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# The $r$-Stirling Genocchi Numbers

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**Abstract.** This paper introduces the $r$-Stirling Genocchi numbers, a new sequence derived by combining Broder's $r$-Stirling numbers with the classical Genocchi numbers, which are closely related to Bernoulli numbers and have notable applications in algebraic combinatorics. While the original $r$-Stirling numbers were developed using combinatorial methods to count set partitions, this study adopts an algebraic approach to explore the properties of the new sequence. Through tools like generating functions, recurrence relations, and algebraic transformations, the paper uncovers deeper structural insights and highlights the broader mathematical connections between partition theory, number theory, and combinatorial analysis.

**2020 Mathematics Subject Classifications**: 11B68, 11B73, 05A15

**Key Words and Phrases**: Genocchi numbers, $r$-Stirling numbers, recurrence relations, generating function, Bernoulli numbers

## 1. Introduction

Stirling numbers were first introduced by James Stirling with pairs of numbers in his book Methodus differentialis usually denoted by $s(n, k)$ and $S(n, k)$ where $s(n, k)$ referred to as Stirling numbers of the first kind and $S(n, k)$ is referred to as Stirling numbers of the second kind. [1]

Over the years, many mathematics enthusiasts have investigated and expanded these two special numbers. One of them is Broder [2], where he developed both types of r-Stirling numbers and imposed a constraint on the first r elements, requiring them to be arranged in different cycles or partition in different subsets. The $r$-Stirling numbers are as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_r$$

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can be represented as the number of permutations of the set $\{1, ..., n\}$ having m cycles, such that the numbers $1, 2, ..., r$ are in distinct.

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r$$

can be represented as the number of partitions of the set $\{1, ..., n\}$ into $m$ non-empty disjoint subsets such that the numbers $1, 2, ..., r$ are in distinct subsets.

Since Broder's formulation was primarily combinatorial, Corcino et.al [3] later introduced an algebraic approach by defining r-Stirling numbers using exponential generating functions:

$$\sum_{n=0}^{\infty} \widehat{\left[ \begin{matrix} n+r \\ k+r \end{matrix} \right]}_r \frac{t^n}{n!} = \left( \frac{1}{1+t} \right)^r \frac{\ln^k(1+t)}{k!}$$

$$\sum_{n=0}^{\infty} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \frac{t^n}{n!} = \frac{e^{rt}(e^t-1)^k}{k!}$$

A natural extension of these ideas leads to Genocchi numbers, first studied by Angelo Genocchi. The Genocchi numbers $G_n$ satisfy the generating function given as follows

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}$$

Moreover, the Genocchi polynomials and Genocchi polynomials of higher order, which are respectively defined by

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{xt}, \quad |t| < \pi, \tag{1.1}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} = \left( \frac{2t}{e^t+1} \right)^k e^{xt}, \tag{1.2}$$

(See [4–6]). It is important to note that the Genocchi polynomials satisfy the following relation

$$G_n(x) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m} x^m \tag{1.3}$$

expressing $G_n(x)$ as polynomial in $x$. In relation to this, we define Genocchi factorial polynomials, denoted by $G_{\underline{n}}(x)$, as follows

$$G_{\underline{n}}(x) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(x)_m. \tag{1.4}$$

Different variations of Genocchi numbers and polynomials and their properties are discussed in [7–9]

In this paper, it presents a new class of numbers called $r$-Stirling Genocchi numbers, created by merging the $r$-Stirling numbers with Genocchi numbers and explore its behavior by generating results.The $r$-Stirling Genocchi numbers defined by means of exponential generating function are as follows:

$$\sum_{n=0}^{\infty} SG_n^1(k;r)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \frac{2t\ln^k(1+t)}{k!(e^t+1)} \tag{1}$$

$$\sum_{n=0}^{\infty} SG_n^2(k;r)\frac{t^n}{n!} = \frac{2te^{rt}(e^t-1)^k}{k!(e^t+1)} \tag{2}$$

where $SG_n^1(k;r)$ referred to as the $r$-Stirling Genocchi number of the First Kind and $SG_n^2(k;r)$ is the $r$-Stirling Genocchi number of the Second Kind.

**Theorem 1.1.** *Convolution Formula of the $r$-Stirling Numbers and Genocchi Numbers are given as follows:*

$$SG_n^1(k;r) = \sum_{j=k}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \binom{n}{j} G_{n-j} \tag{3}$$

$$SG_n^2(k;r) = \sum_{j=k}^{n} \begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \binom{n}{j} G_{n-j} \tag{4}$$

*where $n \geq k$ otherwise $SG_n^1(k;r) = SG_n^2(k;r) = 0$*

*Proof.* The Exponential Generating Function in (1) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} SG_n^1(k;r)\frac{t^n}{n!} &= \left(\frac{1}{1+t}\right)^r \frac{2t\ln^k(1+t)}{k!(e^t+1)} \\
&= \left(\frac{1}{1+t}\right)^r \frac{\ln^k(1+t)}{k!}\frac{2t}{(e^t+1)} \\
&= \left(\sum_{n=0}^{\infty} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \frac{t^j}{j!} G_{n-j}\frac{t^{n-j}}{(n-j)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \frac{n!}{(n-j)!j!} G_{n-j}\right\}\frac{t^n}{n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \binom{n}{j} G_{n-j} \right\} \frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired convolution formula in (3). On the other hand, the exponential generating function in (2) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} SG_n^2(k;r)\frac{t^n}{n!} &= \frac{2te^{rt}(e^t-1)^k}{k!(e^t+1)} \\
&= \frac{e^{rt}(e^t-1)^k}{k!}\frac{2t}{(e^t+1)} \\
&= \left(\sum_{n=0}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n \frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \frac{t^j}{j!} G_{n-j} \frac{t^{n-j}}{(n-j)!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \frac{n!}{(n-j)!j!} G_{n-j} \right\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \binom{n}{j} G_{n-j} \right\} \frac{t^n}{n!}
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired convolution formula in (4).

