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Paley, Cubic Paley, Quadruple Paley, and Generalized Paley Graphs with an Edge Graceful Labeling

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Abstract. The Paley graph $\mathcal{P}_q$ is a simple, connected, strongly regular graph with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$, where $V(\mathcal{P}_q)$ is the finite field F_q of order $q = p^n$, p is an odd prime, $n \in \mathbb{N}$, and $q \equiv 1 \pmod 4$. In Paley graphs, two vertices are adjacent if their difference is a quadratic residue $\pmod q$. The vertex set of the generalized Paley graph $m - \mathcal{P}_q$, where $m \geq 3$ is an odd integer, is $V(m - \mathcal{P}_q) = F_q$ and the set of edges is $E(m - \mathcal{P}_q) = \{(x, y) \Leftrightarrow x - y \in (F_q^*)^m\}$. In 1985, *edge-graceful labeling* was first introduced by Lo [1]. A graph G with order n and size m is called *edge-graceful* if there exists a bijective mapping $f : E(G) \longrightarrow \{1, 2, 3, \ldots, m\}$ such that the weights map $f_w : V(G) \longrightarrow \{0, 1, 2, \ldots, n-1\}$, given by $f_w(u) = \sum_{v \in N(u)} f(uv) \pmod n$, is one-to-one and onto. In this paper, we prove that Paley graphs and generalized Paley graphs of prime order are edge graceful, edge-even graceful, and edge-odd graceful graphs.

2020 Mathematics Subject Classifications: 05C25, 05C78
Key Words and Phrases: Paley graphs, cubic Paley graphs, quadruple Paley graphs, generalized Paley graphs, edge-graceful labeling, edge-even graceful labeling, edge-odd graceful labeling

1. Introduction

A finite field F_q of order q, where q is a prime power congruent to 1 modulo 4, is used in the construction of the Paley graph as its vertex set. Two vertices are adjacent if and only if their difference is a non-zero square in the field.

In this paper, we are concerned with the simple form of Paley graphs where $F_q = \mathbb{Z}_p$ is the field of integers $\pmod p$. We are also interested in the generalized Paley graph $m - \mathcal{P}_q$, where $V(m - \mathcal{P}_q) = F_q$ is the finite field with $q = p^n$ elements, p is an odd prime, and $n \in \mathbb{N}$. The edge set is $E(m - \mathcal{P}_q) = \{(x, y) \Leftrightarrow x - y \in (F_q^*)^m\}$, where $m \geq 3$ is an odd integer.

Paley graphs have several applications, for instance, in network design [2]. Also, Paley graphs form an important family in graph theory due to their rich structural properties. One of their key features is that Paley graphs are strongly regular with parameters

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$(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$. This means that the graph has order q, is $\frac{q-1}{2}-$regular, each pair of adjacent vertices has $\frac{q-5}{4}$ common neighbors, and each pair of non-adjacent vertices has $\frac{q-1}{4}$ common neighbors. Furthermore, Paley graphs are self-complementary, symmetric, and Hamiltonian, for more details (see [3], [4]). Paley graphs possess not only a Hamiltonian property but also a Hamiltonian decomposition; that is, the edge set can be represented as the union of edge-disjoint Hamiltonian cycles, see [5]. Paley graphs provide intriguing characteristics that make them valuable in graph theory and allow the application of graph-theoretic techniques to the number theory involving quadratic residues (see [6], [7], [8], [9]).

Another active area of research in graph theory concerns graph coloring and the clique number. Several papers have investigated the clique number of Paley graphs and their generalizations (see [10], [11], [12], [13], [14]).

Graph labeling is another important branch of graph theory, in which labels are assigned to the vertices, edges, or both, under certain conditions, see [15]. One prominent type of labeling that has attracted significant attention is edge-graceful labeling, which has important applications in various areas, including computer networks, coding theory, radar, circuit design, and communication systems.

In 1967, Alexander Rosa [16] identified three types of labeling: α-labeling, β-labeling, and ρ-labeling. Later, β-labeling was renamed graceful labeling by Solomon Golomb [17]. In an edge-graceful labeling of a graph G distinct integers are assigned to the edges, and each vertex is labeled, using an injective map, by the sum of the labels on its incident edges modulo p. For more information about different types of labeling (see [18], [19], [20], [21],[22]).

In this paper, we prove that Paley graphs and generalized Paley graphs of prime order are edge-graceful.

2. Paley graphs

Paley graphs have many important applications in coding theory and cryptography. For example they are used in quantum secret sharing domain [23] and in construction of codes from their incidence matrices and line graphs [24]. Moreover, a new application of these graphs in cryptography was proposed in [25], introducing an efficient algorithm for the encryption and decryption of sensitive messages by converting them into binary form using Paley graphs.

Definition 1. *Let $q = p^n$ be a prime power, where $q \equiv 1 \ (mod\, 4)$. A Paley graph $\mathcal{P}_q$ of order q is a graph with vertex set $V(\mathcal{P}_q) = \mathcal{F}_q$, where $\mathcal{F}_q$ is the finite field with q elements, and edge set $E(\mathcal{P}_q) = \{(u, v) \mid u - v \in (\mathcal{F}_q^*)^2\}$.*

Note that the condition $q \equiv 1 \ (mod\, 4)$ is necessary for the graph to be undirected. A simple version of the Paley graph is obtained when $n = 1$. That is, when $\mathcal{F}_q = \mathbb{Z}_p$, the field of integers $\pmod p$.

Example 1. *The Paley graph $\mathcal{P}_{13}$ has vertex set $V(\mathcal{P}_{13}) = \mathbb{Z}_{13} = \{0, 1, 2, \ldots, 12\}$, where two vertices $u, v \in \mathbb{Z}_{13}$ are adjacent if $u - v \in \{1, 3, 4, 9, 10, 12\}$.*

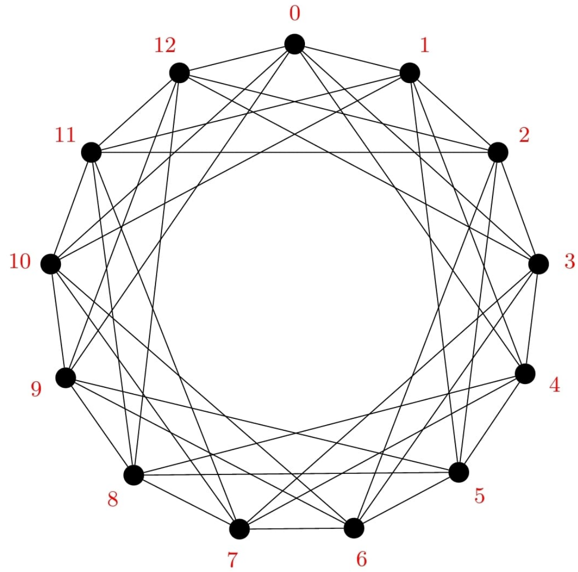

Figure 1: The Paley Graph $\mathcal{P}_{13}$ of Order 13.

Example 2. *Consider the Paley graph $\mathcal{P}_9$ of order 9, then $V(\mathcal{P}_9) = \mathcal{F}_9 = \mathbb{Z}_3[x]/(x^2+1) = \{0, 1, 2, a+2, a, a+1, 2a+1, 2a+2, 2a\}$. Therefore, $(\mathcal{F}_9^*)^2 = \{1, 2, a, 2a\}$, meaning that every vertex in $V(\mathcal{P}_9)$ is adjacent to four vertices. Hence, $E(\mathcal{P}_9) = \{(x_i, x_i + x_j) \mid x_i \in \mathcal{F}_9, x_j \in (\mathcal{F}_9^*)^2\}$.*

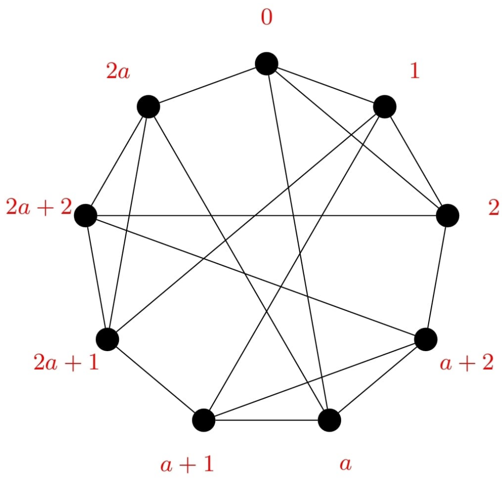

Figure 2: The Paley Graph $\mathcal{P}_9$ of Order 9.

Graph labeling is one of the significant fields in graph theory. It has many applications in coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, and data base management (see [26], [27]).

In [21], Kamaraj and Thangakani investigated edge-even and edge-odd graceful labeling for Paley graphs of prime order. An edge even-graceful labeling assigns distinct even numbers to the edges and ensures that the sum of labels modulo $2k$ assigned to each vertex is unique. On the other hand, an edge-odd graceful labeling assigns distinct odd numbers to the edges and ensures that the sum of labels modulo $2q$ assigned to each vertex is unique. Kamaraj and Thangakani proved the following theorem:

Theorem 1. *Every Paley graph of prime order admits both an edge-even and an edge-odd graceful labeling.*

We introduce another lebeling of Paley graphs of prime order, which is called edge graceful labeling.

Definition 2. *Accordung to [1], a graph* $\mathcal{G}$ *of order* n *and size* m *is called edge-graceful if there exists a bijective mapping* $h : E(\mathcal{G}) \longrightarrow \{1, 2, 3, \ldots, m\}$ *such that the weight function* $h_w : V(\mathcal{G}) \longrightarrow \{0, 1, 2, \ldots, n-1\}$*, given by* $h_w(u) = \sum_{v \in N(u)} h(uv) \pmod{n}$*, is one-to-one and onto.*

For example, consider the Paley graph $\mathcal{P}_9$. As in Figure 3, we can define an edge-graceful labeling to the Paley graph $\mathcal{P}_9$ such that $h : E(\mathcal{P}_9) \longrightarrow \{1, 2, 3, \ldots, 18\}$ and $h_w : V(\mathcal{P}_9) \longrightarrow \{0, 1, 2, \ldots, 8\}$ are bijections.

