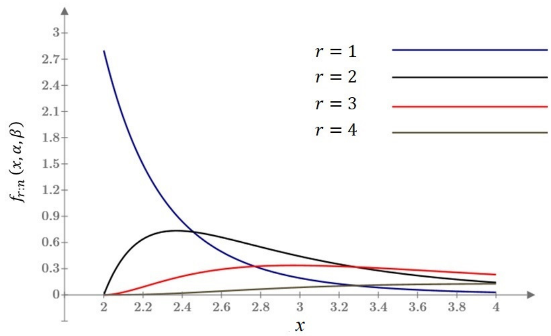

Figure 3: The PDF of rth O.S., when $n = 4, \alpha = 0.7$ and $\beta = 2.0$.

At $r = 1$, we get the PDF of rth smallest O.S.

$$f_{1:n}(x;\alpha,\beta) = \frac{2n\alpha}{x}\left(\frac{\beta}{x}\right)^{2n\alpha}. \tag{11}$$

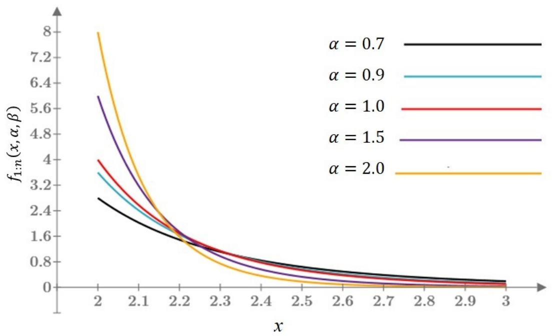

Figure 4: The plots of $f_{1:n}(x;\alpha,\beta)$ at $\beta = 2$ and $n = 4$.

The PDF of rth largest O.S. is obtained at $r = n$, as follows.

$$f_{n:n}(x;\alpha,\beta) = n\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\left(\frac{\beta}{x}\right)^{2(i+1)\alpha}\left(\frac{2\alpha}{x}\right). \tag{12}$$

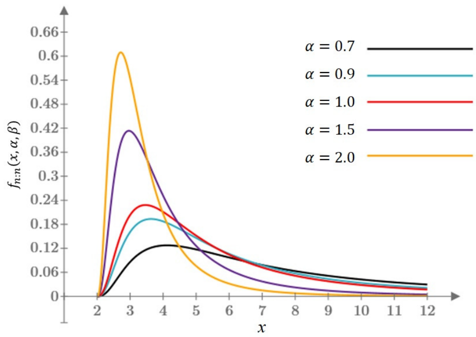

Figure 5: The plots of $f_{n:n}(x;\alpha,\beta)$ at $\beta = 2$ and $n = 4$.

3.1. Single Moments

An efficient mechanism for calculating the statistical features such as mean, dispersion, symmetry, and kurtosis of all order statistics for any distribution, single moments have assumed a significant interest and are often used as a pivotal tool for estimating future events. For example, see David [13] and Arnold et al. [14].

Theorem 1. *Let $X_1, X_2, \cdots, X_n$ be a random sample from SWPD with CDF and PDF denoted by $F(x)$ and $f(x)$ respectively and let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the corresponding O.S. Then the expected value of $X_{r:n}$ which is the k^{th} moments of the r^{th} O.S. for $k = 1, 2, \cdots$ denote $\mu_{r:n}^{(k)}$ is given by.*

$$\mu_{r:n}^{(k)} = C(r,n)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\frac{2\alpha\beta^k}{2(n-r+i+1)\alpha-k}. \tag{13}$$

Proof. We have

$$\begin{aligned}
\mu_{r:n}^{k} &= E(X_{r:n}^k) = \int_{-\infty}^{\infty} x^k f_{r:n}(x;\alpha,\beta)dx \\
&= = C(r,n)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i 2\alpha\beta^{2(n-r+i+1)\alpha}\int_{\beta}^{\infty} x^{k-1-2(n-r+i+1)\alpha}dx. \\
&= C(r,n)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i 2\alpha\beta^{2(n-r+i+1)\alpha}\left[\frac{x^{k-2(n-r+i+1)\alpha}}{k-2(n-r+i+1)\alpha}\right]_{\beta}^{\infty} \\
&= C(r,n)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\frac{2\alpha\beta^{2(n-r+i+1)\alpha}}{k-2(n-r+i+1)\alpha}\left[\frac{1}{x^{2(n-r+i+1)\alpha-k}}\right]_{\beta}^{\infty}
\end{aligned}$$

$$= \quad C(r,n)\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\frac{2\alpha\beta^k}{2(n-r+i+1)\alpha-k},$$

for $2(n-r+i+1)\alpha-k\geq 0$.
This completes the proof.