**Theorem 1.2.** *The Horizontal Generating Function for both kinds of r-Stirling Genocchi Numbers are given as follows:*

$$G_{\underline{n}}(z-r) = \sum_{k=0}^{n} SG_n^1(k;r)z^k \tag{5}$$

$$G_n(z+r) = \sum_{k=0}^{n} SG_n^2(k;r)z^{\underline{k}} \tag{6}$$

*Proof.* The exponential generating function of (1) can be written as

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} SG_n^1(k;r)\frac{t^n}{n!} \right\} z^k &= \left(\frac{1}{1+t}\right)^r \frac{2t}{(e^t+1)} \sum_{k\geq 0} \frac{\ln^k(1+t)}{k!} z^k \\
&= \left(\frac{1}{1+t}\right)^r \frac{2t}{(e^t+1)} \sum_{k\geq 0} \frac{[z\ln(1+t)]^k}{k!}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{1+t}\right)^{r} e^{\ln(1+t)^{z}} \frac{2t}{(e^{t}+1)} \\
&= \sum_{n\geq 0} \binom{z-r}{n} t^{n} \sum_{n=0}^{\infty} G_{n} \frac{t^{n}}{n!} \\
&= \left(\sum_{n\geq 0} (z-r)_{n} \frac{t^{n}}{n!}\right) \left(\sum_{n=0}^{\infty} G_{n} \frac{t^{n}}{n!}\right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (z-r)_{m} \frac{t^{m}}{m!} G_{n-m} \frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} (z-r)_{m} \frac{1}{(n-m)!m!} G_{n-m} \right\} t^{n} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} (z-r)_{m} \binom{n}{m} G_{n-m} \right\} \frac{t^{n}}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} G_{n-m} (z-r)_{m} \right\} \frac{t^{n}}{n!}
\end{aligned}$$

Rewriting

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} SG_{n}^{1}(k;r) z^{k} \right\} \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} G_{n-m} (z-r)_{m} \right\} \frac{t^{n}}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\begin{aligned}
\sum_{k=0}^{n} SG_{n}^{1}(k;r) z^{k} &= \sum_{m=0}^{n} \binom{n}{m} G_{n-m} (z-r)_{m} \\
&= G_{\underline{n}}(z-r)
\end{aligned}$$

Similarly (2) can be written as

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} SG_{n}^{2}(k;r) \frac{t^{n}}{n!} \right\} z^{\underline{k}} &= \sum_{k=0}^{\infty} \left\{ \frac{2te^{rt}(e^{t}-1)^{k}}{k!(e^{t}+1)} \right\} z^{\underline{k}} \\
&= \frac{2te^{rt}}{(e^{t}+1)} \sum_{k=0}^{\infty} \binom{z}{k} (e^{t}-1)^{k} \\
&= \frac{2te^{rt}}{(e^{t}+1)} (1+(e^{t}-1))^{z} \\
&= \frac{2t}{(e^{t}+1)} e^{rt} (1+(e^{t}-1))^{z}
\end{aligned}$$

$$
\begin{aligned}
&= e^{(z+r)t}\frac{2t}{(e^t+1)}\\
&= \left(\sum_{n=0}^{\infty}(z+r)^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(z+r)^m\frac{t^m}{m!}\,G_{n-m}\frac{t^{n-m}}{(n-m)!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z+r)^m\frac{1}{(n-m)!m!}\,G_{n-m}\right\}t^n\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z+r)^m\binom{n}{m}\,G_{n-m}\right\}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}(z+r)^m\right\}\frac{t^n}{n!}
\end{aligned}
$$

Rewriting

$$
\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}SG_n^2(k;r)z^{\underline{k}}\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}(z+r)^m\right\}\frac{t^n}{n!}
$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$
\begin{aligned}
\sum_{k=0}^{n}SG_n^2(k;r)z^{\underline{k}} &= \sum_{m=0}^{n}\binom{n}{m}G_{n-m}(z+r)^m\\
&= G_n(z+r)
\end{aligned}
$$

Hence, we proved the horizontal generating function in (5) and (6).

An essential characteristic of a special number is its explicit formula, which is valuable for directly calculating the number's value for particular parameter inputs. The following theorem presents the explicit formula for the $r$-Stirling Genocchi numbers.

**Theorem 1.3.** *The formula for the first kind of $r$-Stirling Genocchi Numbers is explicitly stated as follows:*

$$
\begin{aligned}
SG_n^1(k;r) = \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{n-j+d+f}G_{j-m}\binom{j}{m}\binom{n}{j}\binom{f}{d}\\
\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}r^{\overline{n-j}}
\end{aligned}
$$

*Proof.* The exponential generating function in (1) composed of three functions. The first function can be expressed as

$$\begin{aligned}\left(\frac{1}{1-t}\right)^r &= (1-t)^{-r}\\ &= \sum_{n\geq 0}\binom{-r}{n}t^n\\ &= \binom{-r}{0}(-t)^0+\sum_{n>0}n\binom{-r}{n}t^n\end{aligned}$$

By Newtons Binomial Theorem,

$$\begin{aligned}\left(\frac{1}{1-t}\right)^r &= \sum_{n\geq 0}\frac{(-r)(-r-1)\dots(-r-n+1)}{n!}(-t)^n\\ &= \sum_{n\geq 0}(-1)^n\frac{(r)(r+1)\dots(r+n-1)}{n!}(-1)^n(t)^n\end{aligned}$$

And by the definition of a rising factorial,

$$\left(\frac{1}{1-t}\right)^r = \sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}$$

where $r^{\overline{n}} = r(r+1)\dots(r+n-1)$
The second function can be expresses as the Genocchi number,

$$\frac{2t}{(e^t+1)} = \sum_{n=0}^{\infty}G_n\frac{t^n}{n!}$$

Lastly the third function can be expressed

$$\frac{1}{k!}\left[\ln 1+t\right]^k = \sum_{n\geq k}s(n,k)\frac{t^n}{n!}$$

Hence, using Cauchy's Rule for the product of power series, we have

$$\sum_{k\geq 0}^{n}SG_n^1(k;r)\frac{t^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}\right)\left(\sum_{n\geq k}s(n,k)\frac{t^n}{n!}\right)$$

$$
\begin{aligned}
&= \left(\sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left\{\sum_{m=k}^{n} s(m,k)\binom{n}{m}G_{n-m}\right\}\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=k}^{j} s(m,k)\binom{j}{m}G_{j-m}\frac{t^j}{j!}(-1)^{n-j}r^{\overline{n-j}}\frac{t^{n-j}}{(n-j)!}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=k}^{j} s(m,k)\binom{j}{m}G_{j-m}\frac{t^{j-j+n}}{(n-j)!j!}(-1)^{n-j}r^{\overline{n-j}}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n} s(m,k)\binom{j}{m}G_{j-m}\frac{1}{(n-j)!j!}(-1)^{n-j}r^{\overline{n-j}}\right\}t^n\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n} s(m,k)\binom{j}{m}G_{j-m}\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}\right\}\frac{t^n}{n!}
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$,

$$
SG_n^1(k;r) = \sum_{m=0}^{n}\sum_{j=m}^{n} s(m,k)\binom{j}{m}G_{j-m}\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}
$$