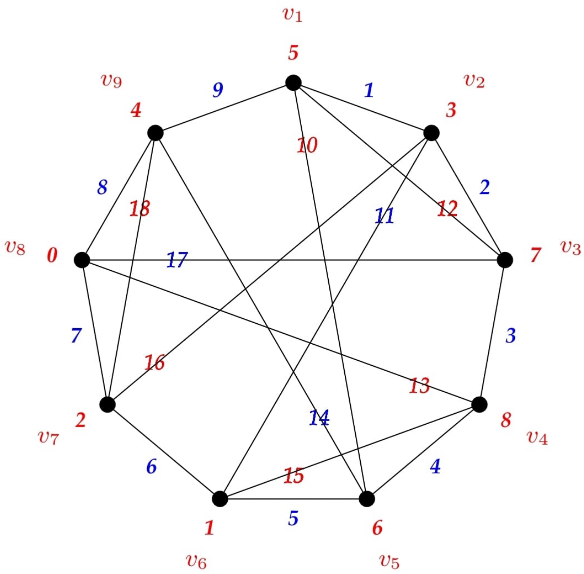

Figure 3: An edge-graceful labeling for $\mathcal{P}_9$

In the following we provide an algorithm which produces an edge-graceful labeling for the prime order Paley graphs. Note that the case of the prime power order is open.

2.1. Edge-graceful labeling Algorithm for Paley graph of prime order

Input: The Paley graph $\mathcal{P}_p$ with $V(\mathcal{P}_p) = \mathbb{Z}_p$ and $E(3 - \mathcal{P}_p) = \{(u, v) \mid u - v \in (\mathbb{Z}_p^*)^2\}$, where $p \equiv 1 \pmod 4$

(1) Rename the vertices of the graph as $0 := v_p, 1 := v_1, 2 := v_2, \ldots, p-1 := v_{p-1}$.
(2) Set $r = \frac{p-1}{4}$, and rewrite $(\mathbb{Z}_p^*)^2$ as
$(\mathbb{Z}_p^*)^2 = S = \{s_1, s_2, s_3, \ldots, s_{2r} : s_1 < s_2 < s_3 < \ldots < s_{2r}\}$.
(3) Partition S into two sets.
Let $S_1 = \{s_1, s_2, s_3, \ldots, s_r\}$ and $S_2 = \{s_{r+1}, s_{r+2}, s_{r+3}, \ldots, s_{2r}\}$.
Note that: $p-1$ is divisible by 4 and for any vertex $v_i \in \mathbb{Z}_p$ the vertex v_{i+s_j} is adjacent to v_i for all $s_j \in S$.
(4) If $s_j \in S_1$, the vertex v_{i+s_j} is placed in clockwise direction of v_i and if $s_j \in S_2$, the vertex v_{i+s_j} is placed in anticlockwise direction of v_i.
(5) Set $f(v_i, v_{i+s_j}) = 0$ for all $i \in \{1, 2, 3, \cdots, p\}, j \in \{1, 2, 3, \cdots, 2r\}$.
(6) Set $i = 1$.
Step 1: If $i \leq p$ then continue to Step 2.
Else jump to Step 4.
Step 2: For each $s_j \in S_1$, $f\left(v_i, v_{i+s_j}\right) = (j-1)\,p + i$.
Step 3: $i = i + 1$, go back to Step 1.
Step 4: For each $k = 1, 2, 3, \ldots, p$, find the weight of the vertex v_k using the following mapping: $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) = \sum_{j=1}^{r} 2[(j-1)p+k]+(p-s_j) = 2kr - L \pmod{p}$, where $L = \sum_{j=1}^{r} s_j$.

Theorem 2. *Every Paley graph of prime order admits an edge-graceful labeling.*

Proof. To prove that the algorithm defines an edge-graceful labeling, we need to prove that both functions f and f_w are bijections.

(i) Consider the function $f_w : V(\mathcal{P}_p) \longrightarrow \mathbb{Z}_p$ defined by $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) = 2kr - L \pmod{p}$. Now, we prove that f_w is one-to-one, which implies tat it is a bijection. Let v_x and v_y be two vertices in $V(\mathcal{P}_p)$, if $f_w(v_x) = f_w(v_y)$ then $2xr - L = 2yr - L \pmod{p}$ which leads to $x = y$.

(ii) The function $f : E(\mathcal{P}_p) \longrightarrow \{1, 2, 3, \ldots, rp\}$ is defined as $f(v_x, v_{x+s_j}) = (j-1)p + x$. To prove that f is a bijection, we need only to prove the injectivity of f. Let (v_x, v_{x+s_i}) and (v_y, v_{y+s_j}) be two edges, with $x, y \in \{1, 2, 3, \cdots, p\}$, $s_i, s_j \in S = (\mathbb{Z}_p^*)^2$, and $f(v_x, v_{x+s_j}) = f(v_y, v_{y+s_i})$, which implies that $(i-1)p + x = (j-1)p + y$. In case of $i = j$ or $x = y$ the proof is trivial. The last case if $x \neq y$ and $j \neq i$, here we will find that $x - y = (j-i)p$ but $|x-y| < p$ and in the same time $|p(i-j)| \geq p$ which is a contradiction. So, from these three cases, we can be sure that the function f is a one-to-one function. □

Example 3. *We apply the previous algorithm to show that: the Paley graph $\mathcal{P}_{13}$ is an edge-graceful graph, where $V(\mathcal{P}_{13}) = \{v_1, v_2, \cdots, v_{13}\}$ and $|E(\mathcal{P}_{13})| = \frac{13 \cdot 12}{4} = 39$. Figure 4 illustrates the edge-graceful labeling for $\mathcal{P}_{13}$.*

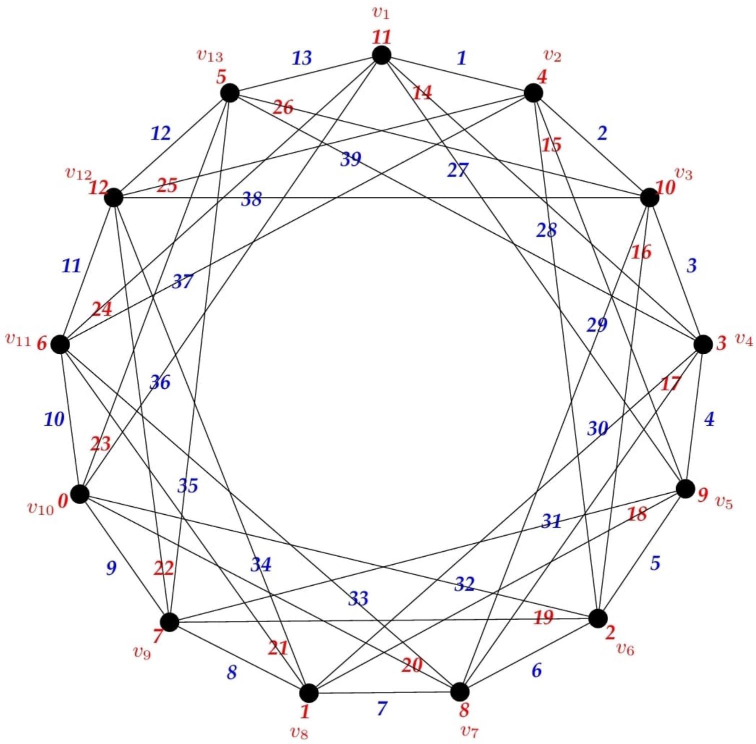

Figure 4: An edge-graceful labeling for $\mathcal{P}_{13}$

Input: The graph $\mathcal{P}_{13}$.

Here we have $p = 13, r = 3$, and $(\mathbb{Z}_{13}^*)^2 = \{1^2, 2^2, 3^2, \cdots, 12^2\} = \{1, 4, 9, 3, 12, 10\}$.

(1) Rename the vertices of the graph as $0 := v_{13}, 1 := v_1, 2 := v_2, \ldots, 12 := v_{12}$.

(2) Rewrite $(\mathbb{Z}_{13}^*)^2$ as

$(\mathbb{Z}_{13}^*)^2 = S = \{s_1, s_2, s_3, s_4, s_5, s_6 : s_1 < s_2 < s_3 < s_4 < s_5 < s_6\} = \{1, 3, 4, 9, 10, 12\}$.

(3) Partition S into two sets.

Let $S_1 = \{s_1, s_2, s_3\} = \{1, 3, 4\}$ and $S_2 = \{s_4, s_5, s_6\} = \{9, 10, 12\}$.

Note that: for any vertex $v_i \in \mathbb{Z}_{13}$ the vertices $v_{i+1}, v_{i+3}, v_{i+4}, v_{i+9}, v_{i+10}, v_{i+12}$ are adjacent to v_i.

(4) The vertices $v_{i+1}, v_{i+3}, v_{i+4}$ are placed in clockwise direction of v_i and the vertices $v_{i+9}, v_{i+10}, v_{i+12}$ are placed in anticlockwise direction of v_i.

(5) Set $f(v_i, v_{i+s_j}) = 0$ for all $i \in \{1, 2, 3, \ldots, 13\}, j \in \{1, 2, 3, \ldots, 6\}$.

(6) Set $i = 1$.

Step 1: $i = 1 \leq 13$, then continue to Step 2.

Step 2: For each $s_1, s_2, s_3 \in S_1$, $f\left(v_1, v_{1+s_j}\right) = (j-1)\,p + 1$.

That is, $f(v_1, v_2) = (1-1)13 + 1 = 1, f(v_1, v_4) = (2-1)13 + 1 = 14, f(v_1, v_5) = (3-1)13 + 1 = 27$.

Step 3: $i = 1 + 1 = 2$, go back to Step 1.

Step 1: $i = 2 \leq 13$, then continue to Step 2.

