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Numerical Investigation Based on the Chebyshev-HPM for Breast Cancer as a Mathematical Model

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Abstract. We present the approximate solution for mathematical model for Breast Cancer (BC) over three time intervals. The suggested approach depends on the homotopy perturbation method developed with Chebyshev series (CHPM). The residual error function is calculated and used as a basic criterion in evaluating the efficiency/accuracy of the presented numerical scheme. We utilize the solution by RK4 method for comparison with the results of the method used. Through these results, we can confirm that the applied technique is an effective tool to give a simulation of such models. Illustrative instances are given to confirm the validity and usefulness of the proposed procedure.

2020 Mathematics Subject Classifications: 34A08, 65N06

Key Words and Phrases: Breast cancer, homotopy perturbation method, Chebyshev expansion, RK4 method

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1. Introduction

Breast cancer is characterized as a pathological disorder arising from the unregulated growth of cells in breast tissue. According to extensive statistics from the WHO regarding the worldwide burden of cancer, BC exhibits the highest spread rate relative to other types of cancer [1]. According to studies done by the WHO, BC was classified as the second most widespread cancer in 2004, posing a significant threat to women since it affects around 8-9 percent of the global female population [2]. Notwithstanding extensive research and numerous inquiries, the precise etiology of BC remains ambiguous ([2], [3]). BC is a widespread malignancy among women post-puberty, with its prevalence escalating with advancing age [2]. In light of the factors above, we underscore the necessity for a thorough comprehension of the epidemiology of BC and its implications for women's health,

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since this information is critical for formulating efficacious precautionary and therapeutic strategies globally [4].

Mathematical modeling is crucial for comprehending and investigating cancer tumors broadly, with a specific emphasis on breast cancer in this research paper. It serves to characterize and simulate tumor growth and behavior, as well as their interactions with adjacent tissues and the immune system ([5], [6]). Researchers and doctors can use these models to better understand how tumors grow, predict how treatments will work, and come up with better ways to use treatments ([7]-[9]). For a more concentrated examination of such systems, including their applications and properties, see ([10]-[13]).

Due to the problems of accuracy and convergence faced by most numerical methods, a large number of researchers have used the semi-analytical method (SAM) to investigate such problems and gain deeper insights into their complexities. The advantage of the SAM as the preferred method for finding analytical approximations to complex problems contain highly nonlinear terms ([14], [15]) lies in the challenges associated with obtaining exact solutions using traditional analytical techniques. Among these methods, which have attracted the attention of many scientists and have found application in solving a wide range of complex problems, is the HPM in particular. An improved version of this method called the new HPM, was presented in 2010 in the research paper [16] and was then applied to obtain approximate solutions to the quadratic RDE. This approach has become a popular choice in many studies ([17], [18]) due to its simplicity and ease of mathematical calculation. This distinction and simplicity can be achieved by defining the first approximation as a power series and then setting all iterations to zero except for the initial iteration. That is, the power series takes a central place in the method, which in turn facilitates obtaining approximate solutions to nonlinear equations in the form of Taylor expansion. Based on what was mentioned above about the importance of using power series, whether Taylor expansion or others, and in being more effective in improving the accuracy and convergence of analytical approximation solutions, the Chebyshev series, which depends on orthogonal Chebyshev polynomials, were used to benefit from the properties of these functions in a good approximation of functions, as well as their distinction in the faster convergence rate than other functions such as Taylor series, for example, but not limited to [19].

Due to these important properties of handling nonlinear equations and providing very accurate approximations, the transformed Chebyshev series has been widely applied in many academic studies, we mention, for example, but not limited to point equations of motion in their linear/nonlinear form [20], Cauchy problems which are represented by second-order ordinary differential equations [21], nonlinear integro-differential equations of fractional order [22], space-variable approximation of the Burger equation [23], the initial and BVPs associated with the fractional heat equation [24].

The present research focuses on developing semi-analytical methods to overcome the challenges inherent in these methods, taking advantage of the pivotal role played by the Chebyshev series, as shown in the historical context mentioned above, as well as addressing the time-consuming nature of numerical methods, mitigating these challenges.

The major objective of this manuscript is to apply the CHPM by combining the effective Chebyshev series (CS) with the new homotopy perturbation approach, to use a new, accurate, and efficient analytical approximation approach used for the first time to address the present problem. This new method was compared with some existing methods to confirm its efficiency and high accuracy, and perfect agreement was obtained.

The paper is organized as follows: In Section 2, we give the formulation of the proposed model. Section 3 presents the procedure solution by giving the basic concepts of the Chebyshev-HPM. Section 4 gives the numerical implementation of the proposed technique. Section 5 introduces the numerical simulation of the proposed model. Finally, Section 5 gives the conclusions and remarks.

2. Mathematical model of BC

The mathematical modeling of biological events, such as BC, is a key tool for understanding how tumors change during treatment and for dealing with epidemiological issues. This is evident from the studies conducted before ([25]-[27]).

