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On Nearly α-Compact Topological Spaces

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Abstract. In this paper, we introduce and investigate the concept of nearly α-compact topological spaces as a natural generalization of α-compact and countably α-compact spaces. We establish fundamental properties and characterizations of nearly α-compact spaces, demonstrating their relationship with various topological properties including α-continuity, separation axioms, and compactness-like properties. Several equivalent conditions for nearly α-compactness are provided, and we prove that the property is preserved under certain types of mappings. The behavior of nearly α-compact spaces under topological operations such as subspaces, products, and sums is thoroughly examined. We also introduce the notion of α-nearness and investigate its connection with nearly α-compact spaces. Additionally, we provide comprehensive examples and establish new theorems that demonstrate the richness and applicability of this concept.

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1. Introduction and Literature Review

The study of generalized forms of compactness has been a central theme in general topology for several decades. The classical notion of compactness, while fundamental,

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often proves too restrictive for many applications, leading researchers to investigate various weakenings and generalizations. Among these, α-compactness and its variants have emerged as particularly fruitful areas of investigation.

The concept of α-open sets was first introduced by Njåstad [1] in 1965, providing a foundation for numerous generalizations of classical topological concepts. A subset A of a topological space (X, τ) is called α-open if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. This seemingly simple modification of the notion of openness has led to rich theoretical developments in topology.

Building upon this foundation, Mashhour et al. [2] introduced α-continuous functions and explored their properties extensively. The study of α-compact spaces naturally followed, with various researchers investigating different aspects and generalizations of this concept. Recent developments in this area include the work of Kumar and Singh [3] on α-compact spaces in digital topology (2021), and the investigations by Chen and Liu [4] on relationships between α-compactness and other covering properties (2022).

Recent work by Oudetallah [5] on nearly metacompact spaces in bitopological settings has opened new avenues for understanding nearness properties in topological contexts. The investigation of pairwise expandable spaces by Oudetallah and AL-Hawari [6] further demonstrates the richness of generalized topological properties. The exploration of h-convexity in metric linear spaces by Oudetallah and Abualigah [7] provides additional geometric insights that complement topological investigations.

The study of various forms of compactness continues to evolve, with recent contributions including work on r-compactness by Oudetallah, Alharbi, and Batiha [8], and investigations of D-metacompactness by Oudetallah, Rousan, and Batiha [9]. Novel results on near Lindelöfness by Oudetallah [10] further expand our understanding of nearness properties in topological spaces. Additionally, recent work by Martinez and Rodriguez [11] (2023) on nearly compact spaces in the context of fuzzy topology, and the study by Wang et al. [12] (2024) on applications of near-compactness in functional analysis have provided new perspectives on these concepts.

The work of Levine [13] on generalized closed sets and the contributions of Dunham [14] on closure operators have provided essential tools for understanding generalized topological structures. These foundations, combined with the separation axiom investigations of Mashhour et al. [15], create a rich framework for the current investigation.

In this context, we introduce the concept of nearly α-compact spaces as a natural bridge between α-compactness and other generalized compactness properties. Our approach differs from previous investigations by focusing on a localized version of α-compactness that captures essential features while allowing for greater flexibility in applications.

The motivation for studying nearly α-compact spaces arises from several considerations. First, classical α-compactness, while elegant, can be quite restrictive in practice. Second, the interplay between nearness properties and α-structures provides new insights into the geometry of topological spaces. Third, the potential applications to function space topology and convergence theory make this investigation particularly worthwhile.

2. Preliminary Concepts

Throughout this paper, (X, τ) denotes a topological space. We begin by recalling essential definitions and establishing notation that will be used throughout our investigation.

Definition 1. *[1] Let (X, τ) be a topological space and $A \subseteq X$. Then:*

(i) A is called α-open if $A \subseteq Int(Cl(Int(A)))$.

(ii) A is called α-closed if $X \setminus A$ is α-open.

(iii) The α-interior of A, denoted $Int_\alpha(A)$, is the union of all α-open sets contained in A.

(iv) The α-closure of A, denoted $Cl_\alpha(A)$, is the intersection of all α-closed sets containing A.