Step 2: For each $s_1, s_2, s_3 \in S_1$, $f\left(v_2, v_{2+s_j}\right) = (j-1)\,p + 2$.

That is, $f(v_2, v_3) = (1-1)13+2 = 2, f(v_2, v_5) = (2-1)13+2 = 15, f(v_2, v_6) = (3-1)13+2 = 28$.

Step 3: $i = 2+1 = 3$, go back to Step 1. We repeat Step1, Step2, and Step3 and get: $f(v_3, v_4) = 3$, $f(v_3, v_6) = 16$, $f(v_3, v_7) = 29$, $f(v_4, v_5) = 4$, $f(v_4, v_7) = 17$, $f(v_4, v_8) = 30$, $f(v_5, v_6) = 5$, $f(v_5, v_8) = 18$, $f(v_5, v_9) = 31$, $f(v_6, v_7) = 6$, $f(v_6, v_9) = 19$, $f(v_6, v_{10}) = 32$, $f(v_7, v_8) = 7$, $f(v_7, v_{10}) = 20$, $f(v_7, v_{11}) = 33$, $f(v_8, v_9) = 8$, $f(v_8, v_{11}) = 21$, $f(v_8, v_{12}) = 34$, $f(v_9, v_{10}) = 9$, $f(v_9, v_{12}) = 22$, $f(v_9, v_{13}) = 35$, $f(v_{10}, v_{11}) = 10$, $f(v_{10}, v_{13}) = 23$, $f(v_{10}, v_1) = 36$, $f(v_{11}, v_{12}) = 11$, $f(v_{11}, v_1) = 24$, $f(v_{11}, v_2) = 37$, $f(v_{12}, v_{13}) = 12$, $f(v_{12}, v_2) = 25$, $f(v_{12}, v_3) = 38$, $f(v_{13}, v_1) = 13$, $f(v_{13}, v_3) = 26$, $f(v_{13}, v_4) = 39$.

Step 4: For each $k = 1, 2, 3, \ldots, 13$, the weight of the vertex v_k can be found by: $f_w(v_k) = \sum_{j=1}^{6} f(v_k, v_{k+s_j}) = 6k - \sum_{j=1}^{3} s_j = 6k - 8 \pmod{p}$. $f_w(v_1) = 6\cdot 1 - 8 = -2 \equiv 11 \pmod{13}$.

Equivalently, $f_w(v_1) = f(v_1, v_2) + f(v_1, v_4) + f(v_1, v_5) + f(v_{10}, v_1) + f(v_{11}, v_1) + f(v_{13}, v_1) = 1 + 14 + 27 + 36 + 24 + 13 \equiv 11 \pmod{13}$.

By the same function modulo 13 , we get: $f_w(v_2) = 4$, $f_w(v_3) = 10$, $f_w(v_4) = 3$, $f_w(v_5) = 9$, $f_w(v_6) = 2$, $f_w(v_7) = 8$, $f_w(v_8) = 1$, $f_w(v_9) = 7$, $f_w(v_{10}) = 0$, $f_w(v_{11}) = 6$, $f_w(v_{12}) = 12$, $f_w(v_{13}) = 5$.

3. Cubic Paley graphs

A natural generalization of Paley graph is the cubic Paley graph, which has the same vertex set, and two vertices are adjacent if their difference is a cubic residue in $\mathcal{F}_q$.

Definition 3. *[28] Let $q = p^n$, where p is an odd prime, and $n \in \mathbb{N}$, such that $q \equiv 1\ (mod\ 3)$. The graph $3{-}\mathcal{P}_q$, with $V(3{-}\mathcal{P}_q) = \mathcal{F}_q$ and $E(3{-}\mathcal{P}_q) = \left\{(u, v) \mid u - v \in (\mathcal{F}_q^*)^3\right\}$ is called the cubic Paley graph of order q.*

Note that: W. Ananchuen and L. Caccetta, in [28], stated that the condition $q \equiv 1\ (mod\ 3)$ is necessary to ensure that -1 is a cubic residue and, consequently, that the graph $3 - \mathcal{P}_q$ is well-defined. However, we disagree with this claim, since $(-1)^3 = -1$ in any field with any order, implying that -1 is always a cubic residue. Therefore, the graph $3 - \mathcal{P}_q$ is well-defined without the condition $q \equiv 1\ (mod\ 3)$.

Example 4. *The cubic Paley graph $3 - \mathcal{P}_{19}$ of order 19 has $V(3 - \mathcal{P}_{19}) = \mathbb{Z}_{19}$ and $E(3 - \mathcal{P}_{19}) = \{(u, v) \mid u - v \in \{1, 7, 8, 11, 12, 18\}\}$. See Figure 5.*

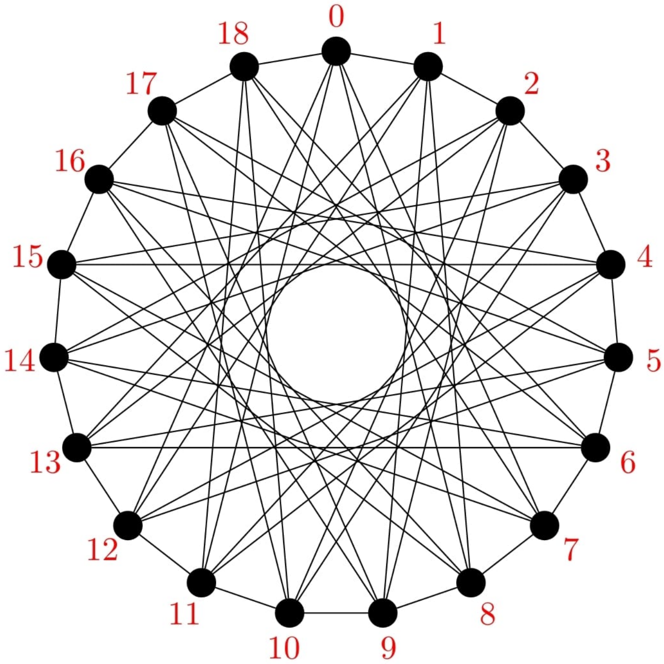

Figure 5: The cubic Paley graph $3-\mathcal{P}_{19}$

Now we provide an algorithm that produces an edge-graceful labeling for cubic Paley graphs of prime order

3.1. Edge-graceful labeling Algorithm for cubic Paley graphs of prime order

Input: The cubic Paley graph $3-\mathcal{P}_p$ with $V(3-\mathcal{P}_p)=\mathbb{Z}_p$ and $E(3-\mathcal{P}_p)=\{(u,v)\mid u-v\in(\mathbb{Z}_p^*)^3\}$, where p is any odd prime.

(1) Rename the vertices of the graph as $0:=v_p, 1:=v_1, 2:=v_2, \ldots, p-1:=v_{p-1}$.

(2) Set $r=\frac{p-1}{2d}$, where $d=gcd(3,p-1)$, and rewrite $(\mathbb{Z}_p^*)^3$ as
$(\mathbb{Z}_p^*)^3 = S = \{s_1, s_2, s_3, \ldots, s_{2r} : s_1 < s_2 < s_3 < \ldots < s_{2r}\}$.

(3) Partition S into two sets.
Let $S_1=\{s_1, s_2, s_3, \ldots, s_r\}$ and $S_2=\{s_{r+1}, s_{r+2}, s_{r+3}, \ldots, s_{2r}\}$.
Note that: $p-1$ is divisible by $2d$ and for any vertex $v_i\in\mathbb{Z}_p$ the vertex v_{i+s_j} is adjacent to v_i for all $s_j\in S$.

(4) If $s_j\in S_1$, the vertex v_{i+s_j} is placed in clockwise direction of v_i and if $s_j\in S_2$, the vertex v_{i+s_j} is placed in anticlockwise direction of v_i.

(5) Set $f(v_i, v_{i+s_j})=0$ for all $i\in\{1,2,3,\cdots,p\}, j\in\{1,2,3,\cdots,2r\}$.

(6) Set $i=1$.
Step 1: If $i\leq p$ then continue to Step 2.
Else jump to Step 4.
Step 2: For each $s_j\in S_1,\ f\left(v_i, v_{i+s_j}\right)=(j-1)\,p+i$.

Step 3: $i = i + 1$, go back to Step 1.

Step 4: For each $k = 1, 2, 3, \ldots, p$, find the weight of the vertex v_k using the following mapping: $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) = \sum_{j=1}^{r} 2[(j-1)p+k]+(p-s_j) = 2kr - L \pmod{p}$, where $L = \sum_{j=1}^{r} s_j$.

Theorem 3. *Every cubic Paley graph of prime order admits an edge-graceful labeling.*

Proof. See Theorem 5 with $m = 3$.