Using the Schlömilch Formula for the Stirling numbers of the first kind,

$$
s(n,k) = \sum_{r=0}^{n-k}\sum_{j=i}^{r}(-1)^{j+r}\binom{r}{j}\binom{n-1+r}{n-k+r}\binom{2n-k}{n-k+r}\frac{(r-j)^{n-k+r}}{r!}
$$

Then the Schlömilch Formula for the $r$-Stirling Genocchi number of the first kind is given by

$$
\begin{aligned}
SG_n^1(k;r) = &\sum_{m=0}^{n}\sum_{j=m}^{n}\binom{j}{m}G_{j-m}\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}\\
&\sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{d+f}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}\\
= &\sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{f=0}^{m-k}\sum_{d=i}^{f}\binom{j}{m}G_{j-m}\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}\\
&(-1)^{d+f}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}
\end{aligned}
$$

$$= \sum_{m=0}^{n} \sum_{j=m}^{n} \sum_{f=0}^{m-k} \sum_{d=i}^{f} (-1)^{n-j+d+f} G_{j-m}$$
$$\binom{j}{m}\binom{n}{j}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!} r^{\overline{n-j}}$$

**Theorem 1.4.** *The formula for the second kind of r-Stirling Genocchi Numbers is explicitly defined by the following formula:*

$$SG_n^2(k;r) = \sum_{m=0}^{n} \left\{ \begin{matrix} m+r \\ k+r \end{matrix} \right\}_r \binom{n}{m} G_{n-m}$$

*Proof.* The exponential generating function in (2) can be written as

$$\begin{aligned}
\sum_{n\geq k}^{n} k! SG_n^2(k;r)\frac{t^n}{n!} &= \frac{2te^{rt}}{(e^t+1}\sum_{i=0}^{k}\binom{k}{i}(e^t)^{k-i}(-1)^i \\
&= \frac{2t}{(e^t+1)}\sum_{i=0}^{k}\binom{k}{i}e^{rt}(e^t)^{k-i}(-1)^i \\
&= \frac{2t}{(e^t+1)}\sum_{i=0}^{k}\binom{k}{i}e^{rt+(k-i)t}(-1)^i \\
&= \sum_{i=0}^{k}\binom{k}{i}e^{rt}(e^t)^{k-i}(-1)^i\sum_{n=0}^{\infty}G_n\frac{t^n}{n!} \\
&= \left(\sum_{i=0}^{k}\binom{k}{i}\sum_{n\geq 0}\frac{[((k-i)+r)t]^n}{n!}(-1)^i\right)\left(\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}\right) \\
&= \left(\sum_{n\geq 0}^{\infty}\left\{\sum_{i=0}^{k}(-1)^i\binom{k}{i}((k-i)+r)^n\right\}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}\right) \\
&= \left(\sum_{n\geq 0}^{\infty}k!\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}k!\left\{ \begin{matrix} m+r \\ k+r \end{matrix} \right\}_r\frac{t^m}{m!}G_{n-m}\frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}k!\left\{ \begin{matrix} m+r \\ k+r \end{matrix} \right\}_r\frac{1}{(n-m)!m!}G_{n-m}\right\}t^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} k!\left\{ m+r \atop k+r \right\}_r \binom{n}{m} G_{n-m}\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\begin{aligned} k!SG_n^2(k;r) &= \sum_{m=0}^{n} k!\left\{ m+r \atop k+r \right\}_r \binom{n}{m} G_{n-m} \\ SG_n^2(k;r) &= \sum_{m=0}^{n} \left\{ m+r \atop k+r \right\}_r \binom{n}{m} G_{n-m} \end{aligned}$$

The following sections investigates the convolutions of $r$-Stirling numbers with the Genocchi polynomials and higher order genocchi polynomials.

## 2. $r$-Stirling Genocchi Polynomials

The Genocchi Polynomials satisfy the relation

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}$$

The $r$-Stirling Genocchi Polynomials defined by means of exponential generating function are as follows:

$$\sum_{n=0}^{\infty} SG_n^1(x;k;r)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \frac{2te^{xt}\ln^k(1+t)}{k!(e^t+1)} \tag{7}$$

$$\sum_{n=0}^{\infty} SG_n^2(x;k;r)\frac{t^n}{n!} = \frac{2te^{xt}e^{rt}(e^t-1)^k}{k!(e^t+1)} \tag{8}$$

**Theorem 2.1.** *Convolution Formula of the $r$-Stirling Numbers and Genocchi Polynomials are given as follows:*

$$SG_n^1(x;k,r) = \sum_{i=0}^{n}\sum_{j=i}^{n} \left[ \widehat{n-j+r} \atop k+r \right]_r \binom{n}{j}\binom{j}{i} G_{j-i}x^i \tag{9}$$

$$SG_n^2(x;k,r) = \sum_{i=0}^{n}\sum_{j=i}^{n} \left\{ n-j+r \atop k+r \right\}_r \binom{n}{j}\binom{j}{i} G_{j-i}x^i \tag{10}$$