The family of all α-open subsets of X is denoted by $\alpha O(X)$, and the family of all α-closed subsets is denoted by $\alpha C(X)$.

Definition 2. *[2] A function $f : X \to Y$ between topological spaces is called α-continuous if for every open set V in Y, the set $f^{-1}(V)$ is α-open in X.*

Definition 3. *[2] A topological space (X, τ) is called α-compact if every α-open cover of X has a finite subcover.*

Definition 4. *[2] A topological space (X, τ) is called countably α-compact if every countable α-open cover of X has a finite subcover.*

We also recall the following separation properties:

Definition 5. *[15] A topological space (X, τ) is called:*

(i) α-T_1 if for any two distinct points $x, y \in X$, there exist α-open sets U and V such that $x \in U$, $y \notin U$, $y \in V$, and $x \notin V$.

(ii) α-T_2 (or α-Hausdorff) if for any two distinct points $x, y \in X$, there exist disjoint α-open sets U and V such that $x \in U$ and $y \in V$.

Definition 6. *[13] A subset A of a topological space (X, τ) is called regular open if $A = Int(Cl(A))$. The collection of all regular open sets forms a complete Boolean algebra under set operations.*

Definition 7. *Let (X, τ) be a topological space. A point $x \in X$ is called a cluster point (or accumulation point) of a subset $A \subseteq X$ if every open neighborhood of x contains a point of A different from x. For α-open neighborhoods, we define α-cluster points analogously.*

Definition 8. *Let* $\{(X_i, \tau_i) : i \in I\}$ *be a family of topological spaces. The product space* $\prod_{i\in I} X_i$ *is endowed with the product topology, which is the coarsest topology making all projection maps* $\pi_j : \prod_{i\in I} X_i \to X_j$ *continuous.*

Definition 9. *Let* X *and* Y *be topological spaces. The set of all continuous functions from* X *to* Y *is denoted by* $C(X, Y)$*. This can be endowed with various topologies, including the compact-open topology and the topology of uniform convergence on compact subsets when* Y *is a metric space.*

3. Nearly α-Compact Spaces

In this section, we introduce the central concept of our investigation and establish its fundamental properties.

Definition 10. *A topological space* (X, τ) *is called nearly* α*-compact if for every* α*-open cover* $\mathcal{U}$ *of* X*, there exists a finite subfamily* $\mathcal{F} \subseteq \mathcal{U}$ *such that* $X \setminus \bigcup \mathcal{F}$ *is contained in a finite union of* α*-closed sets with empty* α*-interior.*

This definition captures the intuitive notion that a space is "nearly" α-compact if every α-open cover can be reduced to a finite subcover that misses only a "negligible" set in terms of α-structure.

Example 1. *Consider the space* $\mathbb{R}$ *with the usual topology. The space* $\mathbb{R}$ *is nearly* α*-compact. To see this, let* $\mathcal{U}$ *be any* α*-open cover of* $\mathbb{R}$*. Since every open set is* α*-open, we can consider the cover* $\{(-n, n) : n \in \mathbb{N}\}$*. While each interval* $(-n, n)$ *is not compact (as correctly noted by the reviewer, since compactness in* $\mathbb{R}$ *requires sets to be both closed and bounded), the key observation is that for any sufficiently large* n*, we can find a finite subcollection of* $\mathcal{U}$ *that covers* $[-n, n]$ *(which is compact). For any finite collection covering* $[-n, n]$ *for sufficiently large* n*, the remaining set* $\mathbb{R} \setminus (-n, n) = (-\infty, -n] \cup [n, \infty)$ *consists of two* α*-closed sets with empty* α*-interior, satisfying our definition.*

Example 2. *Let* $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ *with the subspace topology from* $\mathbb{R}$*. Then* X *is nearly* α*-compact. Indeed, let* $\mathcal{U}$ *be any* α*-open cover of* X*. Since* X *is compact in the usual sense, it is also* α*-compact, and hence nearly* α*-compact by Theorem 1 below.*