Example 5. *We apply the algorithm to show that the cubic Paley graph $3 - \mathcal{P}_{19}$ is edge-graceful with $V(3-\mathcal{P}_{19}) = \{v_1, v_2, \cdots, v_{19}\}$ and $|E(3-\mathcal{P}_{19})| = \frac{19\cdot 18}{6} = 57$. Figure 6 shows the edge-graceful labeling for the cubic Paley graph $3 - \mathcal{P}_{19}$.*

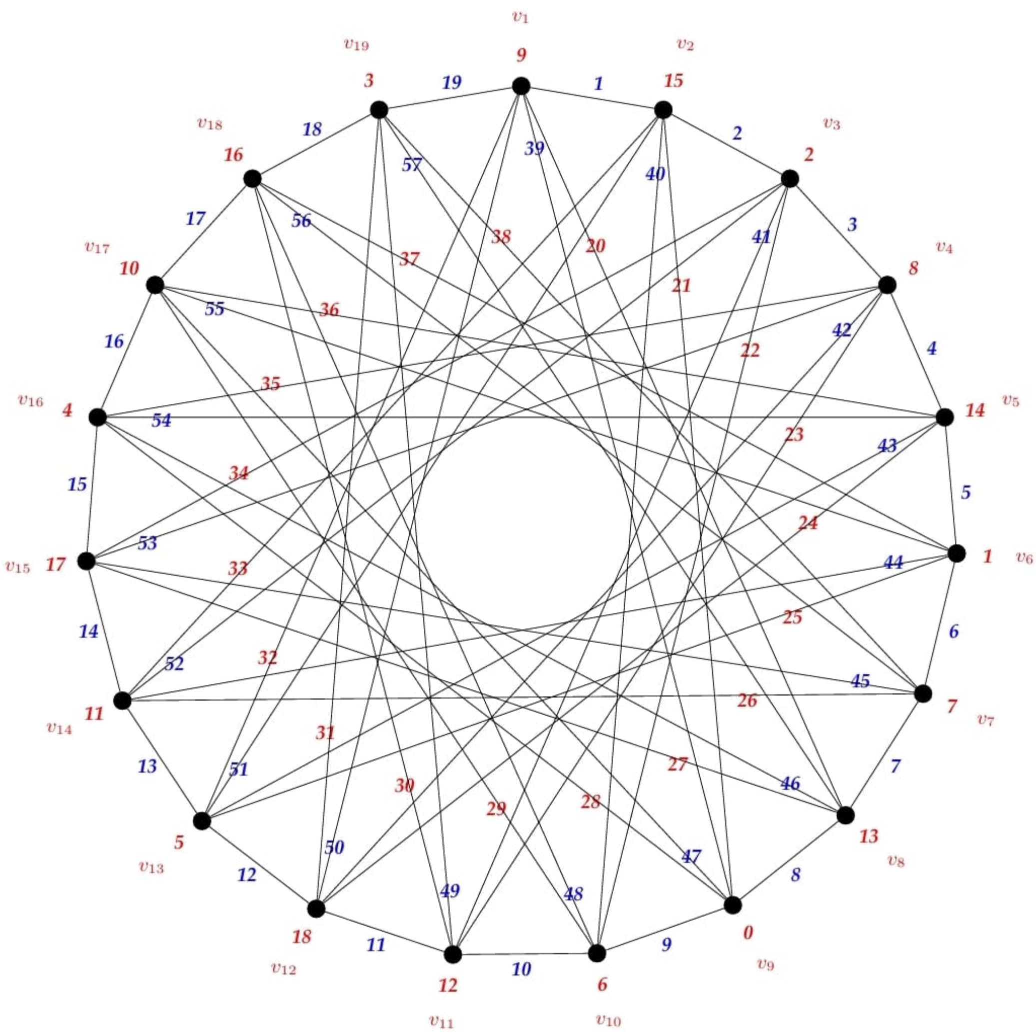

Figure 6: An Edge-Graceful Labeling of $3 - \mathcal{P}_{19}$

Input: The graph $3-\mathcal{P}_{19}$.

We have $p=19, m=3, d=3$, and $(\mathbb{Z}_{19}^{*})^{3}=\{1^{3},2^{3},\cdots,18^{3}\}=\{1,8,7,11,18,12\}$.

(1) Rename the vertices of the graph as $0:=v_{19}, 1:=v_{1}, 2:=v_{2}, \ldots, 18:=v_{18}$.

(2) Rewrite $(\mathbb{Z}_{19}^{*})^{3}$ as

$(\mathbb{Z}_{19}^{*})^{3}=S=\{s_1,s_2,s_3,s_4,s_5,s_6: s_1<s_2<s_3<s_4<s_5<s_6\}=\{1,7,8,11,12,18\}$

(3) Partition S into two sets.

Let $S_1=\{s_1,s_2,s_3\}=\{1,7,8\}$ and $S_2=\{s_4,s_5,s_6\}=\{11,12,18\}$.

Note that: for any vertex $v_i\in\mathbb{Z}_{19}$ the vertices $v_{i+1},v_{i+7},v_{i+8},v_{i+11},v_{i+12},v_{i+18}$ are adjacent to v_i.

(4) The vertices v_{i+1},v_{i+7},v_{i+8} are placed in clockwise direction of v_i and the vertices $v_{i+11},v_{i+12},v_{i+18}$ are placed in anticlockwise direction of v_i.

(5) Set $f(v_i,v_{i+s_j})=0$ for all $i\in\{1,2,3,\cdots,19\}, j\in\{1,2,3,\cdots,6\}$.

(6) Set $i=1$.

Step 1: $i=1\le 19$, then continue to Step 2.

Step 2: For each $s_1,s_2,s_3\in S_1$, $f\left(v_1,v_{1+s_j}\right)=(j-1)\,p+1$.

That is, $f(v_1,v_2)=(1-1)19+1=1, f(v_1,v_8)=(2-1)19+1=20, f(v_1,v_9)=(3-1)19+1=39$.

Step 3: $i=1+1=2$, go back to Step 1.

Step 1: $i=2\le 19$, then continue to Step 2.

Step 2: For each $s_1,s_2,s_3\in S_1$, $f\left(v_2,v_{2+s_j}\right)=(j-1)\,p+2$.

That is, $f(v_2,v_3)=(1-1)19+2=2, f(v_2,v_9)=(2-1)19+2=21, f(v_2,v_{10})=(3-1)19+2=40$.

Step 3: $i=2+1=3$, go back to Step 1.

We repeat Step1, Step2, and Step3 and get:

$f(v_3,v_4)=3$, $f(v_3,v_{10})=22$, $f(v_3,v_{11})=41$, $f(v_4,v_5)=4$, $f(v_4,v_{11})=23$, $f(v_4,v_{12})$ $=42$, $f(v_5,v_6)=5$, $f(v_5,v_{12})=24$, $f(v_5,v_{13})=43$, $f(v_6,v_7)=6$, $f(v_6,v_{13})=25$, $f(v_6,v_{14})=44$, $f(v_7,v_8)=7$, $f(v_7,v_{14})=26$, $f(v_7,v_{15})=45$, $f(v_8,v_9)=8$, $f(v_8,v_{15})=27$, $f(v_8,v_{16})=46$, $f(v_9,v_{10})=9$, $f(v_9,v_{16})=28$, $f(v_9,v_{17})=47$, $f(v_{10},v_{11})=10$, $f(v_{10},v_{17})=29$, $f(v_{10},v_{18})=48$, $f(v_{11},v_{12})=11$, $f(v_{11},v_{18})=30$, $f(v_{11},v_{19})=49$, $f(v_{12},v_{13})=12$, $f(v_{12},v_{19})=31$, $f(v_{12},v_1)=50$, $f(v_{13},v_{14})=13$, $f(v_{13},v_1)=32$, $f(v_{13},v_2)=51$, $f(v_{14},v_{15})=14$, $f(v_{14},v_2)=33$, $f(v_{14},v_3)=52$, $f(v_{15},v_{16})=15$, $f(v_{15},v_3)=34$, $f(v_{15},v_4)=53$, $f(v_{16},v_{17})=16$, $f(v_{16},v_4)=35$, $f(v_{16},v_5)=54$, $f(v_{17},v_{18})=17$, $f(v_{17},v_5)=36$, $f(v_{17},v_6)=55$, $f(v_{18},v_{19})=18$, $f(v_{18},v_6)=37$, $f(v_{18},v_7)=56$, $f(v_{19},v_1)=19$, $f(v_{19},v_7)=38$, $f(v_{19},v_8)=57$.

Step 4: For each $k=1,2,3,\cdots,19$, find the weight of the vertex v_k using the following mapping: $f_w(v_k)=\sum_{j=1}^{6} f(v_k,v_{k+s_j})=6k-\sum_{j=1}^{3} s_j=6k-16 \pmod p$. So $f_w\left(v_1\right)=6\cdot 1-16=-10\equiv 9 \pmod{19}$.

Equivalently, $f_w\left(v_1\right)=f(v_1,v_2)+f(v_1,v_8)+f(v_1,v_9)+f(v_{12},v_1)+f(v_{13},v_1)+f(v_{19},v_1)$ $=1+20+39+50+32+19\equiv 9 \pmod{19}$.

By the same function modulo 19, we get: $f_w(v_2)=15$, $f_w(v_3)=2$, $f_w(v_4)=8$, $f_w(v_5)=14$, $f_w(v_6)=1$, $f_w(v_7)=7$, $f_w(v_8)=13$, $f_w(v_9)=0$, $f_w(v_{10})=6$, $f_w(v_{11})=12$, $f_w(v_{12})=18$, $f_w(v_{13})=5$, $f_w(v_{14})=11$, $f_w(v_{15})=17$, $f_w(v_{16})=4$, $f_w(v_{17})=10$, $f_w(v_{18})=16$, $f_w(v_{19})=3$.

4. Quadruple Paley graphs

Similarly like cubic Paley graphs, quadruple Paley graphs are defined with the same vertex set and two vertices are adjacent if their difference is a quadruple residue in $\mathcal{F}_q$.

Definition 4. *[28] Let $q = p^n$, where p is an odd prime number, and $n \in \mathbb{N}$, such that $q \equiv 1\ (mod\ 8)$. The graph $4{-}\mathcal{P}_q$, with $V(4{-}\mathcal{P}_q) = \mathcal{F}_q$ and $E(4{-}\mathcal{P}_q) = \{(u,v) \mid u-v \in (\mathcal{F}_q^*)^4\}$, is called the quadruple Paley graph of order q.*

Note that: The condition $q \equiv 1\ (mod\ 8)$ is necessary to ensure that -1 is a quadruple and, consequently, the graph $4 - \mathcal{P}_q$ is well-defined.