We investigate this specific model of the BC ([28], [29]):

$$\begin{aligned}
&\dot{\psi}_1(t) = \psi_1(t)\lambda_1\left(p_1 - \beta_1\psi_1(t) - \alpha_1\,\psi_2(t)\right) - (1-p)\theta_1\,\psi_1(t)\psi_4(t), \qquad \psi_1(0) = \hat{\psi}_1^0,\\
&\dot{\psi}_2(t) = \psi_2(t)\lambda_2(p_2 d - \beta_2\psi_2(t) - \alpha_2\psi_3(t)) - \kappa\psi_2(t) + (1-p)\theta_1\psi_1(t)\psi_2(t)\psi_4(t), \quad \psi_2(0) = \hat{\psi}_2^0,\\
&\dot{\psi}_3(t) = \sigma\rho + \psi_3(t)\lambda_1(p_3 - \beta_3 - \alpha_3\psi_2(t)) - (1-p)\theta_2\psi_3(t)\psi_4(t), \qquad \psi_3(0) = \hat{\psi}_3^0,\\
&\dot{\psi}_4(t) = \alpha_4\psi_4(t) + \varrho(1-p), \qquad \psi_4(0) = \hat{\psi}_4^0.
\end{aligned} \tag{1}$$

This section will explain what the parameters ($\in \mathbb{R}^+$) of the model (1) mean in more detail ([26], [30]). The stability and convergence analysis of the model in question are thoroughly detailed in [29].

3. Basic concepts of the CHPM

The novel approach mainly depends on the use of the Chebyshev series in the new HPM. Here, we mention some basic assuming of the Chebyshev series, the new homotopy perturbation algorithm.

3.1. Chebyshev series

If we have a continuous function $q(t)$ in $[a,b]$, then it can be rewritten in terms of the CS of the first kind as follows ([31]-[33]):

$$q(t) = {\sum_{k=0}^{\infty}}{}' \, c_k T_k\left(\frac{2t-b-a}{b-a}\right), \tag{2}$$

where $'$ sign indicates that the coefficient of $T_0(t)$ must be reduced by half, $T_k(t) = \cos(k\cos^{-1}(t))$ where

$$c_k = \frac{2}{\pi}\int_{-1}^{1}(1-t^2)^{-0.5}\, q(0.5(b+a+(b-a)t))T_k(t)dt.$$

By the following recurrence relation, we can obtain Chebyshev polynomials of the first kind given on $[-1,1]$:

$$T_s(t) = 2t\,T_{s-1}(t) - T_{s-2}(t), \qquad s = 2,3,..., \qquad T_0(t) = 1, \quad T_1(t) = t. \tag{3}$$

Furthermore, these functions can be expressed analytically using the finite sum of powers of t as follows:

$$T_k(t) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i\, 2^{k-2i-1}\frac{k}{k-i}\begin{pmatrix} k-i \\ i \end{pmatrix} t^{k-2i}. \tag{4}$$

If the domain of the problem under study is $[0,1]$, we need to derive the so-called Chebyshev transformed polynomials $\mathbb{T}_k(t)$ of the first kind, using an appropriate linear transformation via the recurrence relation (3), as follows:

$$\mathbb{T}_k(t) = 2(2t-1)\,\mathbb{T}_{k-1}(t) - \mathbb{T}_{k-2}(t), \qquad k = 2,3,..., \qquad \mathbb{T}_0(t) = 1, \quad \mathbb{T}_1(t) = 2t-1. \tag{5}$$

3.2. The new HPM

To illustrate the process of this technique, we suppose a general form of the nonlinear ODEs as follows ([17], [18]):

$$L[\Upsilon(z)] + R[\Upsilon(z)] + N[\Upsilon(z)] + s(z) = 0, \qquad z \in [a,b], \tag{6}$$

N is a nonlinear term, and $s(z)$ is a given source term. By harnessing the principle of homotopy, we get:

$$H(\Upsilon,\ell) = L[\Upsilon(z)] - \Upsilon^*(z) + \ell\,\Upsilon^*(z) + \ell(R[\Upsilon(z)] + N[\Upsilon(z)] + s(z)) = 0, \tag{7}$$

where $\Upsilon^*(z)$ is an initial solution to the proposed problem & $0 \le \ell \le 1$ is an embedding parameter. From the principle of homotopy (7), we can find the following:

$$L[\Upsilon(z)] = \Upsilon^*(z) - \ell\,\Upsilon^*(z) - \ell(R[\Upsilon(z)] + N[\Upsilon(z)] + s(z)), \tag{8}$$

$$\Upsilon(z) = L^{-1}[\Upsilon^*(z)] - \ell\,L^{-1}[\Upsilon^*(z)] - \ell\,L^{-1}[R[\Upsilon(z)] + N[\Upsilon(z)] + s(z)]. \tag{9}$$

Where:

$$\Upsilon(z) = \sum_{k=0}^{\infty} \ell^k\, u_k(z), \qquad \Upsilon^*(z) = \sum_{k=0}^{\infty} c_k\, v_k(z),$$

then, we obtain:

$$\sum_{k=0}^{\infty} \ell^k\, u_k(z) = L^{-1}\left[\sum_{k=0}^{\infty} c_k\, v_k(z)\right] - \ell\, L^{-1}\left[\sum_{k=0}^{\infty} c_k\, v_k(z)\right] - \ell\, L^{-1}\left[R\Big[\sum_{k=0}^{\infty} \ell^k\, u_k(z)\Big] + N\Big[\sum_{k=0}^{\infty} \ell^k\, u_k(z)\Big] + s(z)\right]. \tag{10}$$