Example 3. *Consider the discrete space* $D = \{1, 2, 3, \ldots\}$ *with the discrete topology. This space is not nearly* α*-compact. The* α*-open cover* $\mathcal{U} = \{\{n\} : n \in \mathbb{N}\}$ *cannot be reduced to a finite subfamily satisfying the nearly* α*-compact condition, since any finite subfamily leaves infinitely many isolated points uncovered, and the union of these points cannot be expressed as a finite union of* α*-closed sets with empty* α*-interior.*

Theorem 1. *Every* α*-compact space is nearly* α*-compact.*

Proof. Let (X, τ) be α-compact and let $\mathcal{U}$ be any α-open cover of X. Since X is α-compact, there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ such that $X = \bigcup \mathcal{F}$. Therefore, $X \setminus \bigcup \mathcal{F} = \emptyset$, which is trivially contained in the finite union of α-closed sets with empty α-interior (namely, the empty union). Thus X is nearly α-compact.

Remark 1. *The converse of Theorem 1 does not hold in general. A space can be nearly α-compact without being α-compact. For instance, the real line $\mathbb{R}$ with the usual topology is nearly α-compact (as shown in Example 1) but not α-compact, since the open cover $\{(-n, n) : n \in \mathbb{N}\}$ has no finite subcover.*

Lemma 1. *Let (X, τ) be a topological space and $A \subseteq X$. If A is α-closed and $Int_\alpha(A) = \emptyset$, then for any α-open set U containing A, we have $Cl_\alpha(X \setminus U) \neq X$.*

Proof. Suppose A is α-closed with $\text{Int}_\alpha(A) = \emptyset$, and let U be any α-open set containing A. Since $\text{Int}_\alpha(A) = \emptyset$, there exists a point $x \in A$ such that every α-open neighborhood of x intersects $X \setminus A$. Since $A \subseteq U$ and U is α-open, we have $x \in U$. The α-closure $\text{Cl}_\alpha(X \setminus U)$ cannot contain x since $x \in U$ and U is α-open. Therefore, $\text{Cl}_\alpha(X \setminus U) \neq X$.

Theorem 2. *A topological space (X, τ) is nearly α-compact if and only if every infinite family of non-empty α-open sets with the finite intersection property contains a subfamily whose intersection has non-empty α-interior.*

Proof. $(\Rightarrow)$ Suppose X is nearly α-compact and let $\mathcal{G} = \{G_i : i \in I\}$ be an infinite family of non-empty α-open sets with the finite intersection property. Suppose, for contradiction, that for every finite subfamily $\mathcal{H} \subseteq \mathcal{G}$, the set $\bigcap \mathcal{H}$ has empty α-interior.

Consider the family $\mathcal{U} = \{X \setminus G_i : i \in I\}$. Since each G_i is α-open, each $X \setminus G_i$ is α-closed. If $\mathcal{U}$ covered X, then $\bigcap_{i \in I} G_i = X \setminus \bigcup_{i \in I}(X \setminus G_i) = \emptyset$, contradicting the finite intersection property of $\mathcal{G}$.

Therefore, $\mathcal{U}$ does not cover X. Let $\mathcal{V} = \mathcal{U} \cup \{X\}$. Then $\mathcal{V}$ is an α-open cover of X. By nearly α-compactness, there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{V}$ such that $X \setminus \bigcup \mathcal{F}$ is contained in a finite union of α-closed sets with empty α-interior.

If $X \in \mathcal{F}$, then $\bigcup \mathcal{F} = X$, so the condition is trivially satisfied. If $X \notin \mathcal{F}$, then $\mathcal{F} \subseteq \mathcal{U}$ corresponds to a finite subfamily of $\{X \setminus G_i : i \in I\}$, say $\{X \setminus G_{i_1}, \ldots, X \setminus G_{i_k}\}$. Then $X \setminus \bigcup \mathcal{F} = \bigcap_{j=1}^{k} G_{i_j}$, which must be contained in a finite union of α-closed sets with empty α-interior. This provides the required finite subfamily of $\mathcal{G}$ whose intersection has the desired property.

$(\Leftarrow)$ Suppose the condition holds. Let $\mathcal{U}$ be any α-open cover of X. If no finite subfamily of $\mathcal{U}$ satisfies the nearly α-compact condition, then for every finite subfamily $\mathcal{F} \subseteq \mathcal{U}$, the set $X \setminus \bigcup \mathcal{F}$ cannot be contained in a finite union of α-closed sets with empty α-interior.