Example 6. *The quadruple Paley graph $4 - \mathcal{P}_{17}$ of order 17 has $V(4 - \mathcal{P}_{17}) = \mathbb{Z}_{17}$ and $E(4 - \mathcal{P}_{17}) = \{(u,v) \mid u - v \in \{1, 4, 13, 16\}\}$, as in Figure 7.*

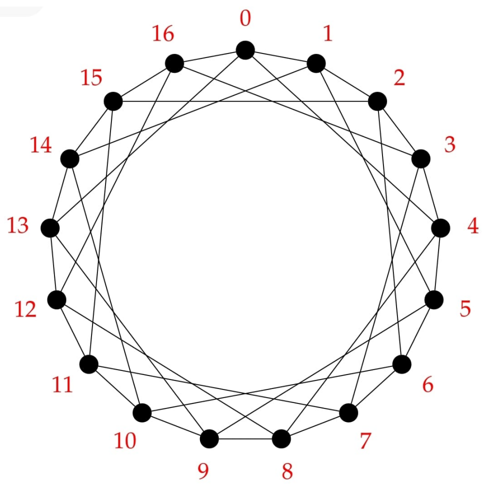

Figure 7: The Quadruple Paley Graph $4 - \mathcal{P}_{17}$

Example 7. *The quadruple Paley graph $4 - \mathcal{P}_9$ with vertex set $V(4 - \mathcal{P}_9) = \mathcal{F}_9$ and $(\mathcal{F}_9^*)^4 = \{1, 2\}$ so the edge set $E(4 - \mathcal{P}_9) = \{(0,1), (0,2), (1,2), (a, a+1), (a, a+2), (a+1, a+2), (2a, 2a+1), (2a, 2a+2), (2a+1, 2a+2)\}$.*

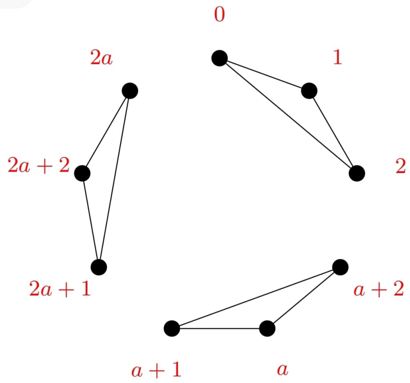

Figure 8: The Quadruple Paley Graph $4-\mathcal{P}_9$

Now we provide an algorithm which produces an edge-graceful labeling for quadruple Paley graphs of prime order.

4.1. Edge-graceful labeling Algorithm for quadruple Paley graphs of prime order

Input: The quadruple Paley graph $4-\mathcal{P}_p$ with $V(4-\mathcal{P}_p)=\mathbb{Z}_p$ and $E(4-\mathcal{P}_p)=\{(u,v) \mid u-v \in (\mathbb{Z}_p^*)^4\}$, where $p \equiv 1\ (mod\ 8)$
(1) Rename the vertices of the graph as $0 := v_p, 1 := v_1, 2 := v_2, \ldots, p-1 := v_{p-1}$.
(2) Set $r = \frac{p-1}{8}$, and rewrite $(\mathbb{Z}_p^*)^4$ as
$(\mathbb{Z}_p^*)^4 = S = \{s_1, s_2, s_3, \ldots, s_{2r} : s_1 < s_2 < s_3 < \ldots < s_{2r}\}$.
(3) Partition S into two sets.
Let $S_1 = \{s_1, s_2, s_3, \ldots, s_r\}$ and $S_2 = \{s_{r+1}, s_{r+2}, s_{r+3}, \ldots, s_{2r}\}$. Note that: $p-1$ is divisible by 8 and for any vertex $v_i \in \mathbb{Z}_p$ the vertex v_{i+s_j} is adjacent to v_i for all $s_j \in S$.
(4) If $s_j \in S_1$, the vertex v_{i+s_j} is placed in clockwise direction of v_i and if $s_j \in S_2$, the vertex v_{i+s_j} is placed in anticlockwise direction of v_i.
(5) Set $f(v_i, v_{i+s_j}) = 0$ for all $i \in \{1,2,3,\cdots,p\}, j \in \{1,2,3,\cdots,2r\}$.
(6) Set $i = 1$.
Step 1: If $i \le p$ then continue to Step 2.
Else jump to Step 4.
Step 2: For each $s_j \in S_1$, $f\left(v_i, v_{i+s_j}\right) = (j-1)\,p + i$.
Step 3: $i = i+1$, go back to Step 1.
Step 4: For each $k = 1,2,3,\ldots,p$, find the weight of the vertex v_k using the following mapping: $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) = \sum_{j=1}^{r} 2[(j-1)p+k]+(p-s_j) = 2kr - L \pmod p$, where $L = \sum_{j=1}^{r} s_j$.

Theorem 4. *Every quadruple Paley graph of prime order admits an edge-graceful labeling.*

Proof. See Theorem 5 with $m = 4$.

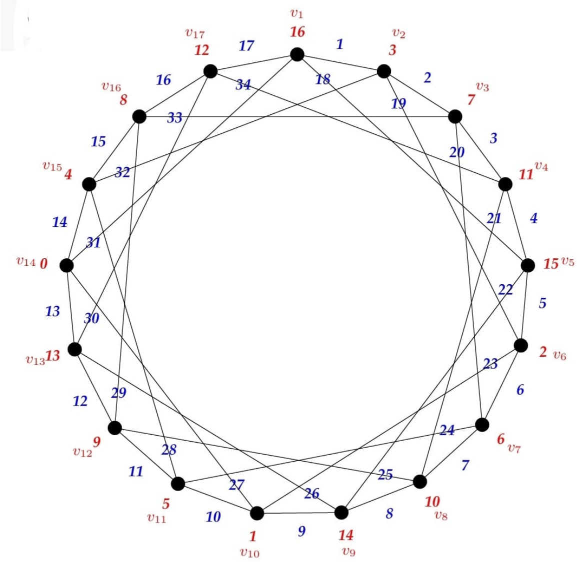

Figure 9: An Edge-Graceful Labeling of $4-\mathcal{P}_{17}$.

Example 8. *We apply the algorithm to show that: the quadruple Paley graph $4-\mathcal{P}_{17}$ is edge-graceful with $V(4-\mathcal{P}_{17})=\{v_1,v_2,\cdots,v_{17}\}$ and $|E(4-\mathcal{P}_{17})|=\frac{17\cdot 16}{8}=34$. Here, $p=17, r=2$, and $S_1=\{1,4\}$. So for each $s_j\in S_1$, $f\left(v_k,v_{k+s_j}\right)=(j-1)\,17+k$, and $f_w(v_k)=4k-\sum_{j=1}^{2}s_j=4k-5 \pmod{17}$. Figure 9 shows the edge-graceful labeling for the cubic Paley graph $4-\mathcal{P}_{17}$.*

5. Generalized Paley graphs

Generalized Paley graphs were first introduced by Cohen [29], and reintroduced by Lim and Praeger [30] in 2009, and by Elsawy [3] in 2009. The generalized Paley graph extends the concept of the Paley graph and its higher-order variants, providing a broader class of graphs based on higher power residues in finite fields. In the following we extend the results of edge-graceful labeling on Paley, cubic Paley, quadruple Paley graphs to generalized Paley graphs

Definition 5. *Let m,n be positive integers and p be an odd prime such that: "if m is even then $q\equiv 1 \pmod{2m}$ and if m is odd then p is any odd prime". The generalized Paley graph $m-\mathcal{P}_q$ has vertex set $V(m-\mathcal{P}_q)=F_q$, where F_q is the finite field of order $q=p^n$, and two vertices are adjacent if their difference belongs to $(F_q^*)^m$.*

Remark 1. *The generalized Paley graphs $m-\mathcal{P}_q$ are non-directed graphs because $(F_q^*)^m=-(F_q^*)^m$, in other words $-1\in(F_q^*)^m$.*

Remark 2. *The generalized Paley graphs $m-\mathcal{P}_q$ are regular of degree $\frac{q-1}{d}$, where d is the greatest common divisor of m and $q-1$ (see [3]).*

Remark 3. *The size of* $m - \mathcal{P}_q$ *is* $\frac{q(q-1)}{2d}$*, and the positive integer* $\frac{q-1}{d}$ *is even.*

Remark 4. *For* $m = 2, 3,$ *or* 4 *we get Paley, cubic Paley, or quadruple Paley graphs respectively.*

Now, we provide an algorithm which produces an edge-graceful labeling for the prime order generalized Paley graphs

5.1. Edge-graceful labeling Algorithm for generalized Paley graphs of prime order

Input: The generalized Paley graph $m - \mathcal{P}_p$ has $V(m - \mathcal{P}_p) = \mathbb{Z}_p$ and $E(m - \mathcal{P}_p) = \{(u, v) \mid u - v \in (\mathbb{Z}_p^*)^m\}$, such that: if m is even then $p \equiv 1 \pmod{2m}$ and if m is odd then p is any odd prime.
(1) Rename the vertices of the graph as $0 := v_p, 1 := v_1, 2 := v_2, \ldots, p-1 := v_{p-1}$.
(2) Set $r = \frac{p-1}{2d}$, where $d = gcd(m, p-1)$, and rewrite $(\mathbb{Z}_p^*)^m$ as
$(\mathbb{Z}_p^*)^m = S = \{s_1, s_2, s_3, \ldots, s_{2r} : s_1 < s_2 < s_3 < \ldots < s_{2r}\}$.
(3) Partition S into two sets.
Let $S_1 = \{s_1, s_2, s_3, \ldots, s_r\}$ and $S_2 = \{s_{r+1}, s_{r+2}, s_{r+3}, \ldots, s_{2r}\}$.
Note that: $p-1$ is divisible by $2d$ and for any vertex $v_i \in \mathbb{Z}_p$ the vertex v_{i+s_j} is adjacent to v_i for all $s_j \in S$.
(4) If $s_j \in S_1$, the vertex v_{i+s_j} is placed in clockwise direction of v_i and if $s_j \in S_2$, the vertex v_{i+s_j} is placed in anticlockwise direction of v_i.
(5) Set $f(v_i, v_{i+s_j}) = 0$ for all $i \in \{1, 2, 3, \cdots, p\}, j \in \{1, 2, 3, \cdots, 2r\}$.
(6) Set $i = 1$.
Step 1: If $i \leq p$ then continue to Step 2.
Else jump to Step 4.
Step 2: For each $s_j \in S_1,\ f\left(v_i, v_{i+s_j}\right) = (j-1)\,p + i$.
Step 3: $i = i + 1$, go back to Step 1.
Step 4: For each $k = 1, 2, 3, \ldots, p$, find the weight of the vertex v_k using the following mapping: $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) = \sum_{j=1}^{r} 2[(j-1)p + k] + (p - s_j) = 2kr - L \pmod{p}$, where $L = \sum_{j=1}^{r} s_j$.