Remark 1. *The k^{th} moments for minimum O.S. is as follows*

$$\mu_{1:n}^{(k)}=\frac{2n\alpha\beta^k}{2n\alpha-k}.$$

Remark 2. *The k^{th} moments for maximum O.S. is as follows.*

$$\mu_{n:n}^{(k)}=n\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\frac{2\alpha\beta^k}{2(k+1)\alpha-k}.$$

Equation (13) enables to perform the higher moments and other statistical features for all O. S. for $1\leq n\leq 4$ and varying values of $\alpha\in(0.6-4.2)$ at $\beta=0.7$.

Table 1: Values of $\mu_{r:n}^{(1)}=\mu_{r:n}$.

	$n=1$	$n=2$		$n=3$			$n=4$			
α	$\mu_{1:1}$	$\mu_{1:2}$	$\mu_{2:2}$	$\mu_{1:3}$	$\mu_{2:3}$	$\mu_{3:3}$	$\mu_{1:4}$	$\mu_{2:4}$	$\mu_{3:4}$	$\mu_{4:4}$
0.6	4.2000	1.2000	7.2000	0.9692	1.6615	9.9692	0.8842	1.2243	2.0988	12.5927
1.2	1.2000	0.8842	1.5158	0.8129	1.0268	1.7603	0.7814	0.9074	1.1462	1.9650
1.8	0.9692	0.8129	1.1256	0.7714	0.8959	1.2404	0.7522	0.8290	0.9627	1.3330
2.4	0.8842	0.7814	0.9870	0.7522	0.8397	1.0607	0.7385	0.7936	0.8858	1.1190
3.0	0.8400	0.7636	0.9164	0.7412	0.8086	0.9703	0.7304	0.7734	0.8437	1.0125
3.6	0.8129	0.7522	0.8736	0.7340	0.7888	0.9160	0.7252	0.7604	0.8171	0.9489
4.2	0.7946	0.7443	0.8449	0.7289	0.7751	0.8798	0.7215	0.7513	0.7988	0.9068

Table 2: Values of $\mu_{r:n}^{(2)}$.

	$n=1$	$n=2$		$n=3$			$n=4$			
α	$\mu_{1:1}^{(2)}$	$\mu_{1:2}^{(2)}$	$\mu_{2:2}^{(2)}$	$\mu_{1:3}^{(2)}$	$\mu_{2:3}^{(2)}$	$\mu_{3:3}^{(2)}$	$\mu_{1:4}^{(2)}$	$\mu_{2:4}^{(2)}$	$\mu_{3:4}^{(2)}$	$\mu_{4:4}^{(2)}$
0.6	-0.7350	2.94 00	-4.4100	1.1025	6.615	-9.9225	0.8400	1.8900	11.34	-17.01
1.2	2.940	0.840	5.040	0.679	1.163	6.979	0.619	0.857	1.469	8.815
1.8	1.103	0.6785	1.527	0.601	0.833	1.874	0.569	0.698	0.967	2.176
2.4	0.840	0.619	1.061	0.569	0.719	1.232	0.547	0.635	0.802	1.376
3.0	0.735	0.588	0.882	0.551	0.662	0.992	0.535	0.601	0.722	1.083
3.6	0.679	0.569	0.788	0.540	0.627	0.868	0.527	0.580	0.674	0.933
4.2	0.643	0.556	0.730	0.532	0.604	0.793	0.521	0.566	0.642	0.843

The numerical values reported in Tables 1-4 are validated by the following relation

$$\sum_{r=1}^{n}\mu_{r:n}=nE(X),\qquad \text{(Arnold et al. [14])}.$$

Table 3: Values of $\mu_{r:n}^{(3)}$.

