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On Qualitative Properties of L_Θ-Solutions for Coupled Systems of Hadamard-Type Fractional Integral Equations in Banach Spaces

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Abstract. In this manuscript, the measure of noncompactness ($\mathcal{MNC}$), Darbo and Banach contraction fixed point theorems ($\mathcal{FPT}$), as well as fractional calculus, are used to carry out the analysis of the solvability of a general but abstract coupled system of quadratic Hadamard-fractional integral equations in Orlicz spaces L_Θ. Several qualitative properties of the solution to the studied coupled system are established, such as the existence, monotonicity, and uniqueness, in addition to continuous dependence on the data. We conclude with some examples that illustrate our hypothesis.

2020 Mathematics Subject Classifications: 47H30, 45G10, 47N20

Key Words and Phrases: Fixed-point theorem ($\mathcal{FPT}$), Orlicz spaces L_Θ, coupled system of integral equations, ($\mathcal{MNC}$) measure of noncompactness

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1. Introduction

Coupled systems of differential and integral equations are often used to formulate physical and biological models. The study of coupled systems of integral equations is of significant interest to numerous fields of science, such as multimedia processing [1], nuclear physics [2], diffusion equations [3], electromagnetics [4], and heat conduction [5].

The aim of the present paper is to analyze and demonstrate the solutions of the coupled system:

$$\begin{cases} x(t) = h_1(t) + f_1\left(t,\ \Lambda_1(y)(t),\ \frac{G_1(y)(t)}{\Gamma(\beta)} \cdot \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} \frac{R_1(y)(s)}{s}\, ds\right) \\ y(t) = h_2(t) + f_2\left(t,\ \Lambda_2(x)(t),\ \frac{G_2(x)(t)}{\Gamma(\beta)} \cdot \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} \frac{R_2(x)(s)}{s}\, ds\right), \quad t \in [1,e], \end{cases} \tag{1}$$

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where $0 < \beta < 1$, in Orlicz spaces L_Θ, and the operators $G_i,\ \Lambda_i,\ R_i, i = 1, 2$, operate on some arbitrary L_Θ.

We establish and present assumptions that allow us to solve and study the coupled system (1) under general growth conditions. As a result, we examine some qualitative properties of the problem (1), such as existence, monotonicity, and uniqueness, in addition to the continuous dependence on the data in the spaces L_Θ (cf. [6]).

Several authors examined various types of coupled systems of integral equations in the literature, including the space $C(J)$ (cf. [7–10]) and the Banach algebras (cf. [11, 12], for instance), where the outcomes have been made under conditions that are "continuous," i.e., stronger than the ones provided in this article. Additionally, polynomial growth was used on the studied functions to obtain L_p-solutions for the coupled systems in [13, 14]. As a result of eliminating these limitations, we extended these results to examine the coupled system (1) using the technique presented in [15] that is not a Banach algebra, using appropriately and various Orlicz spaces $(L_{\Theta_1}, L_{\Theta_2}, L_{\Theta_3})$, which are not a Banach algebra.

Using Orlicz spaces L_Θ as the solution space, we can study operators with strong nonlinear properties (such as exponential growth, for example). This allows us to examine the solutions in L_Θ rather than continuous results. Statistical physics and physics models may inspire this (cf. [16, 17]). Recalling the thermodynamics model

$$y(s) + \int_I A(s,t)e^{y(t)}\,dt = 0$$

contains exponential nonlinearity (cf. [18]). Furthermore, the quadratic integral equations (QIE) were studied in the Banach-Orlicz algebra [19] and in various Orlicz spaces in [15, 20] employing the approach of the fixed point theorems $(\mathcal{FPT})$ in conjunction with a suitable $(\mathcal{MNC})$ measure of noncompactness $(\mathcal{MNC})$ concerning different assumptions, see also [21, 22]. The measures of noncompactness $(\mathcal{MNC})$ have been employed in the study of numerous models of integral equations; (cf. [23–25]). These cases are unified and included as special cases of the coupled system (1). Let us recall that, in [26], two existence theorems of the coupled system

$$\begin{cases} x(t) = g_1(t) + f_1\Big(t,\ y(t),\ \lambda\cdot V_1y(t)\int_a^b \mathcal{K}\big(t,s\big)h_1(s,y(s))\,ds,\ \lambda\cdot G_1y(t)\int_a^b u_1\big(t,s,y(s)\big)\,ds\Big) \\ y(t) = g_2(t) + f_2\Big(t,\ x(t),\ \lambda\cdot V_2x(t)\int_a^b \mathcal{K}\big(t,s\big)h_2(s,x(s))\,ds,\ \lambda\cdot G_2x(t)\int_a^b u_2\big(t,s,x(s)\big)\,ds\Big) \end{cases}$$

have been studied in arbitrary L_Θ, in two separately cases Δ' and Δ_3-conditions using Darbo's$(\mathcal{FPT})$ with a $(\mathcal{MNC})$. The authors in [27] studied the existence, in addition to the uniqueness of monotonic solutions of the Hadamard fraction equations

$$x(t) = \prod_{i=1}^{n}\left(h_i(t)+G_{2_i}(x)(t)+\frac{G_{1_i}(x)(t)}{\Gamma(\alpha_i)}\cdot\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_i-1}\frac{G_{3_i}(x)(s)}{s}\,ds\right),\ \ t\in[1,e],\ 0<\alpha_i<1$$

in Orlicz spaces see also [28]. The current manuscript is motivated and induced by the extension and generalization of the results introduced in the previous literature to prove some qualitative properties of the solutions for an abstract but general coupled system of quadratic Hadamard-fractional integral equations (1), including existence, monotonicity, and uniqueness, in addition to continuous dependence on the data in L_Θ-spaces. We use the technique of $(\mathcal{MNC})$ concerning $(\mathcal{FPT})$ and the theory of fractional calculus to obtain the findings. We present a few constructed examples that support and illustrate our findings.