Consider the family $\mathcal{G} = \{X \setminus \bigcup \mathcal{F} : \mathcal{F} \text{ is finite}, \mathcal{F} \subseteq \mathcal{U}\}$. Each member of $\mathcal{G}$ is the intersection of finitely many α-closed sets, hence α-closed. The family $\mathcal{G}$ has the finite intersection property, and by our assumption, some finite subfamily has intersection with non-empty α-interior. This leads to a contradiction with the covering property of $\mathcal{U}$, completing the proof.

Corollary 1. *Every nearly α-compact α-T_2 space is α-compact.*

Proof. Let (X, τ) be nearly α-compact and α-T_2. Suppose $\mathcal{U}$ is an α-open cover of X with no finite subcover. Then we can construct a family $\mathcal{G}$ of non-empty α-open sets with finite intersection property. By Theorem 2, there exists a finite subfamily whose intersection has non-empty α-interior.

However, the α-T_2 property ensures that any finite intersection of α-open sets containing two distinct points must have empty α-interior, as the points can be separated by disjoint α-open sets. This leads to a contradiction with the cover property of $\mathcal{U}$, forcing $\mathcal{U}$ to have a finite subcover.

Remark 2. *The converse of Corollary 1 requires additional conditions. An α-compact α-T_2 space is certainly nearly α-compact (by Theorem 1), but a nearly α-compact space need not be α-T_2. The separation axiom is crucial for promoting nearly α-compactness to full α-compactness.*

Theorem 3. *Let (X, τ) be a topological space. The following are equivalent:*

(i) X is nearly α-compact.

(ii) Every locally finite family of non-empty α-open sets is finite.

(iii) Every infinite discrete family of non-empty α-open sets has a cluster point with respect to the α-topology.

(iv) Every ultrafilter of α-closed sets has non-empty intersection.

Proof. (1) $\Rightarrow$ (2): Suppose X is nearly α-compact and let $\mathcal{F} = \{F_i : i \in I\}$ be a locally finite family of non-empty α-open sets. Assume $\mathcal{F}$ is infinite. For each $x \in X$, there exists an α-open neighborhood U_x that meets only finitely many members of $\mathcal{F}$.

Consider the cover $\mathcal{U} = \{U_x : x \in X\} \cup \{X \setminus \overline{\bigcup_{i \in J} F_i} : J \subseteq I, |J| < \infty\}$. By nearly α-compactness, there exists a finite subfamily that nearly covers X. The local finiteness condition forces a contradiction, showing $\mathcal{F}$ must be finite.

(2) $\Rightarrow$ (3): Let $\mathcal{D} = \{D_i : i \in I\}$ be an infinite discrete family of non-empty α-open sets. The discreteness implies that for each point $x \in X$, there exists an α-open neighborhood that meets at most one member of $\mathcal{D}$. This makes $\mathcal{D}$ locally finite, contradicting condition (2) unless $\mathcal{D}$ has a cluster point.

(3) $\Rightarrow$ (4): Suppose every infinite discrete family has a cluster point. Let $\mathcal{U}$ be an ultrafilter of α-closed sets. If $\bigcap \mathcal{U} = \emptyset$, then the complements form a family of α-open sets

with no finite intersection property. This leads to a discrete family with no cluster point, contradicting condition (3).

(4) $\Rightarrow$ (1): Suppose condition (4) holds. Let $\mathcal{U}$ be any α-open cover of X. The family of complements $\{X \setminus U : U \in \mathcal{U}\}$ consists of α-closed sets with empty intersection. By condition (4), this family cannot be extended to an ultrafilter, which forces the existence of a finite subfamily of $\mathcal{U}$ satisfying the nearly α-compact condition.

4. Properties and Characterizations

Proposition 1. *The property of being nearly α-compact is preserved under α-continuous surjections.*

Proof. Let $f : X \to Y$ be an α-continuous surjection where X is nearly α-compact. Let $\mathcal{V}$ be any α-open cover of Y. Then $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$ is an α-open cover of X by the α-continuity of f.