α	$n=1$ $\mu_{1:1}^{(3)}$	$n=2$ $\mu_{1:2}^{(3)}$	$\mu_{2:2}^{(3)}$	$n=3$ $\mu_{1:3}^{(3)}$	$\mu_{2:3}^{(3)}$	$\mu_{3:3}^{(3)}$	$n=4$ $\mu_{1:4}^{(3)}$	$\mu_{2:4}^{(3)}$	$\mu_{3:4}^{(3)}$	$\mu_{4:4}^{(3)}$
0.6	-0.229	-1.372	0.915	2.058	-8.232	1 5.488	0.915	5.488	-21.952	14.635
1.2	-1.372	0.915	-3.659	0.588	1.568	-6.272	0.499	0.855	2.281	-9.123
1.8	2.058	0.588	3.528	0.475	0.814	4.885	0.433	0.599	1.028	6.170
2.4	0.915	0.499	1.330	0.433	0.630	1.681	0.407	0.514	0.747	1.992
3.0	0.686	0.457	0.915	0.412	0.549	1.098	0.392	0.470	0.627	1.254
3.6	0.588	0.433	0.743	0.398	0.503	0.863	0.383	0.445	0.562	0.963
4.2	0.534	0.418	0.650	0.389	0.474	0.737	0.377	0.428	0.521	0.809

Table 4: Values of $\mu_{r:n}^{(4)}$.

α	$n=1$ $\mu_{1:1}^{(4)}$	$n=2$ $\mu_{1:2}^{(4)}$	$\mu_{2:2}^{(4)}$	$n=3$ $\mu_{1:3}^{(4)}$	$\mu_{2:3}^{(4)}$	$\mu_{3:3}^{(4)}$	$n=4$ $\mu_{1:4}^{(4)}$	$\mu_{2:4}^{(4)}$	$\mu_{3:4}^{(4)}$	$\mu_{4:4}^{(4)}$
0.6	-0.103	-0.360	0.154	-2.161	3.241	-1.389	1.441	-12.965	19.448	-8.335
1.2	-0.36	1.441	-2.161	0.540	3.241	-4.862	0.412	0.926	5.557	-8.335
1.8	-2.161	0.540	-4.862	0.381	0.858	-7.722	0.332	0.528	1.188	-10.692
2.4	1.441	0.412	2.470	0.332	0.570	3.419	0.303	0.42	0.72	4.319
3.0	0.72	0.360	1.080	0.309	0.463	1.389	0.288	0.37	0.556	1.667
3.6	0.54	0.332	0.748	0.295	0.408	0.918	0.279	0.342	0.474	1.066
4.2	0.458	0.315	0.602	0.285	0.375	0.715	0.273	0.324	0.425	0.812

Based on the numerical results in Table 5, it is clear that the relationship between the variance and the α value is an inverse relationship for all r and n. The variability ordering between O.S. are verified for the sample size $n = 4$ by the following relation

$$Var(X_{i:n}) \leq Var(X_{j:n}) \quad \text{for} \quad 1 \leq i \leq j \leq n \quad \text{(David and Groeneveld [16]).}$$

Based on Table 6:

- When $r = n$, the skewness decreases with increasing α (an inverse relationship).
- When $r < n$, the skewness decreases with increasing α only when $\alpha < 4.2$ (an inverse relationship).
- The skewness is positive for all r and n when $\alpha > 1.2$.

Table 7 reports the leptokurtic and platykurtic kurtoses for all r and n.

Table 5: The variance.