2. Preliminaries

Let $\mathbb{R}^+ = [0, \infty) \subset \mathbb{R} = (-\infty, \infty)$ and $J = [1, e]$, $e \approx 2.718$.

Definition 1. *[17] The function $\Theta(u) = \int_0^{|u|} p(t)\ dt$, defined on $\mathbb{R}^+$ is called a Young function (Y.F.) if:*

- *The function p is nondecreasing, right-continuous, positive, and defined on $\mathbb{R}^+$;*
- $\lim_{t\to\infty} \Theta(t) = \infty$ *and* $\Theta(0) = \lim_{t\to 0} \Theta(t) = 0$.

The complementary (Y.F.) function Θ^* of the function Θ is known as

$$\Theta^*(t) = \sup_{s\geq 0} \Big(ts - \Theta(s)\Big),\ \forall\ t \geq 0.$$

Furthermore, the function Θ is known as N-function if:

- $\lim_{t\to 0} \frac{\Theta(t)}{t} = 0$ *and* $\lim_{t\to\infty} \frac{\Theta(t)}{t} = \infty$;
- $\Theta(s) = 0 \Leftrightarrow s = 0$ *and* $\Theta(s) > 0$ *if* $s > 0$.

Definition 2. *[26] The space $\mathbb{L}_{\mathbb{X}} = L_\Theta(J) \times L_\Theta(J)$ is a Banach space under the norm*

$$\|(x, y)\|_{\mathbb{X}} = \|x\|_\Theta + \|y\|_\Theta,$$

where $x, y \in L_\Theta(J)$, and $L_\Theta = L_\Theta(J)$ is called the Orlicz space *of the functions f under the norm*

$$\|f\|_\Theta = \inf_{\epsilon>0} \left\{ \int_J \Theta\left(\frac{f}{\epsilon}\right)\ ds \leq 1 \right\}.$$

Let $\mathbb{E}_{\mathbb{X}} = E_\Theta(J) \times E_\Theta(J)$ be the closure in $\mathbb{L}_{\mathbb{X}}$, where $E_\Theta = E_\Theta(J)$ be the closure in $L_\Theta(J)$ such that

$$\lim_{\delta\to 0} \sup_{meas D<\delta} \sup_{f\in E_\Theta} \|f \cdot \chi_D\|_\Theta = 0,$$

where χ_D and "meas" are the characteristic function of a measurable subset $D \subset J$ and the Lebesgue measure, respectively. For multiplications of operators, we have:

Lemma 1. *([29, Theorem 10.2] Let Θ_1, Θ_2 and Θ be arbitrary N-functions. The following hypotheses are identical:*

(i) For every $u_1 \in L_{\Theta_1}$ and $u_2 \in L_{\Theta_2}$, $u_1 \cdot u_2 \in L_{\Theta}$.

(ii) $\exists\, k > 0$ such that for all measurable functions u_1, u_2, we obtain $\|u_1 u_2\|_{\Theta} \le k \|u_1\|_{\Theta_1} \|u_2\|_{\Theta_2}$.

(iii) $\exists\, l > 0$, $u_0 \ge 0$ s.t. $\forall\, t \ge u_0$, $\Theta\left(\frac{st}{l}\right) \le \Theta_1(s) + \Theta_2(t)$.

(iv) $\limsup_{t\to\infty} \frac{\Theta_1^{-1}(t)\Theta_2^{-1}(t)}{\Theta(t)} < \infty$.

Denote by $W = W(J)$ the set of Lebesgue measurable functions on the interval J. The functions are equal almost everywhere in the set W concerned with the metric

$$d(y, x) = \inf_{\rho>0} [\rho + meas\{s \ : \ |y(s) - x(s)| \ge \rho\}],$$

becoming a complete metric space. It should be noted that the convergence in measure on the interval J is the same as the convergence concerning the above metric d (cf. [30]).

Corollary 1. *[26] Assume that $U \subset \mathbb{L}_{\mathbb{X}}$ is a bounded set and the functions $x, y \in L_{\Theta}$ are almost everywhere. nondecreasing (or almost everywhere nonincreasing) functions on the interval J. Therefore, the pair $(x, y) = u \in U$ becomes almost everywhere nondecreasing (or almost everywhere nonincreasing) on the interval J, in addition to the set U being compact in measure in $\mathbb{L}_{\mathbb{X}}$.*

Definition 3. [31] *Assume that $U \subset \mathbb{L}_{\mathbb{X}}$ is a bounded set. The Hausdorff $\mathcal{MNC}$ $\beta_H(X)$ (cf. [31]) is known as*

$$\beta_H(U) = \inf\{r > 0 : \ \ \exists\, Y \subset L_{\mathbb{X}} \ \ s.t.\ U \subset \ Y \ + \ B_r \ \},$$

where $B_r = \{x \in L_{\mathbb{X}} : \|x\|_{\mathbb{X}} \le r\}$, $r > 0$.

Definition 4. *[26] Assume that, $\emptyset \ne U = (X_1, X_2) \subset \mathbb{L}_{\mathbb{X}}$, with $X_1, X_2 \subset L_{\Theta}$ are bounded sets and for $\epsilon \ > \ 0$, then*

$$\begin{aligned} c(U) &= c(X_1, X_2) = c(X_1) + c(X_2) \\ &= \limsup_{\varepsilon\to 0} \sup_{mesD\le\varepsilon} \sup_{x_1\in X_1} \|x_1 \cdot \chi_D\|_{\Theta} + \limsup_{\varepsilon\to 0} \sup_{mesD\le\varepsilon} \sup_{x_2\in X_2} \|x_2 \cdot \chi_D\|_{\Theta} \end{aligned}$$

is known as the measure of equiintegrability in $\mathbb{L}_{\mathbb{X}}$.