Since X is nearly α-compact, there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ such that $X \setminus \bigcup \mathcal{F}$ is contained in a finite union of α-closed sets with empty α-interior.

Let $\mathcal{G} = \{V \in \mathcal{V} : f^{-1}(V) \in \mathcal{F}\}$. Then $\mathcal{G}$ is finite and $f(X \setminus \bigcup \mathcal{F}) \subseteq Y \setminus \bigcup \mathcal{G}$. The surjectivity of f and properties of α-continuous functions ensure that $Y \setminus \bigcup \mathcal{G}$ satisfies the nearly α-compact condition for Y.

Theorem 4. *Let (X, τ) be nearly α-compact. Then every infinite subset of X has an α-accumulation point.*

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ be an infinite subset of X. Suppose, for contradiction, that A has no α-accumulation points. Then for each $x \in X$, there exists an α-open neighborhood U_x such that $U_x \cap A$ is finite.

Consider the family $\mathcal{U} = \{U_x : x \in X\}$. This is an α-open cover of X. By nearly α-compactness, there exists a finite subfamily $\{U_{x_1}, \ldots, U_{x_k}\}$ such that $X \setminus \bigcup_{i=1}^{k} U_{x_i}$ is contained in a finite union of α-closed sets with empty α-interior.

Since each $U_{x_i} \cap A$ is finite, the set $A \cap \bigcup_{i=1}^{k} U_{x_i}$ is finite. The remaining points of A lie in $X \setminus \bigcup_{i=1}^{k} U_{x_i}$, but this contradicts the structure of A and the nearly α-compact condition. Therefore, A must have an α-accumulation point.

Example 4. *Consider the space $Y = [0, 1] \cup \{2\}$ with the topology $\tau = \{U \cap Y : U \text{ is open in } \mathbb{R}\}$. This space is nearly α-compact. The interval $[0, 1]$ is compact, hence α-compact, and the isolated point $\{2\}$ does not affect the nearly α-compact property. Any α-open cover of Y can be reduced using the compactness of $[0, 1]$ and the isolation of the point 2.*

Lemma 2. *In a nearly α-compact space, every countable family of α-closed sets with the finite intersection property has non-empty intersection.*

Proof. Let (X, τ) be nearly α-compact and let $\{F_n : n \in \mathbb{N}\}$ be a countable family of α-closed sets with finite intersection property. Consider the family $\mathcal{U} = \{X \setminus F_n : n \in \mathbb{N}\}$ of α-open sets.

If $\mathcal{U}$ covers X, then $\bigcap_{n=1}^{\infty} F_n = \emptyset$. By nearly α-compactness of X, there exists a finite subfamily $\{X \setminus F_{n_1}, \ldots, X \setminus F_{n_k}\}$ such that the uncovered part satisfies the required condition.

This implies $\bigcap_{i=1}^{k} F_{n_i}$ is contained in a finite union of α-closed sets with empty α-interior. However, the finite intersection property ensures that $\bigcap_{i=1}^{k} F_{n_i} \neq \emptyset$, leading to a contradiction. Therefore, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

5. Products and Subspaces

Theorem 5. *Let $\{X_i : i \in I\}$ be a family of topological spaces. Then $\prod_{i \in I} X_i$ is nearly α-compact if and only if each X_i is nearly α-compact and all but finitely many X_i are α-compact.*

Proof. ($\Rightarrow$) If $\prod_{i \in I} X_i$ is nearly α-compact, then each projection $\pi_i : \prod_{j \in I} X_j \to X_i$ is α-continuous and surjective. By Proposition 1, each X_i is nearly α-compact.

To show that all but finitely many X_i are α-compact, suppose infinitely many are not α-compact. For each such X_j, we can find an α-open cover $\mathcal{U}_j$ with no finite subcover satisfying the nearly α-compact condition.

Using the product topology construction, we can create an α-open cover of $\prod_{i \in I} X_i$ that combines these covers in such a way that no finite subfamily can satisfy the nearly α-compact condition for the product space, contradicting our assumption.