α	$X_{1:1}$	$X_{1:2}$	$X_{2:2}$	$X_{1:3}$	$X_{2:3}$	$X_{3:3}$	$X_{1:4}$	$X_{2:4}$	$X_{3:4}$	$X_{4:4}$
0.6	NA	1.50000	NA	0.16315	3.85442	NA	0.0582	0.3911	6.9350	NA
1.2	1.5000	0.0582	2.7424	0.0177	0.1088	3.8798	0.0083	0.0336	0.1553	4.9537
1.8	0.1632	0.0177	0.2595	0.0063	0.0301	0.3349	0.0032	0.0112	0.0402	0.3988
2.4	0.0582	0.0083	0.0869	0.0032	0.0137	0.1071	0.0016	0.0054	0.0178	0.1233
3.0	0.0294	0.0049	0.0422	0.0019	0.0077	0.0508	0.0010	0.0033	0.0098	0.0573
3.6	0.0177	0.0032	0.0247	0.0012	0.0049	0.0292	0.0007	0.0021	0.0063	0.0327
4.2	0.0117	0.0022	0.0161	0.0009	0.0034	0.0189	0.0004	0.0015	0.0043	0.0209

Table 6: The skewness.

α	$X_{1:1}$	$X_{1:2}$	$X_{2:2}$	$X_{1:3}$	$X_{2:3}$	$X_{3:3}$	$X_{1:4}$	$X_{2:4}$	$X_{3:4}$	$X_{4:4}$
0.6	NA	NA	NA	10.2155	NA	NA	4.9204	9.0624	NA	NA
1.2	-4.6268	4.9204	NA	3.2636	4.1901	-4.2155	3.0318	2.6979	3.9328	NA
1.8	10.2155	3.2636	9.2695	2.3671	2.7466	8.9269	2.7208	2.0570	2.4869	8.7628
2.4	4.9204	3.0318	4.3499	2.7208	2.2499	4.1723	2.3219	2.1071	1.9748	4.0743
3.0	3.8103	2.3128	3.3521	1.4999	2.2550	3.1559	3.6217	1.3282	1.9670	3.0766
3.6	3.2636	2.7208	2.8797	2.5938	2.1081	2.7119	0.9045	1.6578	1.5117	2.5963
4.2	3.1420	3.0583	2.6535	5.8943	1.9145	2.4035	7.2964	2.9591	1.5911	2.3333

Table 7: The kurtosis .

α	$X_{1:1}$	$X_{1:2}$	$X_{2:2}$	$X_{1:3}$	$X_{2:3}$	$X_{3:3}$	$X_{1:4}$	$X_{2:4}$	$X_{3:4}$	$X_{4:4}$
0.6	-2.984	11.292	-2.989	-246.927	9.737	NA	92.298	-193.42	9.258	-2.996
1.2	11.292	92.297	9.795	26.214	69.617	9.316	22.314	19.067	62.291	9.082
1.8	-246.93	26.214	-207.24	14.265	19.572	-194.05	-45.123	13.216	17.476	-187.79
2.4	92.298	22.314	75.785	-45.123	14.596	70.553	-69.485	10.129	13.239	68.004
3.0	38.319	15.829	31.150	127.779	9.048	29.129	-193.371	-20.352	12.139	28.049
3.6	26.214	-45.123	21.653	249.789	12.950	19.959	436.840	-14.676	19.807	19.108
4.2	16.969	-44.364	18.678	-765.382	46.872	16.046	2373.51	-0.411	-6.663	16.154

3.2. Joint Probability Density Function of Order Statistics

On using binomial expansion, the joint PDF of $X_{r:n}$ and $X_{s:n}$ of SWPD is given by

$$f_{r,s:n}(x,y) = C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}\beta^{2\alpha(n-r+i+1)}\left(\frac{4\alpha^2}{xy}\right)\times \left(\frac{\beta}{x}\right)^{2\alpha(s+i-r-j)}\left(\frac{\beta}{y}\right)^{2\alpha(n+j-s+1)}, \quad 1\le r\le s\le n. \tag{14}$$

where

$$C_{r,s:n}=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

The sketches of the joint PDF for $r=1,2,3,\ s=2,3,4$ and $n=4$.

4. Recurrence Relations

Calculating the moments of O. S. is a difficult task for some probability models. Recursive computations are considered to meet these difficulties. Applications of recursive computation based on order statistics have been well demonstrated by notable authors see, Balakrishnan and Malik [17], Balakrishnan et al. [18], Ali and Khan [19], Kumar et al. [20, 21], Khan and Mustafa [22], Khan [23] and Akhtar at el. [24, 25] in detail.