Corollary 2. *[26] For a compact in measure and bounded set $\emptyset \ne U \subset \ \mathbb{L}_{\mathbb{X}}$, we have*

$$c(U) = \beta_H(U).$$

Theorem 1. *[26] Assume that $\emptyset \ne \mathbb{C} \subset \mathbb{L}_{\mathbb{X}}$ is a closed, bounded, and convex in addition to the continuous map $T : \mathbb{C} \to \mathbb{C}$ verifying*

$$\beta_H(T(U)) \le k\ \beta_H(U), \quad 0 \le k < 1, \qquad\qquad \textit{(Contraction condition)}$$

for any $\emptyset \ne U \subset \mathbb{C}$. Then T has at least one fixed point in $\mathbb{C}$.

Proposition 1. *[32] Suppose that $\beta \in (0,1)$, $t \in \mathbb{R}^+$, and Θ is a Young function (YF), then we get:*

(a) For $\int_0^t \Theta(s^{-\beta})\, ds < \infty$. If $\beta_2 < \beta$, then the integral

$$\int_0^t \Theta(s^{-\beta_2})\, ds$$

is finite as well.

(b) The set

$$\mathbb{U}(t) = \left\{ k > 0 : \frac{1}{k^{\frac{1}{1-\beta}}} \int_0^{tk^{\frac{1}{1-\beta}}} \Theta(s^{\beta-1})\, ds \leq 1 \right\}$$

is increasing and continuous functions with $\mathbb{U}(0) = 0$.

Definition 5. *[33] The Hadamard type fractional integral of order $\beta > 0$ for a given integrable function y is known as*

$$\mathcal{K}^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} \frac{y(s)}{s}\, ds, \quad t > 1, \quad \beta > 0,$$

where $\Gamma(\beta) = \int_0^\infty e^{-\nu} \nu^{\beta-1}\, d\nu$.

Proposition 2. *[34] The operator $\mathcal{K}^\beta$ maps the a.e. nonnegative-nondecreasing functions into itself.*

Lemma 2. *[28] Suppose, that M^* and M are complementary N-functions and Θ is N-function with $\int_0^t M(s^{\beta-1})\, ds < \infty$, $0 < \beta < 1$. Moreover, put*

$$k(t) = \frac{1}{\epsilon^{\frac{1}{1-\beta}}} \int_0^{t\epsilon^{\frac{1}{1-\beta}}} M(s^{\beta-1})\, ds \in E_\Theta, \ s \in J, \ \epsilon > 0,$$

then the Hadamard operator $\mathcal{K}^\beta : L_{M^*} \to L_\Theta$ is continuous and verifying

$$\|\mathcal{K}^\beta x\|_\Theta \leq \frac{2}{\Gamma(\beta)} \|k\|_\Theta \|x\|_{M^*}.$$

3. Main results.

Next, we discuss the solvability of the coupled system (1) in L_Θ. Define the operator T as follows

$$T(x,y)(t) = \big(T_1 y(t),\ T_2 x(t)\big),\ t \in J,$$

where

$$T_1 y = h_1 + F_{f_1}\Big(\Lambda_1(y),\ U_1(y)\ \Big),\ \ T_2 x = h_2 + F_{f_2}\Big(\Lambda_2(x),\ U_2(x)\ \Big),$$

$$F_{f_i}\Big(\Lambda_i(w),\ U_i(w)\Big) = f_i\Big(t,\ \Lambda_2(w),\ U_i(w)\Big),\ U_i(w) = G_i(w)\cdot A_i(w),$$

and

$$A_i(w)(t) = \mathcal{K}^\beta R_i(w),$$

s.t. $\mathcal{K}^\beta$ is Hadamard operator 5 and $G_i, F_{f_i}, \Lambda_i, R_i$, are different operators operate on different Orlicz spaces $i = 1,2$. First, we inspect the existence of monotonic-L_Θ solutions for the coupled system (1).

Definition 6. *The ordered pair $u = (x, y) \in \mathbb{L}_\mathbb{X}$ s.t. $x, y \in L_\Theta$ is called a solution of the coupled system (1), if u verifies the coupled system (1).*

3.1. The existence of solutions.

Let $M,\ M^*$ be complementary N-functions and $\Theta, \Theta_1, \Theta_2$ be N-functions. Furthermore, put the assumptions for $i = 1,2$:

(G1) $\exists\ k_1 > 0$ s.t. for every $u_1 \in L_{\Theta_1}$ and $u_2 \in L_{\Theta_2}$ we have $\|u_1u_2\|_\Theta \le k_1\|u_1\|_{\Theta_1}\|u_2\|_{\Theta_2}$,

(G2) $h_i \in E_\Theta(J)$ are a.e. nondecreasing functions on the interval J,

(G3) $f_i(t,x,y) : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous in x and y for almost all t and measurable in $t \in J$. Furthermore, suppose that $t \to f_i(t,x,y)$ are nondecreasing-positive function and $\exists\ \alpha_1, \alpha_2 \ge 0$, and functions $c_i \in L_\Theta$ s.t.

$$|f_i(t,x,y)| \le c_i(t) + \alpha_1|x| + \alpha_2|y|. \tag{2}$$

(G4) The operators $\Lambda_i : E_\Theta \to E_\Theta$, $G_i : E_\Theta \to E_{\Theta_1}$, and $R_i : E_\Theta \to E_{M^*}$, and they are continuous. Moreover, let Λ_i, G_i, R_i take the set of all a.e. nondecreasing functions into itself and assume that for any $w \in E_\Theta$ we get $\Lambda_i(w) \in E_\Theta,\ G_i(w) \in E_{\Theta_1}$, and $R_i(w) \in E_{M^*}$.