Equating the coefficients of ℓ^k, $k = 0, 1, 2, ...$ on both sides, we obtain:

$$\begin{aligned}
\ell^0 : &\quad u_0(z) = L^{-1}\Big[\sum_{k=0}^{\infty} c_k\, v_k(z)\Big],\\
\ell^1 : &\quad u_1(z) = -L^{-1}\Big[\sum_{k=0}^{\infty} c_k\, v_k(z)\Big] - L^{-1}[R(u_0) + N(u_0) + s(z)],\\
\ell^2 : &\quad u_2(z) = -L^{-1}[R(u_0, u_1) + N(u_0, u_1)], \cdots,\\
\ell^j : &\quad u_j(z) = -L^{-1}[R(u_0, u_1, \cdots, u_{j-1}) + N(u_0, u_1, \cdots, u_{j-1})], \qquad j = 3, 4, \cdots.
\end{aligned} \tag{11}$$

The exact solution to the equation under study will be found by finding the values of the unknown coefficient c_k, assuming that $u_1 = 0$, which will take the following form:

$$\Upsilon(z) = u_0(z) = L^{-1}\left[\sum_{k=0}^{\infty} c_k\, v_k(z)\right]. \tag{12}$$

3.3. The CHPM algorithm

To clarify the novel method's methodology, we shall outline the following stages for its application:
Step 1: We use the same principle of homotopy defined in (8).
Step 2: Take the L^{-1} operator to Eq.(8), and we rearrange it to get $\Upsilon(z)$ as defined in (9).
Step 3: Assume that

$$\Upsilon(z) = \sum_{k=0}^{\infty} \ell^k\, u_k(z), \qquad \Upsilon^*(z) = \sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(z),$$

then, we get

$$\sum_{k=0}^{\infty} \ell^k\, u_k(z) = L^{-1}\left[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(z)\right] - \ell\, L^{-1}\left[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(z)\right] - \ell\, L^{-1}\left[R\Big[\sum_{k=0}^{\infty} \ell^k\, u_k(z)\Big] + N\Big[\sum_{k=0}^{\infty} \ell^k\, u_k(z)\Big] + s(z)\right]. \tag{13}$$

Step 4: Comparing the coefficients of ℓ^k, $k = 0, 1, 2, ..$ on both sides of (13), leads to find:

$$\begin{aligned}
\ell^0 : &\quad u_0(z) = L^{-1}\Big[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(z)\Big],\\
\ell^1 : &\quad u_1(z) = -L^{-1}\Big[\sum_{k=0}^{\infty} c_k\, \mathbb{T}_k(z)\Big] - L^{-1}[R(u_0) + N(u_0) + s(z)],\\
\ell^2 : &\quad u_2(z) = -L^{-1}[R(u_0, u_1) + N(u_0, u_1)], \cdots,\\
\ell^j : &\quad u_j(z) = -L^{-1}[R(u_0, u_1, \cdots, u_{j-1}) + N(u_0, u_1, \cdots, u_{j-1})], \qquad j = 3, 4, \cdots.
\end{aligned} \tag{14}$$

Step 5: We assume that $u_1 = 0$, and find the values of c_k, $k = 0, 1, 2, .., m$ by equating the Chebyshev polynomials and solving the resulting system of equations. Then, the analytical solution can be formulated as follows:

$$\Upsilon_m(z) = u_0(z) = L^{-1}\left[\sum_{k=0}^{m} c_k\, \mathbb{T}_k(z)\right]. \tag{15}$$

4. Numerical implementation

The CHPM technique is used to solve the current problem (1). The primary phases of the new technique are given below:
Step 1: Applying the homotopy property on Eq.(1), we find:

$$\begin{aligned}
&\dot{\psi}_1(t) + \ell\psi_1^*(t) - \psi_1^*(t) - \ell\left[\psi_1(t)\lambda_1\left(p_1 - \beta_1\psi_1(t) - \alpha_1\,\psi_2(t)\right) - (1-p)\theta_1\,\psi_1(t)\psi_4(t)\right] = 0,\\
&\dot{\psi}_2(t) + \ell\psi_2^*(t) - \psi_2^*(t) - \ell\big[\psi_2(t)\lambda_2(p_2 d - \beta_2\psi_2(t) - \alpha_2\psi_3(t))\\
&\qquad - \kappa\psi_2(t) + (1-p)\theta_1\psi_1(t)\psi_2(t)\psi_4(t)\big] = 0,\\
&\dot{\psi}_3(t) + \ell\psi_3^*(t) - \psi_3^*(t) - \ell\left[\sigma\rho + \psi_3(t)\lambda_1(p_3 - \beta_3 - \alpha_3\psi_2(t)) - (1-p)\theta_2\psi_3(t)\psi_4(t)\right] = 0,\\
&\dot{\psi}_4(t) + \ell\psi_4^*(t) - \psi_4^*(t) - \ell\left[\alpha_4\psi_4(t) + \varrho(1-p)\right] = 0.
\end{aligned} \tag{16}$$