Theorem 2. *As stated in Theorem 1, the relation for single moments as follows*

$$\mu_{r:n}^{(i)}=\frac{2\alpha(n-r+1)}{i}\left[\mu_{r:n}^{(i)}-\mu_{r-1:n}^{(i)}\right] \tag{15}$$

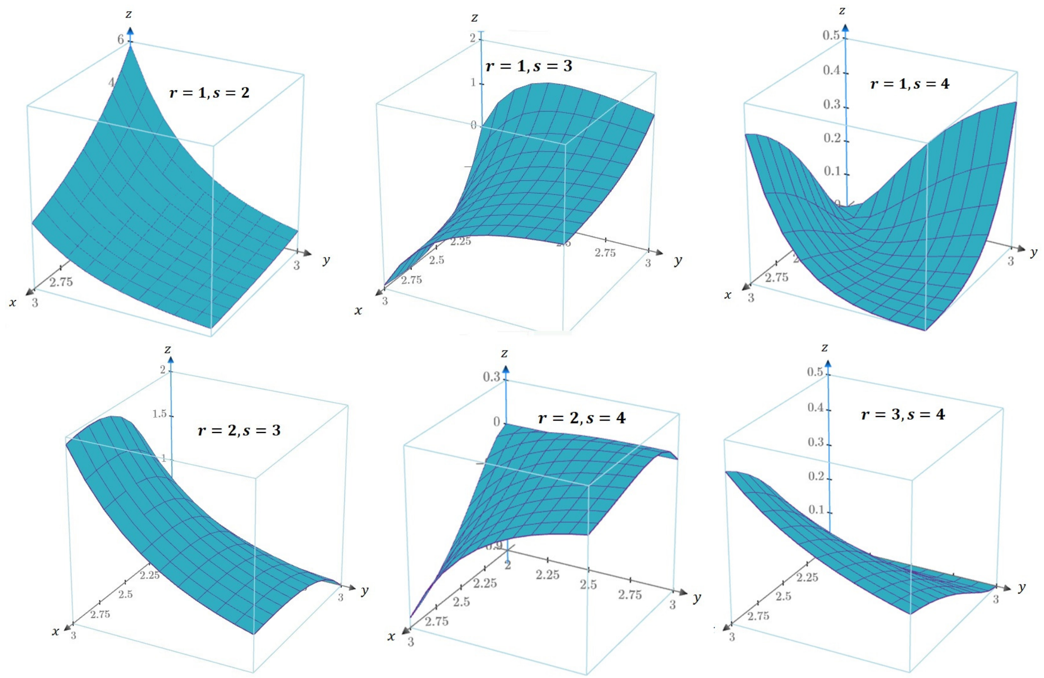

Figure 6: The joint PDF for $r = 1, 2, 3$, $s = 2, 3, 4$ and $n = 4$.

Proof. We know that

$$\begin{aligned} \mu_{r:n}^{(i)} &= \int_{-\infty}^{\infty} x^i f_r(x)dx \\ &= C_{r:n} \int_{\beta}^{\infty} x^i [F(x)]^{r-1}[1 - F(x)]^{n-r} f(x)dx \end{aligned} \tag{16}$$

Using Equation (5) in Equation (16), we have

$$\mu_{r:n}^{(i)} = C_{r:n} 2\alpha \int_{\beta}^{\infty} x^{i-1}[F(x)]^{r-1}[1 - F(x)]^{n-r+1}dx. \tag{17}$$

Integrating Equation (17) by parts and rephrase, the Equation (15) is proved.

5. Characterization

Characterization helps a researcher identify the actual probability distribution under certain conditions. For a comprehensive study, researchers referred to the contributions of Ahsanullah et al. [26, 27] and Hamdani [28]. The hazard rate function is used to analyze extreme values from a probability model of a distribution. An application also presents the behavior of the graph. We characterize the SWPD through HRF, and the conditions for characterization are described as follows.

Lemma 1. *A RV $T : \Omega \to (0, +\infty)$ has a continuous PDF $f(t)$ if and only if the HRF $h(t)$ satisfies the following equations*

$$\frac{f^{'}(t)}{f(t)} = \frac{h^{'}(t)}{h(t)} - h(t). \tag{18}$$

Proof. As stated by the definition of HRF, mentioned in Equation 7, it follows that

$$\begin{aligned} \frac{h^{'}(t)}{h(t)} &= \frac{f^{'}(t)F(t) + f^2(t)}{\overline{F}^2(t)} \times \left(\frac{\overline{F}(t)}{f(t)}\right) \\ &= \frac{f^{'}(t)}{f(t)} + h(t). \end{aligned}$$

Lemma 1 is proved.