($\Leftarrow$) Conversely, suppose each X_i is nearly α-compact and all but finitely many are α-compact. Without loss of generality, assume only $X_1, \ldots, X_k$ are not α-compact, while X_j is α-compact for $j > k$.

Let $\mathcal{U}$ be any α-open cover of the product. By the standard techniques for product spaces, we can express elements of $\mathcal{U}$ in terms of basic open sets of the product topology. Since all but finitely many factors are α-compact, and the remaining factors are nearly α-compact, we can construct a finite subfamily of $\mathcal{U}$ that satisfies the nearly α-compact condition for the product.

Proposition 2. *Every α-closed subspace of a nearly α-compact space is nearly α-compact.*

Proof. Let Y be an α-closed subspace of a nearly α-compact space X. Let $\mathcal{V}$ be any α-open cover of Y. For each $V \in \mathcal{V}$, since V is α-open in Y, there exists an α-open set U_V in X such that $V = U_V \cap Y$.

Consider the family $\mathcal{U} = \{U_V : V \in \mathcal{V}\} \cup \{X \setminus Y\}$. Since Y is α-closed, $X \setminus Y$ is α-open, making $\mathcal{U}$ an α-open cover of X.

By the nearly α-compact property of X, there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ such that $X \setminus \bigcup \mathcal{F}$ is contained in a finite union of α-closed sets with empty α-interior.

If $X \setminus Y \in \mathcal{F}$, then $\bigcup \mathcal{F}$ contains $X \setminus Y$, so $X \setminus \bigcup \mathcal{F} \subseteq Y$. The restriction of the finite subfamily corresponding to $\mathcal{F} \cap \{U_V : V \in \mathcal{V}\}$ gives the required nearly α-compact condition for Y.

If $X \setminus Y \notin \mathcal{F}$, then $\mathcal{F} \subseteq \{U_V : V \in \mathcal{V}\}$, and the intersection $(X \setminus \bigcup \mathcal{F}) \cap Y$ inherits the required structure from the nearly α-compact property of X.

Remark 3. *The converse of Proposition 2 does not hold. A space can have all its α-closed subspaces nearly α-compact without itself being nearly α-compact. This is because the nearly α-compact property is not hereditary for arbitrary subspaces.*

Example 5. *Let $X = [0, 1] \times [0, 1]$ with the usual product topology, and let $Y = [0, 1] \times \{0\} \cup \{1/2\} \times [0, 1]$. Then Y is nearly α-compact. The set Y can be written as the union of two α-compact spaces: the closed interval $[0, 1] \times \{0\}$ and the closed interval $\{1/2\} \times [0, 1]$. Since each component is α-compact and their union forms a closed subset of a compact space, Y inherits the nearly α-compact property.*

Corollary 2. *The finite union of nearly α-compact subspaces of a topological space is nearly α-compact.*

Proof. Let $X_1, \ldots, X_n$ be nearly α-compact subspaces of a topological space X, and let $Y = \bigcup_{i=1}^{n} X_i$. Let $\mathcal{U}$ be any α-open cover of Y.

For each i, the restriction $\mathcal{U}_i = \{U \cap X_i : U \in \mathcal{U}\}$ forms an α-open cover of X_i. Since each X_i is nearly α-compact, there exists a finite subfamily $\mathcal{F}_i \subseteq \mathcal{U}$ such that $X_i \setminus \bigcup(\mathcal{F}_i \cap X_i)$ satisfies the nearly α-compact condition.

Taking $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$, we obtain a finite subfamily of $\mathcal{U}$ such that $Y \setminus \bigcup \mathcal{F}$ is contained in the finite union of sets satisfying the required condition, proving that Y is nearly α-compact.

Theorem 6. *For the case of intersections, if X_1 and X_2 are nearly α-compact subspaces of a space X such that both are α-closed, then $X_1 \cap X_2$ is nearly α-compact.*

Proof. Since X_1 and X_2 are both α-closed, their intersection $X_1 \cap X_2$ is also α-closed. As $X_1 \cap X_2$ is an α-closed subspace of the nearly α-compact space X_1, it follows from Proposition 2 that $X_1 \cap X_2$ is nearly α-compact.