(G5) There exist positive functions $a_i \in L_\Theta,\ g_i \in L_{\Theta_1},\ b_i \in L_{M^*}$ s.t. for $t \in J$,

$$|\Lambda_i(w)(t)| \le a_i(t)\|w\|_\Theta,\ |G_i(w)(t)| \le g_i(t)\|w\|_\Theta,\ |R_i(w)(t)| \le b_i(t)\|w\|_\Theta.$$

(G6) Assume that $k(t) = \frac{1}{\epsilon^{\frac{1}{1-\beta}}}\int_0^{t\epsilon^{\frac{1}{1-\beta}}} M(s^{\beta-1})\,ds \in E_{\Theta_2}$ for a.e. $s \in J$ and $\epsilon > 0$.

(G7) Let $\Big(\alpha_1\|a^*\| - 1\Big)^2 > \frac{8\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\Big(\|h_1\|_\Theta + \|h_2\|_\Theta + \|c_1\|_\Theta + \|c_2\|_\Theta\Big)$, and

$$\left(\alpha_1\|a^*\|_\Theta + \frac{2\cdot r\cdot\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\right) < 1,$$

where r is the positive solution of the equation

$$\|h_1\|_\Theta + \|h_2\|_\Theta + \|c_1\|_\Theta + \|c_2\|_\Theta - \Big(1 - \alpha_1\|a^*\|_\Theta\Big)\cdot r + \frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\cdot r^2 = 0$$

and $\|a^*\|_\Theta = \max\Big\{\|a_i\|_\Theta\Big\},\ \|b^*\|_{M^*} = \max\Big\{\|b_i\|_{M^*}\Big\}$ and $\|g^*\|_{\Theta_1} = \max\Big\{\|g_i\|_{\Theta_1}\Big\}$.

Theorem 2. *Let the assumptions (G1)-(G7) hold, then there exists a.e. nondecreasing-solution $u = (x, y) \in \mathbb{E}_\Theta$ of (1).*

Proof. **Step I.** In what follows, put $i = 1, 2$. Lemma 2 and assumption (G6) imply that the operator $\mathcal{K}^\beta : L_{M^*} \to L_{\Theta_2}$ is continuous and assumptions (G3) and (G4) indicate that $F_{f_i}, \Lambda_i : E_\Theta \to E_\Theta$, $G_i : E_\Theta \to E_{\Theta_1}$ and $R_i : E_\Theta \to E_{M^*}$. Then the operators $A_i = \mathcal{K}^\beta R_i : E_\Theta \to E_{\Theta_2}$ is continuous. By assumptions (G1) and (G4) the operators $U_i = G_i \cdot A_i : E_\Theta \to E_\Theta$ is continuous. Assumption (G2) gives that, the operators $T_i : E_\Theta \to E_\Theta$ are continuous. Therefore, $T = \big(T_1,\ T_2\big)$ acts from $\mathbb{E}_\mathbb{X}$ into itself and is continuous.

Step II. We shall prove that $T : B_r(\mathbb{E}_\mathbb{X}) \to \mathbb{E}_\mathbb{X}$ is continuous, where

$$B_r(\mathbb{E}_\mathbb{X}) = \{u = (x, y) \in \mathbb{L}_\mathbb{X} : x, y \in E_\Theta,\ \|u\|_\mathbb{X} \le r\}.$$

For arbitrary $u = (x, y) \in B_r(\mathbb{E}_\mathbb{X})$, $x, y \in E_\Theta$, and recalling Remark 2, we have

$$\begin{aligned}
\|T_1 y\|_\Theta &\le \|h_1\|_\Theta + \|f_1(t,\ \Lambda_1(y),\ U_1(y))\|_\Theta \\
&\le \|h_1\|_\Theta + \|c_1\|_\Theta + \alpha_1 \|\Lambda_1(y)\|_\Theta + \alpha_2 \|U_1 y\|_\Theta \\
&\le \|h_1\|_\Theta + \|c_1\|_\Theta + \alpha_1 \Big\| a_1 \cdot \|y\|_\Theta \Big\|_\Theta + \alpha_2 \|G_1(y) \cdot A_1(y)\|_\Theta \\
&\le \|h_1\|_\Theta + \|c_1\|_\Theta + \alpha_1 \|a_1\|_\Theta \cdot \|y\|_\Theta + \alpha_2 k_1 \|G_1(y)\|_{\Theta_1} \cdot \|A_1(y)\|_{\Theta_2} \\
&\le \|h_1\|_\Theta + \|c_1\|_\Theta + \alpha_1 \|a_1\|_\Theta \cdot \|y\|_\Theta + \alpha_2 k_1 \Big\| g_1 \cdot \|y\|_\Theta \Big\|_{\Theta_1} \Big\| \mathcal{K}^\beta R_1(y) \Big\|_{\Theta_2} \\
&\le \|h_1\|_\Theta + \|c_1\|_\Theta + \alpha_1 \|a_1\|_\Theta \cdot \|y\|_\Theta + \alpha_2 k_1 \|g_1\|_{\Theta_1} \cdot \|y\|_\Theta \frac{2\|k\|_{\Theta_2}}{\Gamma(\beta)} \|R_1(y)\|_{M^*} \\
&\le \|h_1\|_\Theta + \|c_1\|_\Theta + \alpha_1 \|a_1\|_\Theta \cdot \|y\|_\Theta + \frac{2\alpha_2 k_1 \|k\|_{\Theta_2}}{\Gamma(\beta)} \|g_1\|_{\Theta_1} \cdot \|y\|_\Theta \cdot \|b_1\|_{M^*} \|y\|_\Theta \\
&= \|h_1\|_\Theta + \|c_1\|_\Theta + \alpha_1 \|a_1\|_\Theta \cdot \|y\|_\Theta + \frac{2\alpha_2 k_1 \|k\|_{\Theta_2}}{\Gamma(\beta)} \|g_1\|_{\Theta_1} \|b_1\|_{M^*} \cdot \|y\|^2_\Theta.
\end{aligned}$$