Theorem 3. *A RV $T : \Omega \to (\beta, +\infty)$ has a SWPD (α, β) if and only if the HRF $h(t)$ satisfies the following equations*

$$\frac{h^{'}(t)}{(h(t))^2} = -\frac{1}{2\alpha}. \tag{19}$$

Proof. Necessary part:
The logarithm of $f_{\text{SWP}}(t)$ is noted as

$$\ln(f_{\text{SWP}}(t)) = 2\ln(\alpha) + 2\alpha\ln(\beta) - (2\alpha + 1)\ln(t) \tag{20}$$

Differentiate Equation (20) with respect to t, we obtain

$$\frac{f^{'}(t)}{f(t)} = -\frac{(2\alpha + 1)}{t} \tag{21}$$

Using Equations (7) and (18) the above equation follows that

$$\frac{h^{'}(t)}{h(t)} = \frac{f^{'}(t)}{f(t)} + h(t) = -\frac{(2\alpha + 1)}{t}$$

after simplification yields Equation (19).

Sufficiency part:
Suppose that Equation (19) holds. Then the following integration is considered.

$$\begin{aligned} \int \frac{h^{'}(t)}{(h(t))^2} dt &= -\int \frac{1}{2\alpha} dt \\ \frac{1}{h(x)} &= \frac{t}{2\alpha}. \end{aligned}$$

Which is the same as Equation (7).

Furthermore, by replacing the HRF in Equation (18) and employing integration, we get

$$\begin{aligned}\int \frac{f'(t)}{f(t)}dt &= -\int \frac{(2\alpha+1)}{t}dt + C\\ f(t) &= Ct^{-(2\alpha+1)}\end{aligned}$$

Using boundary condition, value of C is $2\alpha\beta^{2\alpha}$. Thus $f(t)$ is PDF from SWPD(α,β). This completes the proof.

5.1. Application

This subsection provides the application of characterization result. For this purpose, we consider the data set that includes given by Fan and Fan [29]. A random number is used to choose the various random time. For SWPD(α,β), the original dataset is divided by minimum value (0.33) to convert into the interval (1,α).

Table 8: The time to repair data of a piece of construction equipment.

0.33	0.33	0.50	0.50	0.50	0.95	1.00	1.02	1.17	1.72
1.83	3.20	4.35	5.25	6.52	7.25	8.58	10.25	11.58	13.83
15.93	27.40	27.43	31.93	38.37	40.02	62.77	88.27	92.90	

Table 9: Estimated hazard rates of SWPD(α,β) for different parameters .

		α								
r	t	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
4	1.52	1.32	1.97	2.63	3.30	3.95	4.60	5.26	5.92	6.58
8	3.09	0.65	0.97	1.29	1.62	1.942	2.27	2.59	2.91	3.24
13	13.18	0.15	0.23	0.303	0.379	0.455	0.53	0.61	0.68	0.76
21	48.27	0.04	0.062	0.082	0.104	0.124	0.15	0.17	0.19	0.21
27	190.21	0.01	0.015	0.021	0.206	0.032	0.04	0.04	0.05	0.05

Table 9 is endorsed the behavior of hazard rate of SWPD(α,β) as depicted in Figure 2.

6. Measure of Inequality

Since, SWPD(α,β) possess the heavy tail properties. Therefore, it becomes necessary to study its inequality behavior. Assume that the RV X is a non-negative with continuous and twice differentiable CDF.