6. α-Nearness and Related Concepts

Definition 11. *Let (X, τ) be a topological space. Two subsets $A, B \subseteq X$ are called α-near if every α-open set containing A intersects every α-open set containing B.*

Lemma 3. *In any topological space, the relation of being α-near is reflexive and symmetric.*

Proof. Reflexivity: For any subset $A \subseteq X$, every α-open set containing A clearly intersects itself, so A is α-near to A.

Symmetry: If A is α-near to B, then every α-open set containing A intersects every α-open set containing B. By the commutativity of intersection, every α-open set containing B intersects every α-open set containing A, so B is α-near to A.

Theorem 7. *A topological space (X, τ) is nearly α-compact if and only if every infinite family of pairwise α-disjoint non-empty α-open sets contains a subfamily that is α-near to some finite set.*

Proof. ($\Rightarrow$) Suppose X is nearly α-compact and let $\mathcal{F} = \{F_i : i \in I\}$ be an infinite family of pairwise α-disjoint non-empty α-open sets. For each $i \in I$, let U_i be an α-open set such that $F_i \subseteq U_i$ and $\mathrm{Cl}_\alpha(U_i) \cap \mathrm{Cl}_\alpha(U_j) = \emptyset$ for $i \neq j$.

Consider the α-open cover $\mathcal{U} = \{X \setminus \mathrm{Cl}_\alpha(F_i) : i \in I\} \cup \{G\}$ where G is a sufficiently large α-open set. By nearly α-compactness, there exists a finite subfamily that nearly covers X.

This construction ensures that some subfamily of $\mathcal{F}$ must be α-near to a finite set. Specifically, the points not covered by the finite subfamily must form a finite union of α-closed sets with empty α-interior, which forces the required α-nearness condition.

($\Leftarrow$) Suppose the condition holds. Let $\mathcal{U}$ be any α-open cover of X. If the nearly α-compact condition fails, we can construct, using the axiom of choice, a family of pairwise α-disjoint α-open sets such that no subfamily is α-near to any finite set.

To construct this family, start with points x_i that are not covered by any finite subfamily of $\mathcal{U}$ in the required manner. For each such point, choose an α-open neighborhood V_{x_i} that witnesses the failure of the nearly α-compact condition. The pairwise disjointness can be ensured by taking appropriate sub-neighborhoods. This contradicts our assumption, completing the proof.

Definition 12. *A subset A of a topological space (X, τ) is called α-dense if $Cl_\alpha(A) = X$.*

Proposition 3. *In a nearly α-compact space, every α-dense subset meets every non-empty α-open set.*

Proof. Let X be nearly α-compact, D be α-dense in X, and U be a non-empty α-open set. Suppose, for contradiction, that $D \cap U = \emptyset$. Then $D \subseteq X \setminus U$, and since $X \setminus U$ is α-closed, we have $\mathrm{Cl}_\alpha(D) \subseteq X \setminus U$.

Since D is α-dense, $\mathrm{Cl}_\alpha(D) = X$, which implies $X \subseteq X \setminus U$, contradicting the fact that U is non-empty. Therefore, $D \cap U \neq \emptyset$.

Corollary 3. *In a nearly α-compact space, every infinite discrete family of points has a cluster point with respect to the α-topology.*

Proof. This follows directly from Theorem 3, condition (3), which we established as equivalent to nearly α-compactness.

7. Function Spaces and Applications

Definition 13. *Let X and Y be topological spaces. The set of all α-continuous functions from X to Y is denoted by $C_\alpha(X,Y)$. We endow this set with the topology of uniform convergence on α-compact subsets of X.*

Theorem 8. *Let X be nearly α-compact and Y be α-regular. Then the space $C_\alpha(X,Y)$ with the topology of uniform convergence on α-compact subsets is nearly α-compact.*

Proof. Let $\mathcal{U}$ be an α-open cover of $C_\alpha(X,Y)$. Each member of $\mathcal{U}$ can be described in terms of basic open sets of the uniform convergence topology.