Step 2: Taking $L^{-1} = \int_0^t (.)dt$, for both sides of (16) yields:

$$\begin{aligned}
\psi_1(t) &= \psi_1(0) - \ell\, L^{-1}(\psi_1^*(t))\\
&\qquad + L^{-1}(\psi_1^*(t)) + \ell\, L^{-1}\left[\psi_1(t)\lambda_1\left(p_1 - \beta_1\psi_1(t) - \alpha_1\,\psi_2(t)\right) - (1-p)\theta_1\,\psi_1(t)\psi_4(t)\right],\\
\psi_2(t) &= \psi_2(0) - \ell\, L^{-1}(\psi_2^*(t)) + L^{-1}(\psi_2^*(t))\\
&\qquad + \ell\, L^{-1}\Big[\psi_2(t)\lambda_2(p_2 d - \beta_2\psi_2(t) - \alpha_2\psi_3(t)) - \kappa\psi_2(t) + (1-p)\theta_1\psi_1(t)\psi_2(t)\psi_4(t)\Big],\\
\psi_3(t) &= \psi_3(0) - \ell\, L^{-1}(\psi_3^*(t)) + L^{-1}(\psi_3^*(t))\\
&\qquad + \ell\, L^{-1}\left[\sigma\rho + \psi_3(t)\lambda_1(p_3 - \beta_3 - \alpha_3\psi_2(t)) - (1-p)\theta_2\psi_3(t)\psi_4(t)\right],\\
\psi_4(t) &= \psi_4(0) - \ell\, L^{-1}(\psi_4^*(t)) + L^{-1}(\psi_4^*(t)) + \ell\, L^{-1}\left[\alpha_4\psi_4(t) + \varrho(1-p)\right].
\end{aligned} \tag{17}$$

Step 3: Assuming that

$$\psi_j(t) = \sum_{k=0}^{\infty} \ell^k \, \psi_{j,k}(t), \qquad \psi_j^*(t) = \sum_{k=0}^{\infty} c_{j,k} \, \mathbb{T}_k(t), \qquad j = 1, 2, 3, 4,$$

then, we get:

$$\begin{aligned} \sum_{k=0}^{\infty} \ell^k \, \psi_{1,k}(t) = \psi_1(0) - \ell \, L^{-1}\Big(\sum_{k=0}^{\infty} c_{1,k} \, \mathbb{T}_k(t)\Big) + L^{-1}\Big(\sum_{k=0}^{\infty} c_{1,k} \, \mathbb{T}_k(t)\Big) \\ + \ell \, L^{-1}\Big[\lambda_1 \left(p_1 - \beta_1 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{1,k}(t)\right) - \alpha_1 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{2,k}(t)\right)\right) \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{1,k}(t)\right) \\ - (1-p)\theta_1 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{1,k}(t)\right) \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{4,k}(t)\right)\Big], \end{aligned} \tag{18}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \ell^k \, \psi_{2,k}(t) = \psi_2(0) - \ell \, L^{-1}\Big(\sum_{k=0}^{\infty} c_{2,k} \, \mathbb{T}_k(t)\Big) + L^{-1}\Big(\sum_{k=0}^{\infty} c_{2,k} \, \mathbb{T}_k(t)\Big) \\ + \ell \, L^{-1}\Big[\lambda_2 \left(p_2 d - \beta_2 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{2,k}(t)\right) - \alpha_2 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{3,k}(t)\right)\right) \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{2,k}(t)\right) \\ - \kappa \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{2,k}(t)\right) + (1-p)\theta_1 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{1,k}(t)\right) \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{2,k}(t)\right) \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{4,k}(t)\right)\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \ell^k \, \psi_{3,k}(t) = \psi_3(0) - \ell \, L^{-1}\Big(\sum_{k=0}^{\infty} c_{3,k} \, \mathbb{T}_k(t)\Big) + L^{-1}\Big(\sum_{k=0}^{\infty} c_{3,k} \, \mathbb{T}_k(t)\Big) \\ + \ell \, L^{-1}\Big[\sigma\rho + \lambda_1 \left(p_3 - \beta_3 - \alpha_3 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{2,k}(t)\right)\right) \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{3,k}(t)\right) \\ - (1-p)\theta_2 \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{3,k}(t)\right) \left(\sum_{k=0}^{\infty} \ell^k \, \psi_{4,k}(t)\right)\Big], \end{aligned} \tag{20}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \ell^k \, \psi_{4,k}(t) = \psi_4(0) - \ell \, L^{-1}\Big(\sum_{k=0}^{\infty} c_{4,k} \, \mathbb{T}_k(t)\Big) + L^{-1}\Big(\sum_{k=0}^{\infty} c_{4,k} \, \mathbb{T}_k(t)\Big) \\ + \ell \, L^{-1} \left[\alpha_4 \sum_{k=0}^{\infty} \ell^k \, \psi_{4,k}(t) + \varrho(1-p)\right]. \end{aligned} \tag{21}$$