*where $n \geq k$ otherwise $SG_n^1(k;r) = SG_n^2(k;r) = 0$*

*Proof.* The Exponential Generating Function in (7) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} SG_n^1(x;k,r)\frac{t^n}{n!} &= \left(\frac{1}{1+t}\right)^r \frac{2te^{xt}\ln^k(1+t)}{k!(e^t+1)} \\
&= \left(\frac{1}{1+t}\right)^r \frac{\ln^k(1+t)}{k!}\frac{2te^{xt}}{(e^t+1)} \\
&= \left(\sum_{n=0}^{\infty} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \frac{t^j}{j!} G_{n-j}(x)\frac{t^{n-j}}{(n-j)!} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \widehat{\begin{bmatrix} n-j+r \\ k+r \end{bmatrix}}_r \frac{t^{n-j}}{(n-j)!} G_j(x)\frac{t^j}{j!}
\end{aligned}$$

Since $G_j(x) = \sum_{i=0}^{j}\binom{j}{i}G_i x^{j-i}$, then it follows

$$\begin{aligned}
\sum_{n=0}^{\infty} SG_n^1(x;k,r)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\widehat{\begin{bmatrix} n-j+r \\ k+r \end{bmatrix}}_r \sum_{i=0}^{j}\binom{j}{i}G_{j-i}x^i\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\sum_{j=i}^{n}\widehat{\begin{bmatrix} n-j+r \\ k+r \end{bmatrix}}_r \binom{n}{j}\binom{j}{i}G_{j-i}x^i\right\}\frac{t^n}{n!}
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired convolution formula in (9). On the other hand, the exponential generating function in (8) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} SG_n^2(x;k,r)\frac{t^n}{n!} &= \frac{2te^{xt}e^{rt}(e^t-1)^k}{k!(e^t+1)} \\
&= \frac{e^{rt}(e^t-1)^k}{k!}\frac{2te^{xt}}{(e^t+1)} \\
&= \left(\sum_{n=0}^{\infty}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \frac{t^j}{j!}G_{n-j}(x)\frac{t^{n-j}}{(n-j)!} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{Bmatrix} n-j+r \\ k+r \end{Bmatrix}_r \frac{t^{n-j}}{(n-j)!}G_j(x)\frac{t^j}{j!} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{Bmatrix} n-j+r \\ k+r \end{Bmatrix}_r \frac{t^{n-j}}{(n-j)!}\sum_{i=0}^{j}\binom{j}{i}G_i x^{j-i}\frac{t^j}{j!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\sum_{j=i}^{n}\left\{ {n-j+r \atop k+r} \right\}_r \binom{n}{j}\binom{j}{i} G_{j-i}x^i\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired convolution formula in (10).

**Theorem 2.2.** *The Horizontal Generating Function for both kinds of r-Stirling Genocchi Polynomials are given as follows:*

$$G_{\underline{n}}(x+z-r) = \sum_{k=0}^{n} SG_n^1(x;k,r)z^k \tag{11}$$

$$G_n(x+z+r) = \sum_{k=0}^{n} SG_n^2(x;k,r)z^{\underline{k}} \tag{12}$$

*Proof.* The exponential generating function of (7) can be written as

$$\begin{aligned}
\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} SG_n^1(x;k,r)\frac{t^n}{n!}\right\}z^k &= \left(\frac{1}{1+t}\right)^r \frac{2te^{xt}}{(e^t+1)}\sum_{k\geq 0}\frac{\ln^k(1+t)}{k!}z^k \\
&= \left(\frac{1}{1+t}\right)^r \frac{2te^{xt}}{(e^t+1)}\sum_{k\geq 0}\frac{[z\ln(1+t)]^k}{k!} \\
&= \left(\frac{1}{1+t}\right)^r e^{\ln(1+t)^z}\frac{2te^{xt}}{(e^t+1)} \\
&= \sum_{n\geq 0}\binom{z-r}{n}t^n\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} \\
&= \left(\sum_{n\geq 0}(z-r)_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(z-r)_m\frac{t^m}{m!}G_{n-m}(x)\frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z-r)_m\frac{1}{(n-m)!m!}G_{n-m}(x)\right\}t^n \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z-r)_m\binom{n}{m}G_{n-m}(x)\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}(x)(z-r)_m\right\}\frac{t^n}{n!}
\end{aligned}$$

Rewriting

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} SG_n^1(x;k,r)z^k\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}(x)(z-r)_m\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\sum_{k=0}^{n} SG_n^1(x;k,r)z^k = \sum_{m=0}^{n}\binom{n}{m}G_{n-m}(x)(z-r)_m$$

Applying the Addition Formula, we have

$$\sum_{k=0}^{n} SG_n^1(x;k,r)z^k = G_{\underline{n}}(x+z-r)$$

Similarly (8) can be written as

$$\begin{aligned}
\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} SG_n^2(x;k,r)\frac{t^n}{n!}\right\}z^{\underline{k}} &= \sum_{k=0}^{\infty}\left\{\frac{2te^{xt}e^{rt}(e^t-1)^k}{k!(e^t+1)}\right\}z^{\underline{k}} \\
&= \frac{2te^{xt}e^{rt}}{(e^t+1)}\sum_{k=0}^{\infty}\binom{z}{k}(e^t-1)^k \\
&= \frac{2te^{xt}e^{rt}}{(e^t+1)}(1+(e^t-1))^z \\
&= \frac{2te^{xt}}{(e^t+1)}e^{rt}(1+(e^t-1))^z \\
&= e^{(z+r)t}\frac{2te^{xt}}{(e^t+1)} \\
&= \left(\sum_{n=0}^{\infty}(z+r)^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(z+r)^m\frac{t^m}{m!}\,G_{n-m}(x)\frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z+r)^m\frac{1}{(n-m)!m!}\,G_{n-m}(x)\right\}t^n \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z+r)^m\binom{n}{m}\,G_{n-m}(x)\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}(x)(z+r)^m\right\}\frac{t^n}{n!}
\end{aligned}$$

Rewriting

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} SG_n^2(x;k,r)z^{\underline{k}}\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}(x)(z+r)^m\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\sum_{k=0}^{n} SG_n^2(x;k,r)z^{\underline{k}} = \sum_{m=0}^{n}\binom{n}{m}G_{n-m}(x)(z+r)^m$$

Applying the Addition Formula, we have

$$\sum_{k=0}^{n} SG_n^2(x;k,r)z^{\underline{k}} = G_n(x+z+r)$$

Hence, we proved the horizontal generating function in (11) and (12).