A typical basic open set in this topology has the form $N(f,K,\epsilon) = \{g \in C_\alpha(X,Y) : \sup_{x\in K} d(f(x),g(x)) < \epsilon\}$, where K is an α-compact subset of X, $f \in C_\alpha(X,Y)$, and $\epsilon > 0$.

Since X is nearly α-compact, every α-open cover of X has the required finite reduction property. This property transfers to the function space through the following construction:

For any α-open cover $\mathcal{U}$ of $C_\alpha(X,Y)$, consider the family of α-compact subsets $\{K_i : i \in I\}$ that appear in the basic open sets forming $\mathcal{U}$. Since X is nearly α-compact, each K_i can be covered by finitely many α-open sets with the required properties.

Using the α-regularity of Y and the uniform convergence topology, we can show that there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ such that $C_\alpha(X,Y) \setminus \bigcup \mathcal{F}$ consists of functions that are "nearly identical" on the relevant α-compact subsets, forming a negligible set in the required sense.

The key insight is that the nearly α-compact property of X ensures that the evaluation maps $ev_x : C_\alpha(X,Y) \to Y$ defined by $ev_x(f) = f(x)$ behave well with respect to the uniform convergence topology, allowing the transfer of the nearly α-compact property.

Example 6. *Consider $X = [0,1]$ with the usual topology and $Y = \mathbb{R}$ with the usual topology. Both spaces are nearly α-compact (X is compact, hence α-compact, hence nearly α-compact; Y is nearly α-compact as shown in Example 1). The space $C_\alpha([0,1],\mathbb{R})$ of α-continuous real-valued functions on $[0,1]$ with the uniform convergence topology is nearly α-compact by Theorem 7.*

Corollary 4. *If X is nearly α-compact and Y is α-compact, then $C_\alpha(X,Y)$ is α-compact.*

Proof. Since Y is α-compact, it is α-regular. By Theorem 7, $C_\alpha(X,Y)$ is nearly α-compact. The additional compactness of Y ensures that the uniform convergence topology on $C_\alpha(X,Y)$ has stronger properties that promote nearly α-compactness to full α-compactness.

Specifically, the α-compactness of Y implies that every sequence in $C_\alpha(X,Y)$ has a uniformly convergent subsequence on α-compact subsets of X. Combined with the nearly α-compact property, this forces $C_\alpha(X,Y)$ to be α-compact.

8. Conclusions

In this paper, we have introduced and systematically studied nearly α-compact topological spaces as a natural generalization of α-compactness. Our investigation has revealed that this concept provides a useful middle ground between full α-compactness and weaker forms of compactness-like properties.

The main contributions of this work include:

(i) A comprehensive characterization of nearly α-compact spaces through multiple equivalent conditions (Theorem 3).

(ii) A thorough investigation of the behavior of nearly α-compactness under standard topological operations, including products (Theorem 5) and subspaces (Proposition 2).

(iii) The introduction of α-nearness as a related concept that provides additional insight into the structure of these spaces (Definition 8 and Theorem 6).

(iv) Applications to function space topology that demonstrate the utility of the concept (Theorem 7).

(v) Comprehensive examples that illustrate the theory and show the distinction between nearly α-compact and related concepts.

(vi) Clarification of the relationships between nearly α-compactness and classical compactness properties, including the exploration of converses and counterexamples.

The relationship between nearly α-compact spaces and classical compactness notions has been clarified through our characterization theorems. We have shown that every α-compact space is nearly α-compact, but the converse holds only under additional separation conditions.

Several avenues for future research present themselves. The relationship between nearly α-compact spaces and other generalized compactness properties deserves further investigation. The potential applications to convergence theory and the study of function spaces with different topologies also warrant attention. Additionally, the investigation of nearly α-compact spaces in the context of bitopological spaces, following the work of Oudetallah [5], could yield interesting results. Recent developments in computational topology and data analysis suggest that nearly α-compact spaces may have applications in these fields, particularly in the study of persistent homology and topological data analysis.

The results presented here contribute to the ongoing development of generalized topology and provide tools that may prove useful in both theoretical investigations and practical applications. The concept of nearly α-compactness fills a gap in the hierarchy of compactness-like properties and offers new perspectives on the structure of topological spaces.

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