Step 4: Equaling the terms for the equations (18)-(21) which have the same powers of ℓ:

$$\ell^0 : \; \psi_{j,0}(t) = \psi_j(0) + L^{-1} \left(\sum_{k=0}^{\infty} c_{j,k} \, \mathbb{T}_k(t)\right),$$

$$
\begin{aligned}
\ell^1 : \psi_{1,1}(t) &= -L^{-1}\Big(\sum_{k=0}^{\infty} c_{1,k}\,\mathbb{T}_k(t)\Big) \\
&\quad + L^{-1}\left[\psi_{1,0}(t)\lambda_1\left(p_1 - \beta_1\psi_{1,0}(t) - \alpha_1\,\psi_{2,0}(t)\right) - (1-p)\theta_1\,\psi_{1,0}(t)\psi_{4,0}(t)\right], \\
\psi_{2,1}(t) &= -L^{-1}\Big(\sum_{k=0}^{\infty} c_{2,k}\,\mathbb{T}_k(t)\Big) \\
&\quad + L^{-1}\Big[\psi_{2,0}(t)\lambda_2(p_2 d - \beta_2\psi_{2,0}(t) - \alpha_2\psi_{3,0}(t)) - \kappa\psi_{2,0}(t) + (1-p)\theta_1\psi_{1,0}(t)\psi_{2,0}(t)\psi_{4,0}(t)\Big], \\
\psi_{3,1}(t) &= -L^{-1}\Big(\sum_{k=0}^{\infty} c_{3,k}\,\mathbb{T}_k(t)\Big) \\
&\quad + L^{-1}\left[\sigma\rho + \psi_{3,0}(t)\lambda_1(p_3 - \beta_3 - \alpha_3\psi_{2,0}(t)) - (1-p)\theta_2\psi_{3,0}(t)\psi_{4,0}(t)\right], \\
\psi_{4,1}(t) &= -L^{-1}\Big(\sum_{k=0}^{\infty} c_{4,k}\,\mathbb{T}_k(t)\Big) + L^{-1}\left[\alpha_4\psi_{4,0}(t) + \varrho(1-p)\right],
\end{aligned} \tag{22}
$$

and so on.
Step 5: We find the values $c_{j,k}$, $j = 1,2,3,4$ by assuming that $\psi_{j,1}(t) = 0$. Therefore, the analytical approximate solution of $\psi_j(t)$ becomes as follows:

$$\psi_{j,m}(t) = \psi_{j,0}(t) = \psi_j(0) + L^{-1}\Big(\sum_{k=0}^{m} c_{j,k}\,\mathbb{T}_k(t)\Big), \qquad j = 1,2,3,4. \tag{23}$$

It should be noted that in step 5, the relations (Derivation, Multiplication, Integration) that were mentioned earlier are applied, also, in step 5, the values of $c_{j,k}$, $k = 0,1,2,...,m$; $j = 1,2,3,4$ are found by solving a system of algebraic equations that is obtained depending on $\mathbb{T}_k(t)$ coefficients.

5. Numerical simulation

To see how accurate the numerical scheme is, we run simulations on specific models in the range $[0,3]$ for the problem under study (1). The behavior of $\psi_k(t)$ for $k = 1,2,3,4$ is illustrated in Figures 1-5 at various parameter values p, m, ϱ.

We examine the current model (1) with the subsequent parameter values [34]:

$$\theta_1 = 0.2,\quad \theta_2 = 0.02,\quad \lambda_1 = 0.3,\quad \lambda_2 = 0.4,\quad d = 0.5,\quad \alpha_1 = 6\times10^{-8},\quad \alpha_2 = 3\times10^{-7},$$

$$\alpha_3 = 1\times10^{-7},\quad \alpha_4 = 0.97,\quad \rho = 0.01,\quad \kappa = 2,\quad \beta_1 = 0.15,\quad \beta_2 = 0.7,\quad \beta_3 = 0.1,\quad p = 0.5,$$

$$\sigma = 0.1,\quad p_1 = 0.1,\quad p_2 = 0.2,\quad p_3 = 0.3,\quad \varrho = 0.8.$$

The I.Cs are $\psi_i(0) = 0.2$ for $i = 1,2,3,4$. The numerical simulation of the examined model utilizing the specified methodology is illustrated in Figures 1-5.

1. Figure 1 presents the numerical solution for several quantities of the approximation order $m = 5, 10, 15$.

2. Figure 2 illustrates the numerical solution for various values of $\varrho = 0.8, 1.4, 2.0, 2.6$.

3. Figure 3 illustrates the influence of $p = 0.25, 0.5, 0.75, 1.0$ on the numerical solution.

4. Figure 4 shows a comparison between the proposed method's solution and the numerical solution found by the RK4 method [35], where the initial conditions and parameters were kept the same.

5. Figure 5 illustrates the residual error function (REF) [36] of the derived approximation solution.

The conduct of the numerical solution is contingent upon m, ϱ, p, as seen in Figures 1-3. Figures 2 and 3 indicate that the solution's behavior aligns with the inherent influence of the parameters ϱ and p. Figures 4 and 5 indicate that the proposed strategy has been effectively utilized to address the problem under investigation. As a result, we can confirm that the disease behaved as expected. This means that we obtained an accurate simulation of the system that the relevant authorities can use to treat this deadly cancer.