**Theorem 2.3.** *The formula for the first kind of r-Stirling Genocchi Polynomials is explicitly stated as follows:*

$$SG_n^1(x;k,r) = \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{i=0}^{j-m}\sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{n-j+d+f}G_{j-m-i}x^i\binom{j}{m}\binom{j-m}{i}\binom{n}{j}\binom{f}{d}$$
$$\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}r^{\overline{n-j}}$$

*Proof* The exponential generating function in (7) composed of three functions.
The first function can be expressed as

$$\left(\frac{1}{1-t}\right)^r = \sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}$$

where $r^{\overline{n}} = r(r+1)\ldots(r+n-1)$
The second function can be expresses as the Genocchi Polynomial,

$$\frac{2t}{(e^t+1)}e^{xt} = \sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}$$

Lastly the third function can be expressed

$$\frac{1}{k!}\left[\ln(1+t)\right]^k = \sum_{n\geq k}s(n,k)\frac{t^n}{n!}$$

Hence, using Cauchy's Rule for the product of power series, we have

$$\sum_{k\geq 0}^{n} SG_n^1(x;k,r)\frac{t^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right)\left(\sum_{n\geq k}s(n,k)\frac{t^n}{n!}\right)$$

$$
\begin{aligned}
&= \left(\sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left\{\sum_{m=k}^{n} s(m,k)\binom{n}{m}G_{n-m}(x)\right\}\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=k}^{j} s(m,k)\binom{j}{m}G_{j-m}(x)\frac{t^j}{j!}(-1)^{n-j}r^{\overline{n-j}}\frac{t^{n-j}}{(n-j)!}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=k}^{j} s(m,k)\binom{j}{m}G_{j-m}(x)\frac{t^{j-j+n}}{(n-j)!j!}(-1)^{n-j}r^{\overline{n-j}}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n} s(m,k)\binom{j}{m}G_{j-m}(x)\frac{1}{(n-j)!j!}(-1)^{n-j}r^{\overline{n-j}}\right\}t^n\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n} s(m,k)\binom{j}{m}\sum_{i=0}^{j-m}\binom{j-m}{i}G_{j-m-i}x^i\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}\right\}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{i=0}^{j-m} s(m,k)\binom{j}{m}\binom{j-m}{i}G_{j-m-i}x^i\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}\right\}\frac{t^n}{n!}
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$,

$$
SG_n^1(x;k,r) = \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{i=0}^{m}\binom{j}{m}\binom{m}{i}\binom{n}{j}G_{m-i}x^i s(m,k)(-1)^{n-j}r^{\overline{n-j}}
$$

Using the Schlömilch Formula for the Stirling numbers of the first kind,

$$
s(n,k) = \sum_{r=0}^{n-k}\sum_{j=i}^{r}(-1)^{j+r}\binom{r}{j}\binom{n-1+r}{n-k+r}\binom{2n-k}{n-k+r}\frac{(r-j)^{n-k+r}}{r!}
$$

Then the Schlömilch Formula for the $r$-Stirling Genocchi Polynomial of the first kind is given by

$$
\begin{aligned}
SG_n^1(x;k,r) &= \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{i=0}^{j-m}\binom{j}{m}\binom{j-m}{i}\binom{n}{j}G_{j-m-i}x^i(-1)^{n-j}r^{\overline{n-j}}\\
&\quad \sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{d+f}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}\\
&= \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{i=0}^{j-m}\sum_{f=0}^{m-k}\sum_{d=i}^{f}\binom{j-m}{i}\binom{j}{m}G_{j-m-i}x^i\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}\\
&\quad (-1)^{d+f}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}
\end{aligned}
$$

$$= \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{i=0}^{j-m}\sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{n-j+d+f}G_{j-m-i}x^i$$

$$\binom{j}{m}\binom{j-m}{i}\binom{n}{j}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}\overline{r^{n-j}} \quad \square$$

**Theorem 2.4.** *The formula for the second kind of r-Stirling Genocchi Polynomials is explicitly defined by the following formula:*

$$SG_n^2(x;k,r) = \sum_{m=0}^{n}\sum_{i=0}^{j-m}\left\{ {m+r \atop k+r} \right\}_r \binom{n}{m}\binom{j-m}{i}G_{m-i}x^i$$

*Proof* The exponential generating function in (8) can be written as

$$\begin{aligned}
\sum_{n\geq k}^{n} k!SG_n^2(x;k,r)\frac{t^n}{n!} &= \frac{2te^{xt}e^{rt}}{(e^t+1)}\sum_{i=0}^{k}\binom{k}{i}(e^t)^{k-i}(-1)^i \\
&= \frac{2te^{xt}}{(e^t+1)}\sum_{i=0}^{k}\binom{k}{i}e^{rt}(e^t)^{k-i}(-1)^i \\
&= \frac{2te^{xt}}{(e^t+1)}\sum_{i=0}^{k}\binom{k}{i}e^{rt+(k-i)t}(-1)^i \\
&= \sum_{i=0}^{k}\binom{k}{i}e^{rt}(e^t)^{k-i}(-1)^i\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} \\
&= \left(\sum_{i=0}^{k}\binom{k}{i}\sum_{n\geq 0}\frac{[((k-i)+r)t]^n}{n!}(-1)^i\right)\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right) \\
&= \left(\sum_{n\geq 0}^{\infty}\left\{\sum_{i=0}^{k}(-1)^i\binom{k}{i}((k-i)+r)^n\right\}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right) \\
&= \left(\sum_{n\geq 0}^{\infty}k!\left\{ {n+r \atop k+r} \right\}_r\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}k!\left\{ {m+r \atop k+r} \right\}_r\frac{t^m}{m!}G_{n-m}(x)\frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}k!\left\{ {m+r \atop k+r} \right\}_r\frac{1}{(n-m)!m!}G_{n-m}(x)\right\}t^n \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}k!\left\{ {m+r \atop k+r} \right\}_r\binom{n}{m}\sum_{i=0}^{j-m}\binom{j-m}{i}G_{m-i}x^i\right\}\frac{t^n}{n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{i=0}^{j-m} k!\left\{\begin{matrix} m+r \\ k+r \end{matrix}\right\}_r \binom{n}{m}\binom{j-m}{i} G_{m-i}x^i\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\begin{aligned} k!SG_n^2(x;k,r) &= \sum_{m=0}^{n}\sum_{i=0}^{j-m} k!\left\{\begin{matrix} m+r \\ k+r \end{matrix}\right\}_r \binom{n}{m}\binom{j-m}{i} G_{m-i}x^i \\ SG_n^2(x;k,r) &= \sum_{m=0}^{n}\sum_{i=0}^{j-m} \left\{\begin{matrix} m+r \\ k+r \end{matrix}\right\}_r \binom{n}{m}\binom{j-m}{i} G_{m-i}x^i \quad \square \end{aligned}$$