Theorem 5. *Paley graphs and their generalizations of prime order are edge-graceful graphs.*

Proof. To prove that the algorithm defines an edge-graceful labeling, we need to prove that both functions f and f_w are bijections.

(i) Consider the function $f_w : V(m - \mathcal{P}_p) \longrightarrow \mathbb{Z}_p$ such that $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) = 2kr - L \pmod{p}$. Now, we prove that f_w is one-to-one, which implies tat it is a bijection. Let v_x and v_y be two vertices in $V(m - \mathcal{P}_p)$, if $f_w(v_x) = f_w(v_y)$ then $2xr - L = 2yr - L \pmod{p}$ which leads to $x = y$.

(ii) The function $f : E(m-\mathcal{P}_p) \longrightarrow \{1,2,3,\ldots,rp\}$ is defined as $f(v_x, v_{x+s_j}) = (j-1)p + x$. To prove that f is a bijection, we need only to prove that it is one-to-one. Let (v_x, v_{x+s_i}) and (v_y, v_{y+s_j}) be two edges, with $x, y \in \{1,2,3,\cdots,p\}$, $s_i, s_j \in S = (\mathbb{Z}_p^*)^m$, and $f(v_x, v_{x+s_j}) = f(v_y, v_{y+s_i})$, which implies that $(i-1)p+x = (j-1)p+y$. In case of $i = j$ or $x = y$ the proof is trivial. The last case if $x \neq y$ and $j \neq i$, here we will find that $x - y = (j-i)p$ but $|x-y| < p$ and in the same time $|p(i-j)| \geq p$ which is a contradiction. So, from these three cases, we can be sure that the function f is a one-to-one function. □

Example 9. *Consider the cubic Paley graph $3-\mathcal{P}_{13}$. Here, $p = 13, m = 3, d = 3, r = 2$, and $S_1 = \{1,5\}$. So for each $s_j \in S_1$, $f\left(v_k, v_{k+s_j}\right) = (j-1)\,13 + k$, and $f_w(v_k) = 4k - \sum_{j=1}^{2} s_j = 4k - 6 \pmod{13}$. The edge-graceful labeling of the graph $3-\mathcal{P}_{13}$ is shown in Figure 10.*

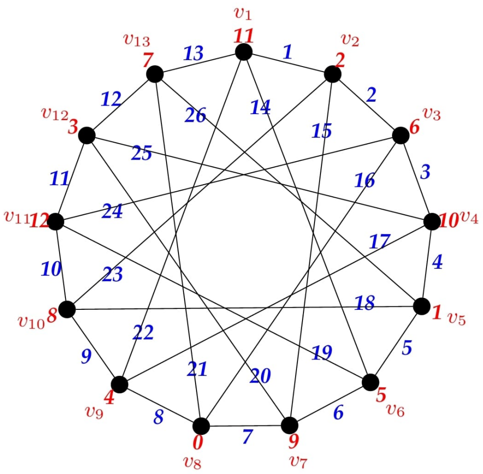

Figure 10: An Edge-Graceful Labeling of $3-\mathcal{P}_{13}$.

Example 10. *Consider the generalized Paley graph $5-\mathcal{P}_7$. Here, $p = 7, m = 5, 7,$ or 11, $d = 1, r = 3$, and $S_1 = \{1,2,3\}$. So for each $s_j \in S_1$, $f\left(v_k, v_{k+s_j}\right) = (j-1)\,7 + k$, and $f_w(v_k) = 6k - \sum_{j=1}^{3} s_j = 6k - 6 \pmod{7}$. The edge-graceful labeling of the graph $5-\mathcal{P}_7$ is shown in Figure 11.*

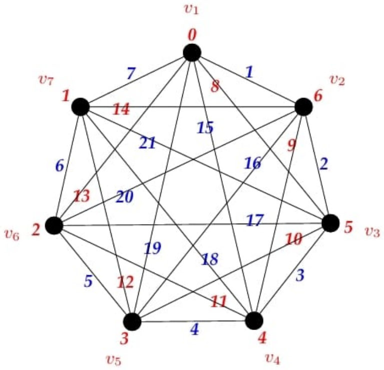

Figure 11: An Edge-Graceful Labeling of $5-\mathcal{P}_7$, $7-\mathcal{P}_7$, $11-\mathcal{P}_7$.

Example 11. *Consider the Paley graph* $\mathcal{P}_{17}$*. Here,* $p = 17, m = d = 2, r = 4$*, and* $S_1 = \{1,2,4,8\}$*. So for each* $s_j \in S_1$*,* $f\left(v_k, v_{k+s_j}\right) = (j-1)\,17 + k$*, and* $f_w(v_k) = 8k - \sum_{j=1}^{4} s_j = 8k - 15 \pmod{17}$*. The edge-graceful labeling of the graph* $\mathcal{P}_{17}$ *is shown in Figure 12.*

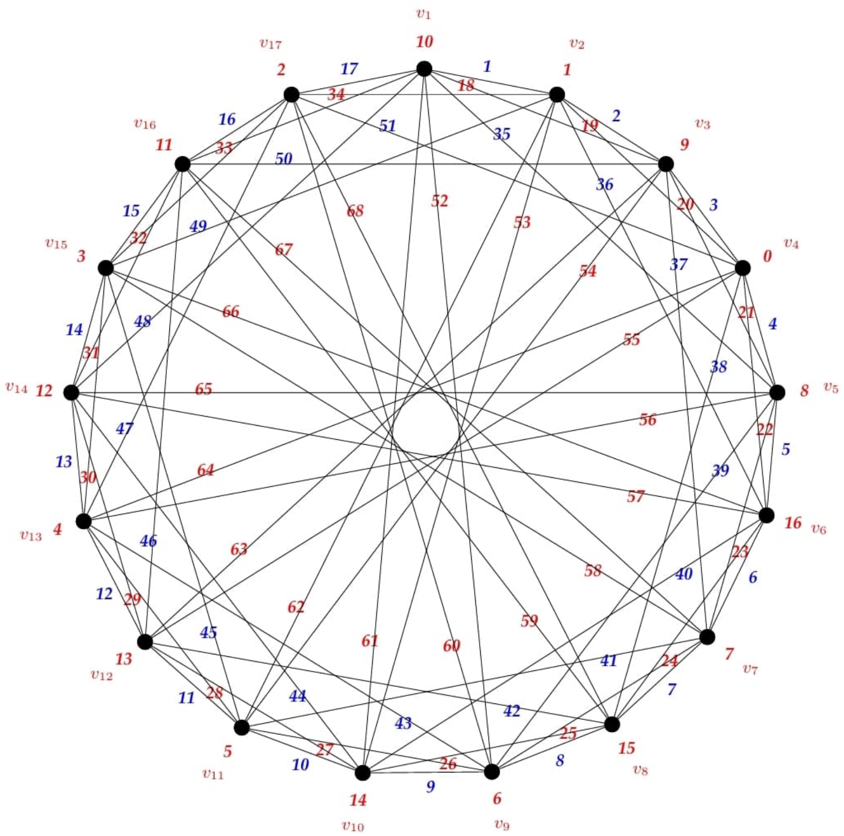

Figure 12: An Edge-Graceful Labeling of $\mathcal{P}_{17}$.

Example 12. *Consider the generalized Paley graph* $7-\mathcal{P}_{29}$. *Here,* $p=29, m=7, d=7, r=2$, *and* $S_1=\{1,12\}$. *So for each* $s_j \in S_1$, $f\left(v_k, v_{k+s_j}\right)=(j-1)\,29+k$, *and* $f_w(v_k)=4k-\sum_{j=1}^{2} s_j=4k-13 \pmod{29}$. *The edge-graceful labeling of the graph* $7-\mathcal{P}_{29}$ *is shown in Figure 13.*

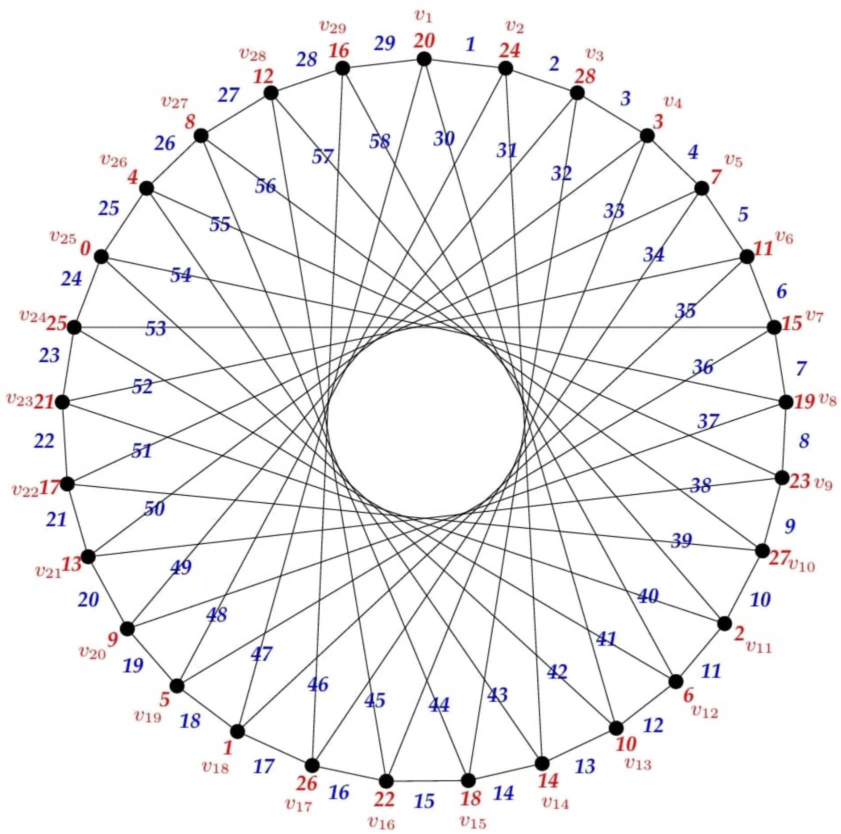

Figure 13: An Edge-Graceful Labeling of $7-\mathcal{P}_{29}$.