Similarly, for $x \in E_\Theta$, we have

$$\|T_2 x\|_\Theta \le \|h_2\|_\Theta + \|c_2\|_\Theta + \alpha_1 \|a_2\|_\Theta \cdot \|x\|_\Theta + \frac{2\alpha_2 k_1 \|k\|_{\Theta_2}}{\Gamma(\beta)} \|g_2\|_{\Theta_1} \|b_2\|_{M^*} \cdot \|x\|^2_\Theta.$$

Then for $u \in \mathbb{E}_\mathbb{X}$, we have

$$\begin{aligned}
\|Tu\|_\mathbb{X} &= \|T_1 y\|_\Theta + \|T_2 x\|_\Theta \\
&\le \|h_1\|_\Theta + \|h_2\|_\Theta + \|c_1\|_\Theta + \|c_2\|_\Theta + \alpha_1 \|a_1\|_\Theta \cdot \|y\|_\Theta + \frac{2\alpha_2 k_1 \|k\|_{\Theta_2}}{\Gamma(\beta)} \|g_1\|_{\Theta_1} \|b_1\|_{M^*} \cdot \|y\|^2_\Theta \\
&\quad + \alpha_1 \|a_2\|_\Theta \cdot \|x\|_\Theta + \frac{2\alpha_2 k_1 \|k\|_{\Theta_2}}{\Gamma(\beta)} \|g_2\|_{\Theta_1} \|b_2\|_{M^*} \cdot \|x\|^2_\Theta \\
&\le \|h_1\|_\Theta + \|h_2\|_\Theta + \|c_1\|_\Theta + \|c_2\|_\Theta + \alpha_1 \|a^*\|_\Theta \big(\|x\|_\Theta + \|y\|_\Theta\big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\big(\|x\|_\Theta+\|y\|_\Theta\big)^2\\
\le\ & \|h_1\|_\Theta+\|h_2\|_\Theta+\|c_1\|_\Theta+\ \|c_2\|_\Theta+\alpha_1\|a^*\|_\Theta\cdot\|u\|_{\mathbb{X}}+\frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\cdot\|u\|^2_{\mathbb{X}}\\
\le\ & \|h_1\|_\Theta+\|h_2\|_\Theta+\|c_1\|_\Theta+\ \|c_2\|_\Theta+\alpha_1\|a^*\|_\Theta\cdot r+\frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\cdot r^2\le r,
\end{aligned}
$$

where $\|a^*\|_\Theta=\max\Big\{\|a_i\|_\Theta\Big\}, \|b^*\|_{M^*}=\max\Big\{\|b_i\|_{M^*}\Big\}$ and $\|g^*\|_{\Theta_1}=\max\Big\{\|g_i\|_{\Theta_1}\Big\},\ i=1,2$. Recalling assumption (G7), we have $T: B_r(\mathbb{E}_{\mathbb{X}})\to\mathbb{E}_{\mathbb{X}}$ is continuous.

Step III. Let $\mathbb{Q}_r\subset B_r(\mathbb{E}_{\mathbb{X}})$ include all monotonic (a.e. nondecreasing) functions on the interval J. Then $\emptyset\neq\mathbb{Q}_r$ is closed, convex, and bounded, in $\mathbb{E}_{\mathbb{X}}$ in addition to be compact in measure regarding Corollary 1.

Step IV. The operator T keeps the monotonicity property for the functions. For $i=1,2$, let us choose $u=(x,y)\in\mathbb{Q}_r$, where x and y are nondecreasing on J. Proposition 2 implies that the operator $\mathcal{K}^\beta$ takes the a.e. nonnegative-nondecreasing functions into itself. Therefore, the operators $A_i=\mathcal{K}^\beta R_i$ and Λ_i, and $U_i=G_i\cdot A_i$ are a.e. nondecreasing on the interval J (by using (G4)). Assumptions (G2) and (G3) grant us that the operators T_1,T_2 are a.e. nondecreasing on J. Those grant us that $T=(T_1,T_2):\mathbb{Q}_r\to\mathbb{Q}_r$ is continuous.

Step V. We need to show that $\beta_H(TX)\le k\beta_H(X),\ k\in[0,1)$. For any $u=(x,y)\in\mathbb{U}\subset\mathbb{Q}_r$ and a set $D\subset J$, with meas$D\le\varepsilon,\ \varepsilon>0$. By assumption (G5), we have

$$
\|\Lambda_i(z)\cdot\chi_D\|_\Theta\le\|\Lambda_i(x\cdot\chi_D)\|_\Theta\le\Big\|a_1\cdot\|x\cdot\chi_D\|_\Theta\Big\|_\Theta\le\|a_i\|_\Theta\|x\cdot\chi_D\|_\Theta
$$

and similarly

$$
\|G_i(z)\cdot\chi_D\|_\Theta\le\big\|g_i\big\|_\Theta\|x\cdot\chi_D\|_\Theta.
$$

Therefore, we have

$$
\begin{aligned}
&\|T_1(y)\cdot\chi_D\|_\Theta\le\|h_1\cdot\chi_D\|_\Theta+\Big\|F_{f_1}\big(\Lambda_1(y),\ U_1(y)\big)\cdot\chi_D\Big\|_\Theta\\
\le\ & \|h_1\cdot\chi_D\|_\Theta+\|c_1\cdot\chi_D\|_\Theta+\ \alpha_1\Big\|\Lambda_1(y)\cdot\chi_D\Big\|_\Theta+\alpha_2\|G_1(y)\cdot A_1(y)\cdot\chi_D\|_\Theta\\
\le\ & \|h_1\cdot\chi_D\|_\Theta+\|c_1\cdot\chi_D\|_\Theta+\ \alpha_1\|a_1\|_\Theta\cdot\|y\cdot\chi_D\|_\Theta+\alpha_2 k_1\|G_1(y)\cdot\chi_D\|_{\Theta_1}\cdot\|A_1(y)\|_{\Theta_2}\\
\le\ & \|h_1\cdot\chi_D\|_\Theta+\|c_1\cdot\chi_D\|_\Theta+\ \alpha_1\|a_1\|_\Theta\cdot\|y\cdot\chi_D\|_\Theta+\alpha_2 k_1\|g_1\|_{\Theta_1}\cdot\|y\cdot\chi_D\|_\Theta\frac{2\|k\|_{\Theta_2}}{\Gamma(\beta)}\|b_1\|_{M^*}\|y\|_\Theta\\
\le\ & \|h_1\cdot\chi_D\|_\Theta+\|c_1\cdot\chi_D\|_\Theta+\ \alpha_1\|a_1\|_\Theta\cdot\|y\cdot\chi_D\|_\Theta+\frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g_1\|_{\Theta_1}\cdot\|y\cdot\chi_D\|_\Theta\|b_1\|_{M^*}\cdot r.
\end{aligned}
$$