If $X \sim$ SWPD(α,β), then the Lorenz curve $L(p)$, Bonferroni curve $B(p)$ and Gini index are defined as

$$L_{\text{SWPD}}(p) = \frac{1}{\mu}\int_0^y xf(x)dx, \quad 0<p<1$$

$$= \quad 1-(1-p)^{1-\frac{1}{2\alpha}}.$$

Bonferroni curve $B(p)$ is given as

$$B_{\text{SWPD}}(p) = \frac{1}{p\mu}\int_0^y xf(x)dx, \quad 0<p<1$$

Gini index:

$$\begin{aligned} G_{\text{SWPD}}(x) &= 1-\frac{1}{\mu}\int_0^\infty (1-F(x))^2 dx = \frac{1}{\mu}\int_0^\infty F(x)(1-F(x))dx \\ &= \frac{1}{4\alpha-1}, \quad 4\alpha-1>0. \end{aligned}$$

Table 10: Measures of inequality.

α		$p=0.10$	$p=0.25$	$p=0.50$	$p=0.75$	$p=0.95$	$G(x)$
1	$L(p)$	0.0513	0.1340	0.2980	0.5000	0.7764	0.33
	$B(p)$	0.513	0.5359	0.5856	0.6667	0.8172	
1.5	$L(p)$	0.0678	0.1745	0.3700	0.6031	0.8643	0.20
	$B(p)$	0.678	0.6980	0.7400	0.8041	0.9098	
2.0	$L(p)$	0.0759	0.1941	0.4054	0.6464	0.8943	0.14
	$B(p)$	0.759	0.7763	0.8108	0.8618	0.9413	
2.5	$L(p)$	0.0808	0.2056	0.4257	0.6701	0.9090	0.11
	$B(p)$	0.808	0.8223	0.8513	0.8935	0.9568	
3.0	$L(p)$	0.0840	0.2132	0.4388	0.6850	0.9176	0.09
	$B(p)$	0.840	0.8527	0.8775	0.9133	0.9659	
3.5	$L(p)$	0.0864	0.2185	0.4480	0.6952	0.9233	0.076
	$B(p)$	0.864	0.8741	0.8959	0.9269	0.9719	
4.0	$L(p)$	0.0881	0.2225	0.4548	0.7026	0.9273	0.066
	$B(p)$	0.881	0.8901	0.9095	0.9368	0.9761	
4.5	$L(p)$	0.0894	0.2256	0.4510	0.7084	0.9303	0.058
	$B(p)$	0.894	0.9025	0.9199	0.9445	0.9792	
5.0	$L(p)$	0.0990	0.2281	0.4641	0.7128	0.9325	0.053
	$B(p)$	0.990	0.9124	0.9282	0.9504	0.9816	

We notice from Table 10 both $L(p)$ and $B(p)$ are increasing as α increases but Gini coefficient is decreasing for the same value.

7. Conclusion

In this article, we study some features of order statistics for SWPD. We obtained the recurrence relation for single moments and calculated the expected values, variances, skewness, and kurtosis for different values of parameters. The model is characterized through the hazard rate function and supported with real data. The measure of inequality is also considered. The results contained in this paper will provide theoretical and practical insights into the fields of mathematical statistics, finance, actuarial sciences, risk analysis and medical sciences. The future work related to this research, one may generate new model using transmutation map and stress- strength of SWPD.

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References

[1] M. Nagy M. Ahmad R. Jabeen A. Zaka Adel F. Alrasheedi and A. H. Mansi. Evaluation of survival weighted pareto distribution: Analytical properties and applications to industrial and aeronautics data. *AIP Advances*, 14:045–140, 2024.

[2] B. C. Arnold. *Pareto Distribution*. International Cooperative Publishing House, 1983, Fairland, MD.

[3] N. M. Kilany. Weighted lomax distribution. *Springer Plus*, 5:1862, 2016.

[4] A. I. Al-Omari and I. K. Alsmairan. Length-biased suja distribution and its applications. *Journal of Applied Probability and Statistics*, 14(3):95–116, 2019.

[5] I. K. Alsmairan and A. I. Al-Omari. Weighted suja distribution with an application to ball bearings data. *Life Cycle Reliability and Safety Engineering*, 9:195–211, 2020.

[6] M. A. Haq R. M. Usman S. Hashmi and A. I. Al-Omeri. The marshall-olkin length-biased exponential distribution and its applications. *Journal of King Saud University-Science*, 31:246–251, 2019.