## 3. $r$-Stirling Genocchi Polynomials of Higher Order

The Genocchi Polynomials of Higher Order satisfy the relation

$$\sum_{n=0}^{\infty} G_n^w(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^w e^{xt},$$

(see [4]). Now, we define the $r$-Stirling Genocchi Polynomials of higher order by means of exponential generating function are as follows:

$$\sum_{n=0}^{\infty} SG_n^{1w}(x;k,r)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \left(\frac{2t}{e^t+1}\right)^w \frac{e^{xt}\ln^k(1+t)}{k!} \tag{13}$$

$$\sum_{n=0}^{\infty} SG_n^{2w}(x;k,r)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^w \frac{e^{xt}e^{rt}(e^t-1)^k}{k!(e^t+1)} \tag{14}$$

We can also establish some properties parallel to those of $r$-Stirling Genocchi polynomials.

**Theorem 3.1.** *Convolution Formula of the $r$-Stirling Numbers and Genocchi Polynomials of higher order are given as follows:*

$$SG_n^{1w}(x;k,r) = \sum_{j=k}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \binom{n}{j} G_{n-j}^w(x) \tag{15}$$

$$SG_n^{2w}(x;k,r) = \sum_{j=k}^{n} \left\{\begin{matrix} j+r \\ k+r \end{matrix}\right\}_r \binom{n}{j} G_{n-j}^w(x) \tag{16}$$

*where $n \geq k$ otherwise $SG_n^{1w}(x;k,r) = SG_n^{2w}(x;k,r) = 0$*

*Proof.* The Exponential Generating Function in (13) can be written as

$$\sum_{n=0}^{\infty} SG_n^{1w}(x;k,r)\frac{t^n}{n!} \quad = \quad \left(\frac{1}{1+t}\right)^r \left(\frac{2t}{e^t+1}\right)^w \frac{e^{xt}\ln^k(1+t)}{k!}$$

$$
\begin{aligned}
&= \left(\frac{1}{1+t}\right)^r \frac{\ln^k(1+t)}{k!}\left(\frac{2t}{e^t+1}\right)^w e^{xt} \\
&= \left(\sum_{n=0}^{\infty} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n^w(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \frac{t^j}{j!} G_{n-j}^w(x)\frac{t^{n-j}}{(n-j)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \frac{n!}{(n-j)!j!} G_{n-j}^w(x)\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n} \widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r \binom{n}{j} G_{n-j}^w(x)\right\}\frac{t^n}{n!}
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired convolution formula in (15). On the other hand, the exponential generating function in (14) can be written as

$$
\begin{aligned}
\sum_{n=0}^{\infty} SG_n^{2w}(x;k,r)\frac{t^n}{n!} &= \left(\frac{2t}{e^t+1}\right)^w \frac{e^{xt}e^{rt}(e^t-1)^k}{k!(e^t+1)} \\
&= \frac{e^{rt}(e^t-1)^k}{k!}\left(\frac{2t}{e^t+1}\right)^w e^{xt} \\
&= \left(\sum_{n=0}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n^w(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \frac{t^j}{j!} G_{n-j}^w(x)\frac{t^{n-j}}{(n-j)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n} \begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \frac{n!}{(n-j)!j!} G_{n-j}^w(x)\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n} \begin{Bmatrix} j+r \\ k+r \end{Bmatrix}_r \binom{n}{j} G_{n-j}^w(x)\right\}\frac{t^n}{n!}
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired convolution formula in (16).

**Theorem 3.2.** *The Horizontal Generating Function for both kinds of r-Stirling Genocchi Polynomials of higher order are given as follows:*

$$
G_{\underline{n}}^w(x+z-r) = \sum_{k=0}^{n} SG_n^{1w}(x;k,r)z^k \tag{17}
$$

$$G_n^w(x+z+r) = \sum_{k=0}^{n} SG_n^{2w}(x;k,r)z^{\underline{k}} \tag{18}$$

*Proof.* The exponential generating function of (13) can be written as

$$\begin{aligned}
\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} SG_n^{1w}(x;k,r)\frac{t^n}{n!}\right\} z^k &= \left(\frac{1}{1+t}\right)^r \left(\frac{2t}{e^t+1}\right)^w e^{xt}\sum_{k\geq 0}\frac{\ln^k(1+t)}{k!}z^k \\
&= \left(\frac{1}{1+t}\right)^r \left(\frac{2t}{e^t+1}\right)^w e^{xt}\sum_{k\geq 0}\frac{[z\ln(1+t)]^k}{k!} \\
&= \left(\frac{1}{1+t}\right)^r e^{\ln(1+t)^z}\left(\frac{2t}{e^t+1}\right)^w e^{xt} \\
&= \sum_{n\geq 0}\binom{z-r}{n}t^n \sum_{n=0}^{\infty} G_n^w(x)\frac{t^n}{n!} \\
&= \left(\sum_{n\geq 0}(z-r)_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} G_n^w(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(z-r)_m\frac{t^m}{m!}\,G_{n-m}^w(x)\frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z-r)_m\frac{1}{(n-m)!m!}\,G_{n-m}^w(x)\right\}t^n \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z-r)_m\binom{n}{m}\,G_{n-m}^w(x)\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}^w(x)(z-r)_m\right\}\frac{t^n}{n!}
\end{aligned}$$

Rewriting

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} SG_n^{1w}(x;k,r)z^k\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}^w(x)(z-r)_m\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\sum_{k=0}^{n} SG_n^{1w}(x;k,r)z^k = \sum_{m=0}^{n}\binom{n}{m}G_{n-m}^w(x)(z-r)_m$$