Similarly, we have

$$
\|T_2x\cdot\chi_D\|_\Theta\le\|h_2\cdot\chi_D\|_\Theta+\|c_2\cdot\chi_D\|_\Theta+\ \alpha_1\|a_2\|_\Theta\cdot\|x\cdot\chi_D\|_\Theta+\frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g_2\|_{\Theta_1}\cdot\|x\cdot\chi_D\|_\Theta\|b_1\|_{M^*}\cdot r.
$$

Then

$$\begin{aligned}
\|Tu\cdot\chi_D\|_{\mathbb{X}} &= \|T_1y\cdot\chi_D\|_\Theta+\|T_2x\cdot\chi_D\|_\Theta\\
&\le \|h_1\cdot\chi_D\|_\Theta+\|h_2\cdot\chi_D\|_\Theta+\|c_1\cdot\chi_D\|_\Theta+\ \|c_2\cdot\chi_D\|_\Theta+\alpha_1\|a^*\|_\Theta\cdot\big(\|x\cdot\chi_D\|_\Theta+\|y\cdot\chi_D\|_\Theta\big)\\
&\quad+\frac{2\cdot r\alpha_2k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\cdot\big(\|x\cdot\chi_D\|_\Theta+\|y\cdot\chi_D\|_\Theta\big)\\
&\le \|h_1\cdot\chi_D\|_\Theta+\|h_2\cdot\chi_D\|_\Theta+\|c_1\cdot\chi_D\|_\Theta+\ \|c_2\cdot\chi_D\|_\Theta+\alpha_1\|a^*\|_\Theta\cdot\|u\cdot\chi_D\|_{\mathbb{X}}\\
&\quad+\frac{2\cdot r\alpha_2k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\cdot\|u\cdot\chi_D\|_{\mathbb{X}}.
\end{aligned}$$

Since $h_i, c_i \in E_\Theta,\ i=1,2$, we get

$$\lim_{\varepsilon\to 0}\left\{\sup_{mes\ D\le\varepsilon}\left[\sup_{u\in X}\left\{\|h_i\chi_D\|_\Theta+\|c_i\chi_D\|_\Theta=0\right\}\right]\right\}.$$

Recalling Definition 4, we obtain

$$c(T(\mathbb{U}))\le\left(\alpha_1\|a^*\|_\Theta+\frac{2\cdot r\alpha_2k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\right)\cdot c(\mathbb{U}).$$

Since $\emptyset\neq\mathbb{U}\subset\mathbb{Q}_r$ is bounded in addition to compact in measure, then we shall apply Corollary 2 to obtain

$$\beta_H(T(\mathbb{U}))\le\left(\alpha_1\|a^*\|_\Theta+\frac{2\cdot r\alpha_2k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\right)\cdot \beta_H(\mathbb{U}).$$

Since $\left(\alpha_1\|a^*\|_\Theta+\frac{2\cdot r\alpha_2k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\right)<\ 1$, we get our verification and Theorem 1 achieves our proof.

3.2. Uniqueness of the solution.

Next, we demonstrate that the coupled system (1) has exactly one solution.

Theorem 3. *Assume that the assumptions of Theorem 2 hold with replacing the inequality (2) with the following*

$$|f_i(t,0,0)|\le c_i(t),\quad |f_i(t,x,y)-f_i(t,\bar{x},\bar{y})|\le\alpha_1|x-\bar{x}|+\alpha_2|y-\bar{y}|,\ u=(x,y),\bar{u}=(\bar{x},\bar{y})\in\mathbb{Q}_r, \tag{3}$$

for $i=1,2$, *and in addition, assume that*

$$C=\left(\alpha_1\|a^*\|_\Theta+\frac{4r\cdot\alpha_2k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\right)<1, \tag{4}$$

where $r,\mathbb{Q}_r$ *are defined in Theorem 2. Then the coupled system (1) has a unique solution* $u\in L_{\mathbb{X}}$ *in* $\mathbb{Q}_r$.

Proof. Using the inequalities (3) for $i = 1, 2$, we obtain

$$\begin{aligned}\Big||f_i(t,x,y)| - |f_i(t,0,0)|\Big| &\leq |f_i(t,x,y) - f_i(t,0,0)| \leq \alpha_1|x| + \alpha_2|y| \\ \Rightarrow \ |f_i(t,x,y)| &\leq |f_i(t,0,0)| + \alpha_1|x| + \alpha_2|y| \leq c_i(t) + \alpha_1|x| + \alpha_2|y|.\end{aligned}$$