Example 13. *Consider the generalized Paley graph* $9-\mathcal{P}_{31}$. *Here,* $p=31, m=9, d=3, r=5$, *and* $S_1=\{1,2,4,8,15\}$. *So for each* $s_j \in S_1$, $f\left(v_k, v_{k+s_j}\right)=(j-1)\,31+k$, *and* $f_w(v_k)=10k-\sum_{j=1}^{5} s_j=10k-30 \pmod{31}$. *The edge-graceful labeling of the graph* $9-\mathcal{P}_{31}$ *is shown in Figure 14.*

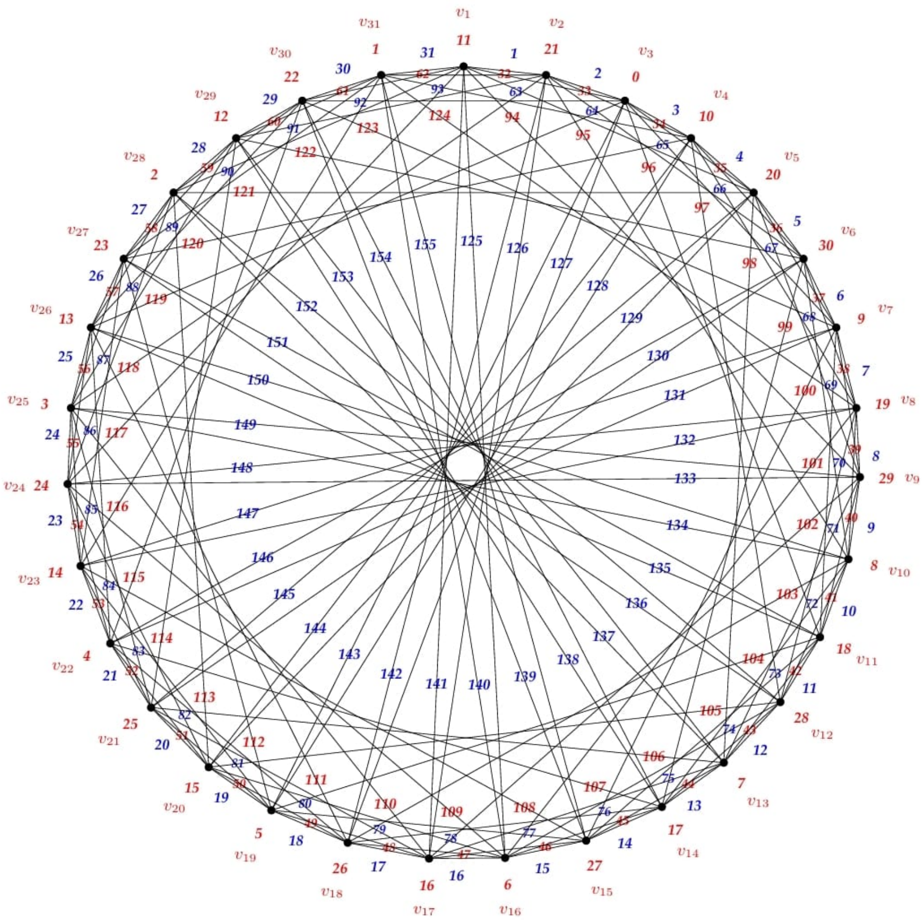

Figure 14: An Edge-Graceful Labeling of $9-\mathcal{P}_{31}$.

6. Edge even graceful labeling of generalized Paley graphs

Definition 6. *A graph $\mathcal{G}$ of order n and size m is said to be edge-even graceful if there exists a bijection $f : E(\mathcal{G}) \longrightarrow \{2, 4, 6, \ldots, 2m\}$ such that the function $f_w : V(\mathcal{G}) \longrightarrow \{0, 2, 4, \ldots, 2k-2\}, k = \max(n, m)$ given by $f_w(u) = \sum_{uv \in N(u)} f(uv) \pmod{2k}$ is an injection.*

Using the following algorithm, we show that the generalized Paley graphs $m - \mathcal{P}_p$ of prime order are edge-even graceful.

6.1. Edge-even graceful labeling Algorithm for generalized Paley graph of prime order

Input: The generalized Paley graph $m - \mathcal{P}_p$ with $\mathbb{Z}_p$ as its vertices and two vertices are joined by an edge if their difference belongs to $(\mathbb{Z}_p^*)^m$ such that: if m is even then $p \equiv 1$

(mod $2m$) and if m is odd then p is any odd prime.

(1) Rename the vertices of the graph as $0 := v_p, 1 := v_1, 2 := v_2, \ldots, p-1 := v_{p-1}$.

(2) Set $r = \frac{p-1}{2d}$ where $d = gcd(m, p-1)$, and rewrite $(\mathbb{Z}_p^*)^m$ as $(\mathbb{Z}_p^*)^m = S = \{s_1, s_2, s_3, \ldots, s_{2r} : s_1 > s_2 > s_3 > \ldots > s_{2r}\}$.

(3) Partition S into two sets.

Let $S_1 = \{s_1, s_2, s_3, \ldots, s_r\}$ and $S_2 = \{s_{r+1}, s_{r+2}, s_{r+3}, \ldots, s_{2r}\}$.

(4) If $s_j \in S_1$, the vertex v_{i+s_j} is placed in anticlockwise direction of v_i and if $s_j \in S_2$, the vertex v_{i+s_j} is placed in clockwise direction of v_i.

(5) Set $f(v_i, v_{i+s_j}) = 0$ for all $i \in \{1, 2, 3, \ldots, p\}, j \in \{1, 2, 3, \ldots, 2r\}$.

(6) Set $i = 1$.

Step 1: If $i \leq p$ then continue to Step 2.

Else jump to Step 4.

Step 2: For each $s_j \in S_1$, $f\left(v_i, v_{i+s_j}\right) = 2[(j-1)\,p + i]$.

Step 3: $i = i + 1$, go back to Step 1.

Step 4: For each $k = 1, 2, 3, \ldots, p$, find the weight of the vertex v_k using the following mapping: $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) \pmod{2t}$, where $t = \max(p, pr)$. Take $M = \sum_{j=1}^{r} s_j$, then

$f_w(v_k) = \sum_{j=1}^{r} 4[(j-1)\,p + k] + 2(p - s_j) = 4rk - 2M \pmod{2rp}$.

Theorem 6. *Paley graphs and their generalizations of prime order are edge-even graceful graphs.*

Proof. To prove that the algorithm defines an edge-even graceful labeling, we need to prove that both functions f and f_w are injection.

(i) Consider the function $f_w : V(m - \mathcal{P}_p) \longrightarrow \{0, 2, 4, \ldots, 2rp-2\}$ defined by $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) = 4kr - 2M \pmod{2rp}$. Let v_x and v_y be two vertices in $V(m - \mathcal{P}_p)$, if $f_w(v_x) = f_w(v_y)$ then $4xr - 2M = 4yr - 2M \pmod{2rp}$ which leads to $x = y$.

(ii) The function $f : E(m - \mathcal{P}_p) \longrightarrow \{2, 4, 6, \ldots, 2rp\}$ is defined as $f(v_x, v_{x+s_j}) = 2[(j-1)p + x]$. To prove that f is a bijection, we need only to prove that it is one-to-one. Let (v_x, v_{x+s_i}) and (v_y, v_{y+s_j}) be two edges, with $x, y \in \{1, 2, 3, \cdots, p\}$, $s_i, s_j \in S_1$, and $f(v_x, v_{x+s_j}) = f(v_y, v_{y+s_i})$, which implies that $2[(i-1)p + x] = 2[(j-1)p + y]$. In case of $i = j$ or $x = y$ the proof is trivial. The last case if $x \neq y$ and $j \neq i$, here we will find that $x - y = (j-i)p$ but $|x - y| < p$ and in the same time $|p(i-j)| \geq p$ which is a contradiction. So, from these three cases, we can be sure that the function f is a one-to-one function and because $|E(m - \mathcal{P}_p)| = rp$, f is a bijection. □

Example 14. *Consider the quadruple Paley graph* $4 - \mathcal{P}_{17}$*. Here,* $p = 17, m = 4, d = 4, r = 2$*, and* $S_1 = \{16, 13\}$*. So for each* $s_j \in S_1$, $f\left(v_k, v_{k+s_j}\right) = 2[(j-1)\,17 + k]$*, and* $f_w(v_k) = 8k - 2\sum_{j=1}^{2} s_j = 8k - 58 \pmod{68}$*. The edge-even graceful labeling of the graph* $4 - \mathcal{P}_{17}$ *is shown in Figure 15.*

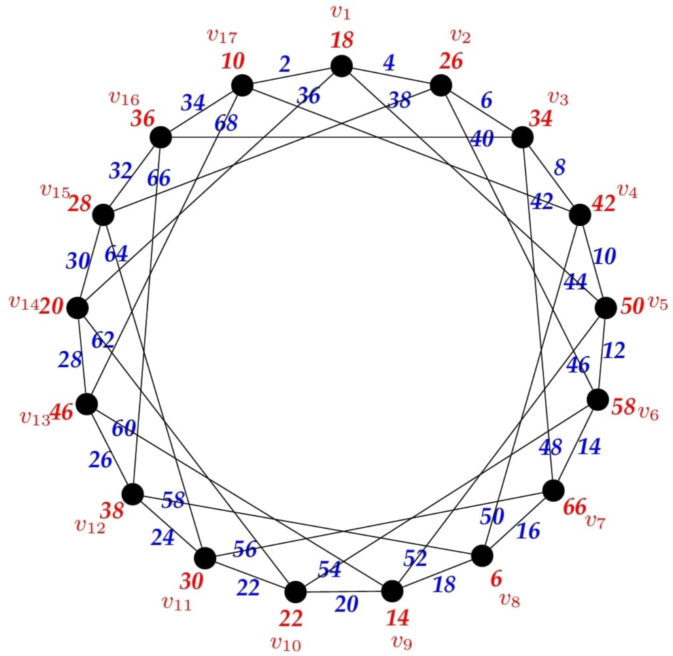

Figure 15: An Edge-Even Graceful Labeling of $4-\mathcal{P}_{17}$.