[7] A. Mustafa and M. I. Khan. The length-biased power hazard rate distribution with applications. *Statistics in Transition New Series*, 23(2):1–16, 2022.

[8] A. Mustafa and M. I. Khan. The length- biased powered inverse rayleigh distribution with applications. *Journal of Applied Mathematics and Informatics*, 40 (1-2):1–13, 2022.

[9] A. Mustafa and M. I. Khan. Length-biased extended rayleigh distribution and its applications. *Journal of Modern Applied Statistical Methods*, 22(2):doi.org/10.56801/Jmasm.V22. i2.6, 2023.

[10] M. I. Khan and A. Mustafa. Some properties of the weighted power hazard rate distribution with application. *Pakistan Journal of Statistics*, 38(2):219–234, 2022.

[11] M. I. Khan and A. Mustafa. Length- biased power rayleigh distribution: Properties and applications. *Journal of Applied Mathematics and Informatics*, 43(2):455–473, 2025.

[12] R. A. Fisher. The effects of methods of ascertainment upon the estimation of frequencies. *Annals of Eugenics*, 6:13–25, 1934.

[13] H. A. David. *Order Statistics, Second Edition* . John Willey and Sons, New York, 1981.

[14] B. C. Arnold N. Balakrishnan and H. N. Nagaraja. *A First Course in Order Statistics*. John Wiley, New York, 1992.

[15] H. A. David and H. N. Nagaraja. *Order Statistics, Third edition*. John Wiley, New York, 2003.

[16] H. A. David and R. A. Groeneveld. Measures of local variation in a distribution: expected length of spacings and variances of order statistics. *Biometrika*, 69:227–232, 1982.

[17] N. Balakrishnan and H. J. Malik. Order statistics from the linear- exponential distribution, part ii: Increasing hazard rate case. *Communications in Statistics – Theory and Methods*, 15:179–203, 1986.

[18] N. Balakrishnan H. J. Malik and S. E. Ahmed. Recurrence relations and identities for moments of order statistics, ii: Specific distributions. *Communications in Statistics—Theory and Methods*, 17:2657– 2694, 1988.

[19] M. A. Ali and A. H. Khan. Recurrence relations for the expected values of the certain function of two order statistics. *Metron*, LVI:107–119, 1998.

[20] D. Kumar S. Dey and S. Nadarajah. Extended exponential distribution based on order statistics. *Communications in Statistics- Theory and Methods*, 46:9166–9184, 2017.

[21] D. Kumar M. Kumar and J. P. S. Joorel. Estimation with modified power function distribution based on order statistics with application to evaporation data. *Annals of Data Science*, 9:723–748, 2022.

[22] M. I. Khan and A. Mustafa. Power hazard distribution through order statistics and its applications. *International Journal of Intelligent Systems and Applications in Engineering*, 12(4):600–608, 2024.

[23] M. I. Khan. Order statistics and actuarial measures from powered inverse rayleigh distribution. *European Journal of Pure and Applied Mathematics*, 17(3):2182–2195, 2024.

[24] Z. Akhter J. Saran and K. Verma. Moments of order statistics from length-biased exponential distribution and associated inference. *Annals of Data Science*, 9:1257–1282, 2022.

[25] Z. Akhter S. MirMostafaee A. Bakar and E. Ormoz. On the order statistics from the xlindley distribution and associated inference with an application to fatigue data. *arXiv preprint*, arXiv:2502.05659, 2025.

[26] M.E. Ghitany M. Ahsanullah and D.K. Al-Mutairi. Characterization of lindley distribution by truncated moments. *Communications in Statistics-Theory and Methods*, 46:6222–6227, 2017.

[27] M. Shakil M. Ahsanullah and B. M. G. Kibria. Characterizations of continuous distributions by truncated moment. *Journal of Modern Applied Statistical Methods*, 15:316–331, 2016.

[28] G. G. Hamedani. Characterizations of univariate continuous distributions based on truncated moments of functions of order statistics. *Studia Scientiarum Mathematicarum Hungarica*, 47:462–468, 2010.

[29] Q. Fan and H. Fan. Reliability analysis and failure prediction of construction equip-

ment with timeseries models. *Journal of Advanced Management Science*, 3(3):203–210, 2015.