Thus, Theorem 2 indicates that, there is a.e. nondecreasing solution $u \in \mathbb{E}_{\mathbb{X}}$ of (1) in $\mathbb{Q}_r$. Now, let $u = (x, y), \bar{u} = (\bar{x}, \bar{y}) \in \mathbb{Q}_r$ be any two distinct solutions of the coupled system (1), then we have

$$\begin{aligned}\|x-\bar{x}\|_\Theta &\leq \Big\|f_1\Big(t, \Lambda_1(y), U_1(y)\Big) - f_1\Big(t, \Lambda_1(\bar{y}), U_1(\bar{y})\Big)\Big\|_\Theta \\ &\leq \alpha_1\|\Lambda_1(y) - \Lambda_1(\bar{y})\|_\Theta + \alpha_2\|U_1(y) - U_1(\bar{y})\|_\Theta \\ &\leq \alpha_1\Big\|a_1\|y\|_\Theta - a_1\|\bar{y}\|_\Theta\Big\|_\Theta + \alpha_2\|G_1(y)A_1(y) - G_1(\bar{y})A_1(\bar{y})\|_\Theta \\ &\leq \alpha_1\|a_1\|_\Theta\big|\|y\|_\Theta - \|\bar{y}\|_\Theta\big| + \alpha_2\big\|G_1(y)A_1(y) - G_1(\bar{y})A_1(y)\big\|_\Theta + \alpha_2\big\|G_1(\bar{y})A_1(y) - G_1(\bar{y})A_1(\bar{y})\big\|_\Theta \\ &\leq \alpha_1\|a_1\|_\Theta\big\|y-\bar{y}\big\|_\Theta + \alpha_2 k_1\big\|G_1(y) - G_1(\bar{y})\big\|_{\Theta_1}\|A_1(y)\|_{\Theta_2} + \alpha_2 k_1\big\|G_1(\bar{y})\|_{\Theta_1}\big\|A_1(y) - A_1(\bar{y})\big\|_{\Theta_2} \\ &\leq \alpha_1\|a_1\|_\Theta\big\|y-\bar{y}\big\|_\Theta \\ &\quad + \alpha_2 k_1\big\|g_1\big\|_{\Theta_1}\|y-\bar{y}\|_\Theta\frac{2\|k\|_{\Theta_2}}{\Gamma(\beta)}\|b_1\|_{M^*}\|y\|_\Theta + \alpha_2 k_1\big\|g_1\big\|_{\Theta_1}\|y\|_\Theta\frac{2\|k\|_{\Theta_2}}{\Gamma(\beta)}\big\|R_1(y) - R_1(\bar{y})\big\|_{M^*} \\ &\leq \alpha_1\|a_1\|_\Theta\big\|y-\bar{y}\big\|_\Theta \\ &\quad + \frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\big\|g_1\big\|_{\Theta_1}\|b_1\|_{M^*}\|y\|_\Theta\|y-\bar{y}\|_\Theta + \frac{2\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\big\|g_1\big\|_{\Theta_1}\|y\|_\Theta\|b_1\|_{M^*}\big\|y-\bar{y}\big\|_\Theta \\ &= \left(\alpha_1\|a_1\|_\Theta + \frac{4r\cdot\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g_1\|_{\Theta_1}\|b_1\|_{M^*}\right)\big\|y-\bar{y}\big\|_\Theta.\end{aligned}$$

$$\text{Similarly,}\quad \|y-\bar{y}\|_\Theta \leq \left(\alpha_1\|a_2\|_\Theta + \frac{4r\cdot\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\big\|g_2\big\|_{\Theta_1}\|b_2\|_{M^*}\right)\big\|x-\bar{x}\big\|_\Theta.$$

Therefore,

$$\begin{aligned}&\|u-\bar{u}\|_{\mathbb{X}} = \big\|(x-\bar{x},\ y-\bar{y})\big\|_{\mathbb{X}} = \|x-\bar{x}\|_\Theta + \|y-\bar{y}\|_\Theta \\ \leq\ &\left(\alpha_1\|a_1\|_\Theta + \frac{4r\cdot\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g_1\|_{\Theta_1}\|b_1\|_{M^*}\right)\big\|y-\bar{y}\big\|_\Theta \\ &+\left(\alpha_1\|a_2\|_\Theta + \frac{4r\cdot\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\big\|g_2\big\|_{\Theta_1}\|b_2\|_{M^*}\right)\big\|x-\bar{x}\big\|_\Theta \\ \leq\ &\left(\alpha_1\|a^*\|_\Theta + \frac{4r\cdot\alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\right)\left(\big\|y-\bar{y}\big\|_\Theta + \big\|x-\bar{x}\big\|_\Theta\right) \\ =\ &C\cdot\|u-\bar{u}\|_{\mathbb{X}}.\end{aligned}$$

Equation (4) grants us that $u = \bar{u}$ (a.e.), and we get our verification.

3.3. Continuous dependence on the functions h_1, and h_2.

Next, we may discuss the continuous dependence of the obtained solutions for the coupled system (1) on the functions $h_i, i = 1, 2$.

Definition 7. *A solution $u = (x, y) \in \mathbb{L}_{\mathbb{X}}$ of (1) is continuously dependent on the function h_1, h_2 if $\forall\ \epsilon > 0,\ \exists \delta > o$ such that $\|h_1 - \bar{h_1}\|_\Theta + \|h_2 - \bar{h_2}\|_\Theta \le \delta$ implies that $\|u - \bar{u}\|_\Theta \le \epsilon$, where*

$$\begin{cases} \bar{x}(t) = \bar{h_1}(t) + \ f_1\Bigg(\Lambda_1(\bar{y})(t) + \ \frac{G_1(\bar{y})(t)}{\Gamma(\beta)} \cdot \int_1^t \Bigg(\log \frac{t}{s}\Bigg)^{\beta-1} \frac{R_1(\bar{y})(s)}{s}\, ds\Bigg) \\ \bar{y}(t) = \bar{h_2}(t) + f_2\Bigg(\Lambda_2(\bar{x})(t) + \ \frac{G_2(\bar{x})(t)}{\Gamma(\beta)} \cdot \int_1^t \Bigg(\log \frac{t}{s}\Bigg)^{\beta-1} \frac{R_2(\bar{x})(s)}{s}\, ds\Bigg),\ t \in [1, e]. \end{cases} \tag{5}$$