Applying the Addition Formula, we have

$$\sum_{k=0}^{n} SG_n^{1w}(x;k,r)z^k = G_{\underline{n}}^w(x+z-r)$$

Similarly (14) can be written as

$$\begin{aligned}
\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} SG_n^{2w}(x;k,r)\frac{t^n}{n!}\right\} z^{\underline{k}} &= \sum_{k=0}^{\infty}\left\{\left(\frac{2t}{e^t+1}\right)^w e^{xt}\frac{e^{rt}(e^t-1)^k}{k!}\right\} z^{\underline{k}} \\
&= \left(\frac{2t}{e^t+1}\right)^w e^{xt}e^{rt}\sum_{k=0}^{\infty}\binom{z}{k}(e^t-1)^k \\
&= \left(\frac{2t}{e^t+1}\right)^w e^{xt}e^{rt}(1+(e^t-1))^z \\
&= e^{(z+r)t}\left(\frac{2t}{e^t+1}\right)^w e^{xt} \\
&= \left(\sum_{n=0}^{\infty}(z+r)^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n^w(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(z+r)^m\frac{t^m}{m!}G_{n-m}^w(x)\frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z+r)^m\frac{1}{(n-m)!m!}G_{n-m}^w(x)\right\}t^n \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z+r)^m\binom{n}{m}G_{n-m}^w(x)\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}^w(x)(z+r)^m\right\}\frac{t^n}{n!}
\end{aligned}$$

Rewriting

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} SG_n^{2w}(x;k,r)z^{\underline{k}}\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}G_{n-m}^w(x)(z+r)^m\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\sum_{k=0}^{n} SG_n^{2w}(x;k,r)z^{\underline{k}} = \sum_{m=0}^{n}\binom{n}{m}G_{n-m}^w(x)(z+r)^m$$

Applying the Addition Formula, we have

$$\sum_{k=0}^{n} SG_n^{2w}(x;k,r)z^{\underline{k}} = G_n^w(x+z+r)$$

Hence, we proved the horizontal generating function in (17) and (18).

**Theorem 3.3.** *The formula for the first kind of r-Stirling Genocchi Polynomials of higher order is explicitly stated as follows:*

$$SG_n^{1w}(x;k,r) = \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{n-j+d+f}G_{j-m}^{w}(x)\binom{j}{m}\binom{n}{j}\binom{f}{d}$$
$$\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}r^{\overline{n-j}}$$

*Proof.* The exponential generating function in (13) composed of three functions. The first function can be expressed as

$$\left(\frac{1}{1-t}\right)^{r} = \sum_{n\geq 0}(-1)^{n}r^{\overline{n}}\frac{t^{n}}{n!}$$

where $r^{\overline{n}} = r(r+1)\ldots(r+n-1)$ The second function can be expresses as the Genocchi Polynomial of higher order,

$$\left(\frac{2t}{e^{t}+1}\right)^{w} e^{xt} = \sum_{n=0}^{\infty}G_n^{w}(x)\frac{t^{n}}{n!}$$

Lastly the third function can be expressed

$$\frac{1}{k!}\left[\ln(1+t)\right]^{k} = \sum_{n\geq k}s(n,k)\frac{t^{n}}{n!}$$

Hence, using Cauchy's Rule for the product of power series, we have

$$\begin{aligned}
\sum_{k\geq 0}^{n} SG_n^{1w}(x;k,r)\frac{t^{n}}{n!} &= \left(\sum_{n\geq 0}(-1)^{n}r^{\overline{n}}\frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}G_n^{w}(x)\frac{t^{n}}{n!}\right)\left(\sum_{n\geq k}s(n,k)\frac{t^{n}}{n!}\right)\\
&= \left(\sum_{n\geq 0}(-1)^{n}r^{\overline{n}}\frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}\left\{\sum_{m=k}^{n}s(m,k)\binom{n}{m}G_{n-m}^{w}(x)\right\}\frac{t^{n}}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=k}^{j}s(m,k)\binom{j}{m}G_{j-m}^{w}(x)\frac{t^{j}}{j!}(-1)^{n-j}r^{\overline{n-j}}\frac{t^{n-j}}{(n-j)!}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=k}^{j}s(m,k)\binom{j}{m}G_{j-m}^{w}(x)\frac{t^{j-j+n}}{(n-j)!j!}(-1)^{n-j}r^{\overline{n-j}}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n}s(m,k)\binom{j}{m}G_{j-m}^{w}(x)\frac{1}{(n-j)!j!}(-1)^{n-j}r^{\overline{n-j}}\right\}t^{n}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n}s(m,k)\binom{j}{m}G_{j-m}^{w}(x)\binom{n}{j}(-1)^{n-j}r^{\overline{n-j}}\right\}\frac{t^{n}}{n!}
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$,

$$SG_n^{1w}(x;k,r) = \sum_{m=0}^{n}\sum_{j=m}^{n} s(m,k)\binom{j}{m}G_{j-m}^{w}(x)\binom{n}{j}(-1)^{n-j}\overline{r^{n-j}}$$

Using the Schlömilch Formula for the Stirling numbers of the first kind,

$$s(n,k) = \sum_{r=0}^{n-k}\sum_{j=i}^{r}(-1)^{j+r}\binom{r}{j}\binom{n-1+r}{n-k+r}\binom{2n-k}{n-k+r}\frac{(r-j)^{n-k+r}}{r!}$$

Then the Schlömilch Formula for the $r$-Stirling Genocchi number of the first kind is given by

$$\begin{aligned}
SG_n^{1w}(x;k,r) &= \sum_{m=0}^{n}\sum_{j=m}^{n}\binom{j}{m}G_{j-m}^{w}(x)\binom{n}{j}(-1)^{n-j}\overline{r^{n-j}} \\
&\quad \sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{d+f}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!} \\
&= \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{f=0}^{m-k}\sum_{d=i}^{f}\binom{j}{m}G_{j-m}^{w}(x)\binom{n}{j}(-1)^{n-j}\overline{r^{n-j}} \\
&\quad (-1)^{d+f}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!} \\
&= \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{n-j+d+f}G_{j-m}^{w}(x) \\
&\quad \binom{j}{m}\binom{n}{j}\binom{f}{d}\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}\overline{r^{n-j}}
\end{aligned}$$