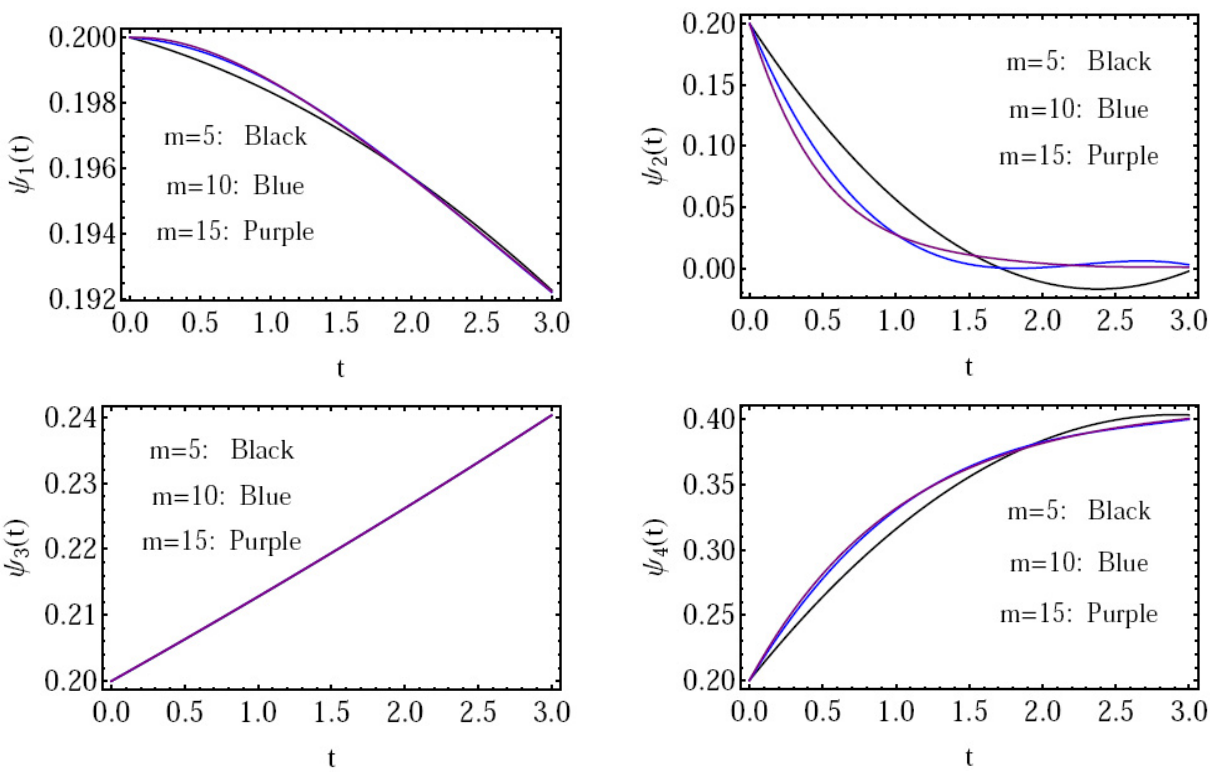

Figure 1. The approximate solution via various values of m.

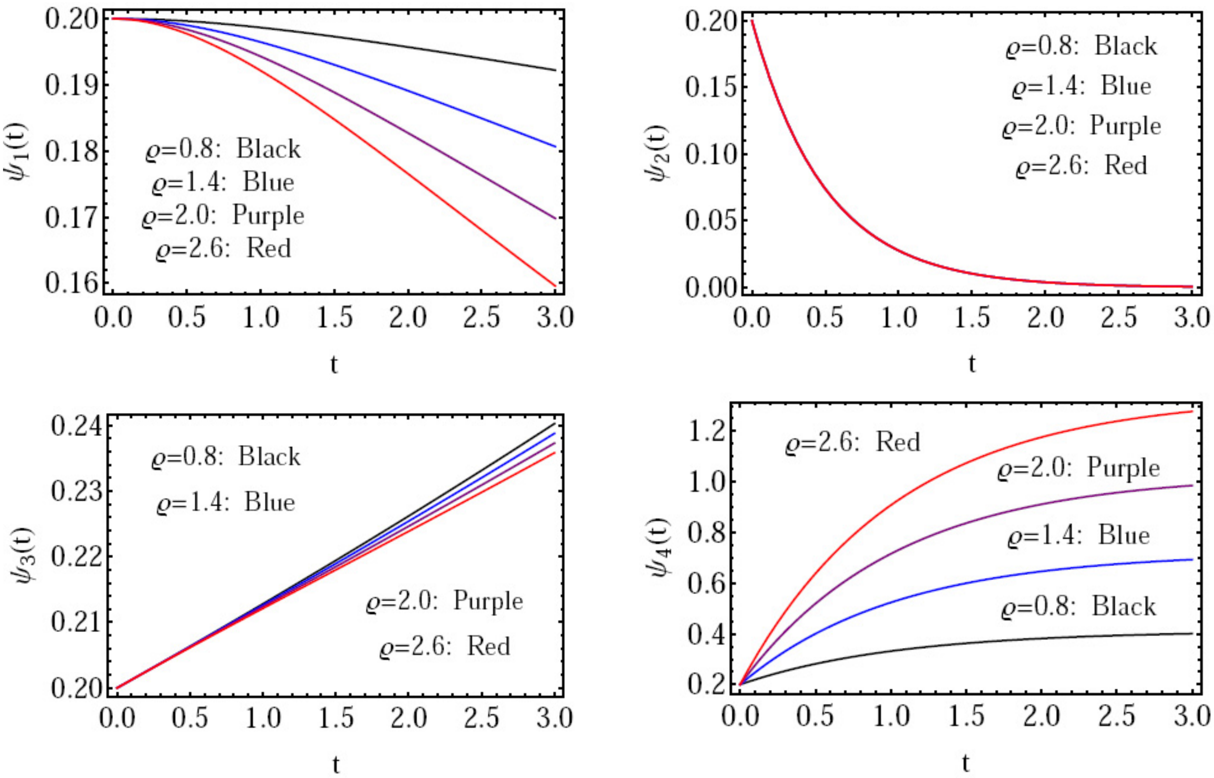

Figure 2. The approximate solution via various values of ϱ.