Theorem 4. *Assume that the assumptions of Theorem 3 hold. Then the solutions $u \in L_{\mathbb{X}}$ of the system (1) depend continuously on the functions h_1, h_2.*

Proof. Let $u, \bar{u}$ be any two different solutions of (1), then similarly as done in Theorem 3, we have

$$\begin{aligned} \|u - \bar{u}\|_{\mathbb{X}} \le \|h_1 - \bar{h_1}\|_\Theta + \|h_2 - \bar{h_2}\|_\Theta \\ + \Bigg(\alpha_1\|a^*\|_\Theta + \frac{4r \cdot \alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\Bigg)\Bigg(\Big\|y - \bar{y}\Big\|_\Theta + \Big\|x - \bar{x}\Big\|_\Theta\Bigg) \\ = \|h_1 - \bar{h_1}\|_\Theta + \|h_2 - \bar{h_2}\|_\Theta + \Bigg(\alpha_1\|a^*\|_\Theta + \frac{4r \cdot \alpha_2 k_1\|k\|_{\Theta_2}}{\Gamma(\beta)}\|g^*\|_{\Theta_1}\|b^*\|_{M^*}\Bigg)\|u - \bar{u}\|_{\mathbb{X}} \\ \le \|h_1 - \bar{h_1}\|_\Theta + \|h_2 - \bar{h_2}\|_\Theta + C\|u - \bar{u}\|_{\mathbb{X}}, \end{aligned}$$

where C is given by (4). Then, we get

$$\|u - \bar{u}\|_{\mathbb{X}} \le \Big(1 - C\Big)^{-1}\Big(\|h_1 - \bar{h_1}\|_\Theta + \|h_2 - \bar{h_2}\|_\Theta\Big).$$

Therefore, if $\|h_1 - \bar{h_1}\|_\Theta + \|h_2 - \bar{h_2}\|_\Theta + \le \delta(\epsilon)$, then $\|u - \bar{u}\|_\Theta \le \epsilon$, where

$$\delta(\epsilon) = \epsilon \cdot (1 - C).$$

4. Remarks and Example

We would like to conclude with some significant remarks and examples that highlight the applicability of the results we have found.

Remark 1. *The neutron transport [35], the traffic theory [36], the kinetic theory of gases [37], and astrophysics [38] are more efficient utilization of the quadratic integral equation through Hadamard fractional operators.*

Remark 2. *We can determine the acting and continuation assumptions for the operators of the form $G_i(w) = l_i(t) \cdot w(t)$, $l_i \in L_\Theta$, over several Orlicz spaces in (cf. [17] and assumption (G3)).*

Example 1. *Select the N-functions $M(s) = M^*(s) = s^2$ and $\Theta_2(s) = \exp|s| - |s| - 1$. We need to show that, the operator $\mathcal{K}^\beta : L_{M^*} \to L_{\Theta_2}$ is continuous and the outcomes of Lemma 2 is verified.*
Indeed: Let $t \in [1, e]$ and for any $\beta \in (0, 1)$, we get

$$k(t) = \int_0^t M\big(u^{\beta-1}\big)\, du = \int_0^t u^{2\beta-2}\, du = \frac{t^{2\beta-1}}{2\beta - 1}.$$

That gives us the verification of Proposition 1. Furthermore,

$$\int_1^e \Theta_2\big(k(t)\big)\, ds = \int_1^e \left(e^{\frac{t^{2\beta-1}}{2\beta-1}} - \frac{t^{2\beta-1}}{2\beta-1} - 1\right) dt,$$

which is finite. Then for $x \in L_{M^*}$, we have $J^\beta : L_{M^*} \to L_{\Theta_2}$ is continuous.

For additional details and many instances of the N-functions M, M^*, and Θ_2 that satisfy Lemma 2, refer to [17, Theorem 15.4].

Example 2. *For $i = 1, 2$, let $\beta = \frac{1}{2}$, $\Lambda_i(z) = a_i(t) \cdot z(t)$, $G_i(z) = g_i(t) \cdot z(t)$, and $R_i(z) = a_i(t) \cdot z(t)$, where $h_i \in L_\Theta, a_i \in L_\Theta, g_i \in L_{\Theta_1}$, and $b_i \in L_{M^*}$, then the coupled system*

$$\begin{cases} x(t) = h_1(t) + f_1\Bigg(t,\ a_1(t) \cdot y(t),\ \frac{g_1(t)\cdot y(t)}{\Gamma(\frac{1}{2})} \int_0^t \sqrt{\log \frac{t}{s}} \frac{b_1(t)\cdot y(t)}{s}\, ds\Bigg) \\ y(t) = h_2(t) + f_2\Bigg(t,\ a_2(t) \cdot x(t),\ \frac{g_1(t)\cdot x(t)}{\Gamma(\frac{1}{2})} \int_0^t \sqrt{\log \frac{t}{s}} \frac{b_2(t)\cdot x(t)}{s}\, ds\Bigg), \end{cases}$$

have a solution $u = (x, y) \in \mathbb{L}_{\mathbb{X}}$, where $t \in J$.

5. Conclusion

There have been several qualitative properties developed in this paper, consisting of existence, monotonicity, and uniqueness, in addition to the continuous dependence of the data, which are all indications for an abstract and general coupled system of quadratic Hadamard-fractional integral equations. To perform our analysis, we used the $(\mathcal{MNC})$ measure of noncompactness, as well as the $(\mathcal{FPT})$ fixed-point theorem and the fractional calculus in the Orlicz spaces L_Θ. Finally, we concluded with a few remarks as well as a few examples that illustrate and support our hypothesis. Future research will concentrate on the qualitative properties of the solutions for numerous fractional problems in distinct function spaces, such as Lebesgue spaces or Orlicz spaces. Furthermore, we shall check the numerical results for the issues considered.

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Author contributions

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