**Theorem 3.4.** *The formula for the second kind of $r$-Stirling Genocchi Polynomials of higher order is explicitly defined by the following formula:*

$$SG_n^{2w}(x;k,r) = \sum_{m=0}^{n}\left\{ {m+r \atop k+r} \right\}_r \binom{n}{m}G_{n-m}^{w}(x)$$

*Proof.* The exponential generating function in (14) can be written as

$$\sum_{n\geq k}^{n} k!SG_n^{2w}(x;k,r)\frac{t^n}{n!} \quad = \quad \left(\frac{2t}{e^t+1}\right)^{w} e^{xt}e^{rt}\sum_{i=0}^{k}\binom{k}{i}(e^t)^{k-i}(-1)^i$$

$$
\begin{aligned}
&= \left(\frac{2t}{e^t+1}\right)^w e^{xt}\sum_{i=0}^{k}\binom{k}{i}e^{rt}(e^t)^{k-i}(-1)^i \\
&= \left(\frac{2t}{e^t+1}\right)^w e^{xt}\sum_{i=0}^{k}\binom{k}{i}e^{rt+(k-i)t}(-1)^i \\
&= \sum_{i=0}^{k}\binom{k}{i}e^{rt}(e^t)^{k-i}(-1)^i\sum_{n=0}^{\infty}G_n^w(x)\frac{t^n}{n!} \\
&= \left(\sum_{i=0}^{k}\binom{k}{i}\sum_{n\geq 0}\frac{[((k-i)+r)t]^n}{n!}(-1)^i\right)\left(\sum_{n=0}^{\infty}G_n^w(x)\frac{t^n}{n!}\right) \\
&= \left(\sum_{n\geq 0}^{\infty}\left\{\sum_{i=0}^{k}(-1)^i\binom{k}{i}((k-i)+r)^n\right\}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n^w(x)\frac{t^n}{n!}\right) \\
&= \left(\sum_{n\geq 0}^{\infty}k!\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n^w(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}k!\left\{\begin{matrix} m+r \\ k+r \end{matrix}\right\}_r\frac{t^m}{m!}G_{n-m}^w(x)\frac{t^{n-m}}{(n-m)!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}k!\left\{\begin{matrix} m+r \\ k+r \end{matrix}\right\}_r\frac{1}{(n-m)!m!}G_{n-m}^w(x)\right\}t^n \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}k!\left\{\begin{matrix} m+r \\ k+r \end{matrix}\right\}_r\binom{n}{m}G_{n-m}^w(x)\right\}\frac{t^n}{n!}
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

When $x = 0$ in the Genocchi polynomials of higher order , we have the following corollaries.

**Corollary 3.5.** *Convolution Formula of the r-Stirling Numbers and Genocchi numbers of higher order are given as follows:*

$$SG_n^{1w}(k;r) = \sum_{j=k}^{n}\widehat{\begin{bmatrix} j+r \\ k+r \end{bmatrix}}_r\binom{n}{j}G_{n-j}^w$$

$$SG_n^{2w}(k;r) = \sum_{j=k}^{n}\left\{\begin{matrix} j+r \\ k+r \end{matrix}\right\}_r\binom{n}{j}G_{n-j}^w$$

*where* $n \geq k$.

*Proof.* Setting $x = 0$ of Theorem 9, the proof of this theorem follows immediately.

**Corollary 3.6.** *The Horizontal Generating Function for both kinds of r-Stirling Genocchi Numbers of higher order are given as follows:*

$$G^w_{\underline{n}}(z-r) = \sum_{k=0}^{n} SG^{1w}_n(k,r)z^k$$

$$G^w_n(z+r) = \sum_{k=0}^{n} SG^{2w}_n(k,r)z^{\underline{k}}$$

*Proof.* Setting $x = 0$ of Theorem 10, the proof of this theorem follows immediately.

**Corollary 3.7.** *The formula for the first kind of r-Stirling Genocchi Numbers of higher order is explicitly stated as follows:*

$$SG^{1w}_n(k;r) = \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{f=0}^{m-k}\sum_{d=i}^{f}(-1)^{n-j+d+f}G^w_{j-m}\binom{j}{m}\binom{n}{j}\binom{f}{d}$$
$$\binom{m-1+f}{m-k+f}\binom{2m-k}{m-k+f}\frac{(f-d)^{m-k+f}}{f!}r^{\overline{n-j}}$$

*Proof.* Setting $x = 0$ of Theorem 11, the proof of this theorem follows immediately.

**Corollary 3.8.** *The formula for the second kind of r-Stirling Genocchi Numbers of higher order is explicitly defined by the following formula:*

$$SG^{2w}_n(k;r) = \sum_{m=0}^{n} \left\{ {m+r \atop k+r} \right\}_r \binom{n}{m} G^w_{n-m}$$

*Proof.* Setting $x = 0$ of Theorem 12, the proof of this theorem follows immediately.

## 4. Conclusion and Recommendation

In this research, we introduced the novel concept of r-Stirling Genocchi numbers, expanding the field's understanding of combinatorial number theory by drawing connections between r-Stirling and Genocchi numbers. We derived significant results that include a convolution formula, which establishes structural relations among these numbers, as well as a horizontal generating function, which provides insight into the sequence's behavior and recursive properties. Additionally, we formulated explicit expressions for both the first and second kinds of r-Stirling Genocchi numbers, offering concrete tools for calculating these values directly. The study was also extended to encompass Genocchi polynomials and higher-order Genocchi polynomials, broadening the scope of applicability.

These findings contribute to the theoretical framework and may open up avenues for further exploration into generalized Stirling and Genocchi number applications, particularly in areas involving combinatorial identities, partition theory, and potentially in solving specific recurrence relations. Furthermore, the results of this study, particularly the use of exponential generating functions, convolution identities, and explicit formulas parallel the methods employed in recent work on degenerate $r$-Whitney numbers and polynomials in [10] . This alignment suggests that the framework developed here can be naturally extended to define and investigate $r$-Whitney Genocchi numbers, potentially including degenerate versions.

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