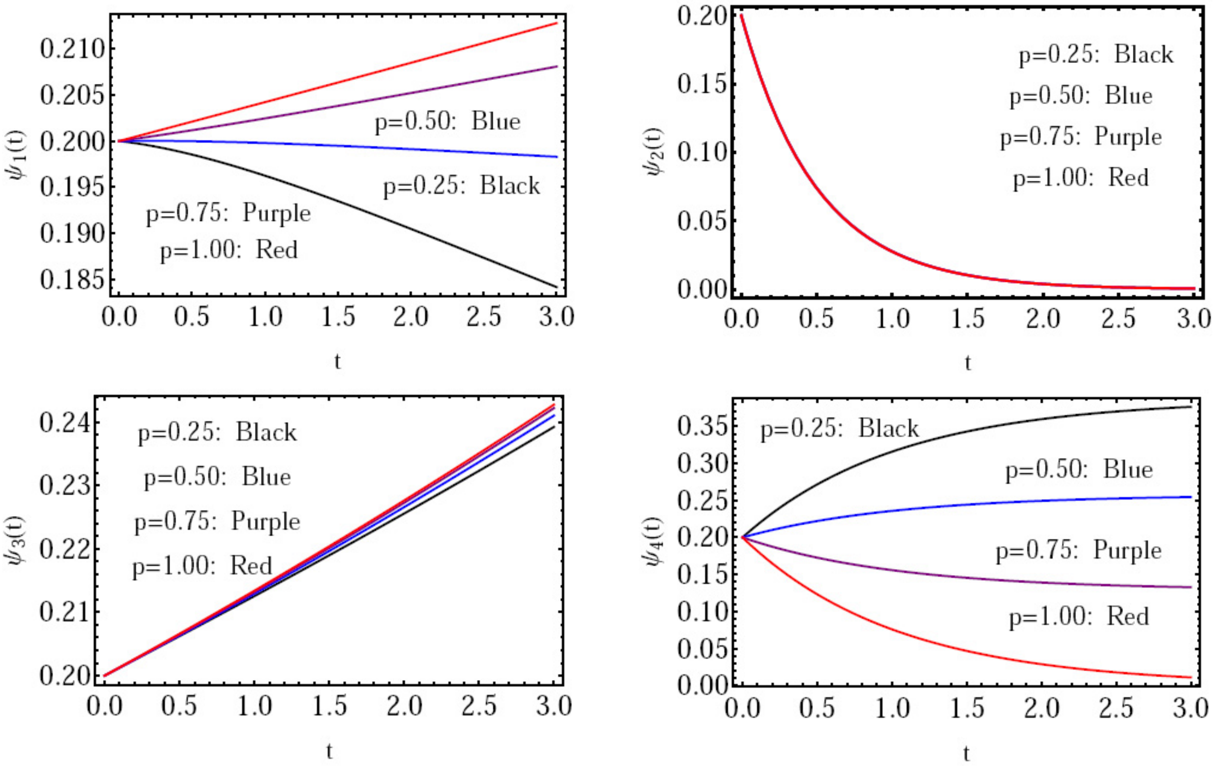

Figure 3. The approximate solution via various values of p.

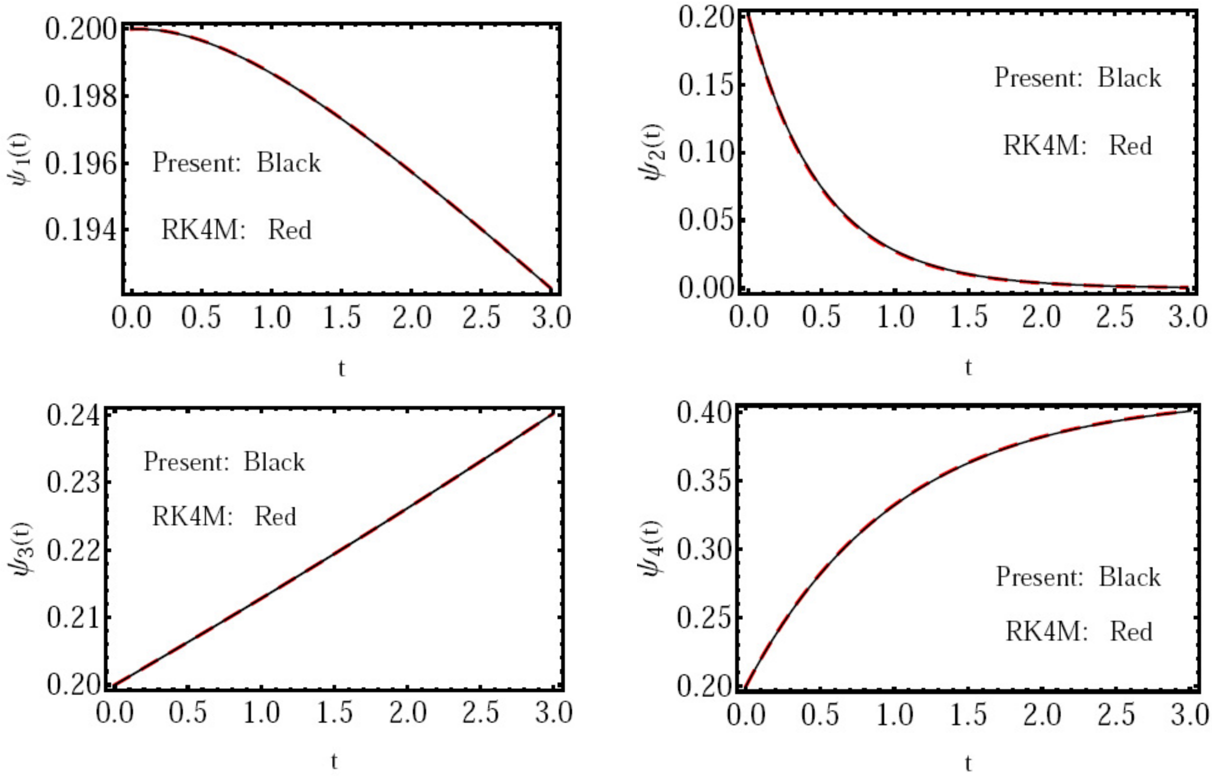

Figure 4. Comparison of the solution obtained by the proposed method and RK4M.

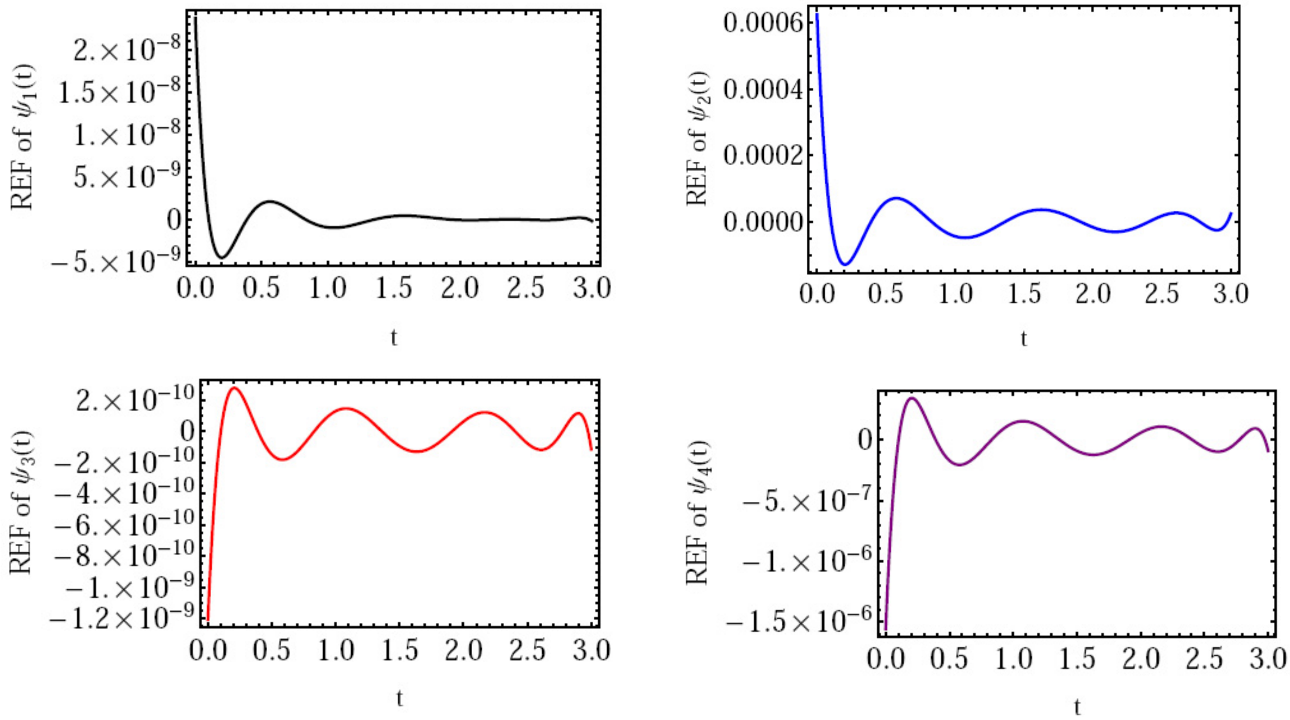

Figure 5. The REF of the obtained approximate solution.

6. Conclusions

In this research, CHPM is applied to obtain numerical solutions for the BC with different initial solutions, and some parameters. By comparing the approximate solutions and the RK4 method 's solution of the model under study, we were able to conclude that the approximate solutions obtained by applying the given technique are in excellent agreement with the RK4 method's solution. Also through the resulting numerical results, we found to a large extent how effective this approach is in solving the problem under study and highlights the validity and potential of the proposed technique. Finally, this view of analytical and numerical solutions of dynamic variables was due to their being present and effectively influencing various models and fields of applied mathematics. Finally, the present study may contribute to providing more robust physical explanations for future theoretical and computational studies on the same topic.

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