Example 15. *Consider the generalized Paley graph* $5-\mathcal{P}_7$. *Here,* $p=7, m=5, d=1, r=3$, *and* $S_1=\{6,5,4\}$. *So for each* $s_j \in S_1$, $f\left(v_k, v_{k+s_j}\right)=2[(j-1)\,7+k]$, *and* $f_w(v_k)=12k-2\sum_{j=1}^{2} s_j=12k-30 \pmod{42}$. *The edge-even graceful labeling of the graph* $5-\mathcal{P}_7$ *is shown in Figure 16.*

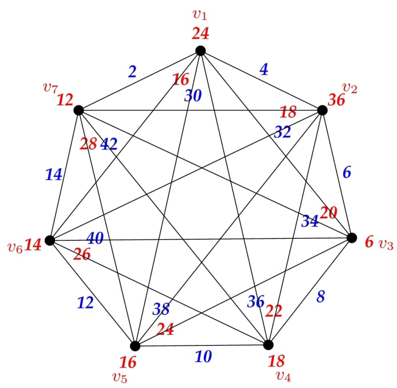

Figure 16: An Edge-Even Graceful Labeling of $7-\mathcal{P}_7$, $5-\mathcal{P}_7$, $11-\mathcal{P}_7$.

Example 16. *Consider the cubic Paley graph* $3-\mathcal{P}_{19}$. *Here,* $p=19, m=3, d=3, r=3$, *and* $S_1=\{18,12,11\}$. *So for each* $s_j \in S_1$, $f\left(v_k, v_{k+s_j}\right)=2[(j-1)\,19+k]$, *and* $f_w(v_k)=12k-2\sum_{j=1}^{2} s_j=12k-82 \pmod{114}$. *The edge-even graceful labeling of the graph* $3-\mathcal{P}_{19}$ *is shown in Figure 17.*

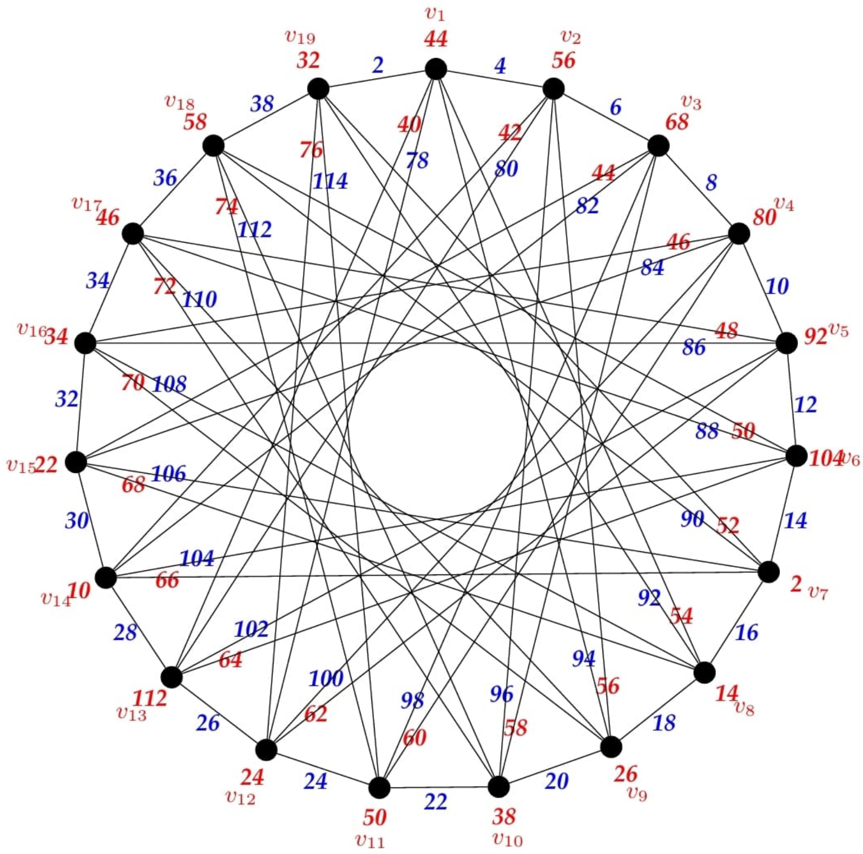

Figure 17: An Edge-Even Graceful Labeling of $3 - \mathcal{P}_{19}$.

7. Edge odd graceful labeling of generalized Paley graphs

Definition 7. *A graph $\mathcal{G}$ of order n and size m is said to be edge-odd graceful if there exists a bijection $f : E(\mathcal{G}) \longrightarrow \{1, 3, 5, \dots, 2m-1\}$ such that the function $f_w : V(\mathcal{G}) \longrightarrow \{0, 1, 2, \dots, 2m-1\}$, given by $f_w(u) = \sum_{uv \in N(u)} f(uv) \pmod{2m}$ is an injection.*

7.1. Edge-odd graceful labeling Algorithm for generalized Paley graph of prime order

Input: The generalized Paley graph $m - \mathcal{P}_p$ with $\mathbb{Z}_p$ as its vertices and two vertices are joined by an edge if their difference belongs to $(\mathbb{Z}_p^*)^m$ such that: if m is even then $p \equiv 1 \pmod{2m}$ and if m is odd then p is any odd prime.

(1) Rename the vertices of the graph as $0 := v_p, 1 := v_1, 2 := v_2, \dots, p-1 := v_{p-1}$.

(2) Set $r = \frac{p-1}{2d}$ where $d = gcd(m, p-1)$, and rewrite $(\mathbb{Z}_p^*)^m$ as $(\mathbb{Z}_p^*)^m = S = \{s_1, s_2, s_3, \dots, s_{2r} : s_1 > s_2 > s_3 > \dots > s_{2r}\}$.

(3) Partition S into two sets.

Let $S_1 = \{s_1, s_2, s_3, \dots, s_r\}$ and $S_2 = \{s_{r+1}, s_{r+2}, s_{r+3}, \dots, s_{2r}\}$.

(4) If $s_j \in S_1$, the vertex v_{i+s_j} is placed in anticlockwise direction of v_i and if $s_j \in S_2$, the vertex v_{i+s_j} is placed in clockwise direction of v_i.

(5) Set $f(v_i, v_{i+s_j}) = 0$ for all $i \in \{1, 2, 3, \ldots, p\}, j \in \{1, 2, 3, \ldots, 2r\}$.

(6) Set $i = 1$.

Step 1: If $i \leq p$ then continue to Step 2.

Else jump to Step 4.

Step 2: For each $s_j \in S_1,\ f\left(v_i, v_{i+s_j}\right) = 2[(j-1)\,p + i] - 1$.

Step 3: $i = i + 1$, go back to Step 1.

Step 4: For each $k = 1, 2, 3, \ldots, p$, find the weight of the vertex v_k using the following mapping: $f_w(v_k) = \sum_{j=1}^{2r} f(v_k, v_{k+s_j}) \pmod{2rp}$. Take $M = \sum_{j=1}^{r} s_j$, then

$f_w(v_k) = \sum_{j=1}^{r} 4[(j-1)\,p + k] - 2 + 2(p - s_j) = 4rk - 2(M + r) \pmod{2rp}$.

Theorem 7. *Paley graphs and their generalizations of prime order are edge-odd graceful graphs.*

Proof. Following the same steps in the previews theorem we can easily prove that the algorithm defines an edge-odd graceful labeling. □

Example 17. *Consider the cubic Paley graph* $3 - \mathcal{P}_{13}$*. Here,* $p = 13, m = 3, d = 3, r = 2,$ *and* $S_1 = \{12, 8\}$*. So for each* $s_j \in S_1,\ f\left(v_k, v_{k+s_j}\right) = 2[(j-1)\,13 + k] - 1,$ *and* $f_w(v_k) = 8k - 2r - 2\sum_{j=1}^{2} s_j = 8k - 44 \pmod{52}$*. The edge-odd graceful labeling of the graph* $3 - \mathcal{P}_{13}$ *is shown in Figure 18.*

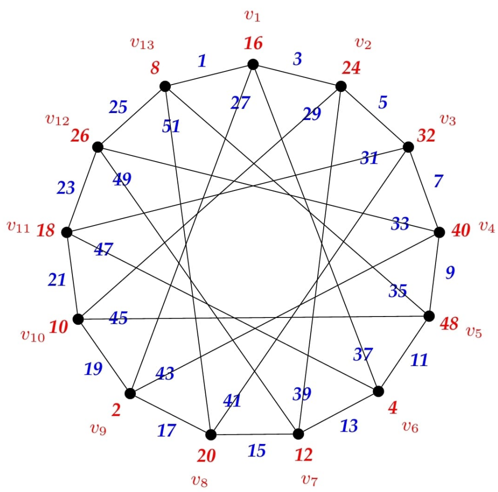

Figure 18: An Edge-Odd Graceful Labeling of $3 - \mathcal{P}_{13}$.

Example 18. *Consider the quadruple Paley graph* $4 - \mathcal{P}_{17}$*. Here,* $p = 17, m = 4, d = 4, r = 2,$ *and* $S_1 = \{16, 13\}$*. So for each* $s_j \in S_1,\ f\left(v_k, v_{k+s_j}\right) = 2[(j-1)\,17 + k] - 1,$ *and* $f_w(v_k) = 8k - 2r - 2\sum_{j=1}^{2} s_j = 8k - 62 \pmod{68}$*. The edge-odd graceful labeling of the graph* $4 - \mathcal{P}_{17}$ *is shown in Figure 19.*

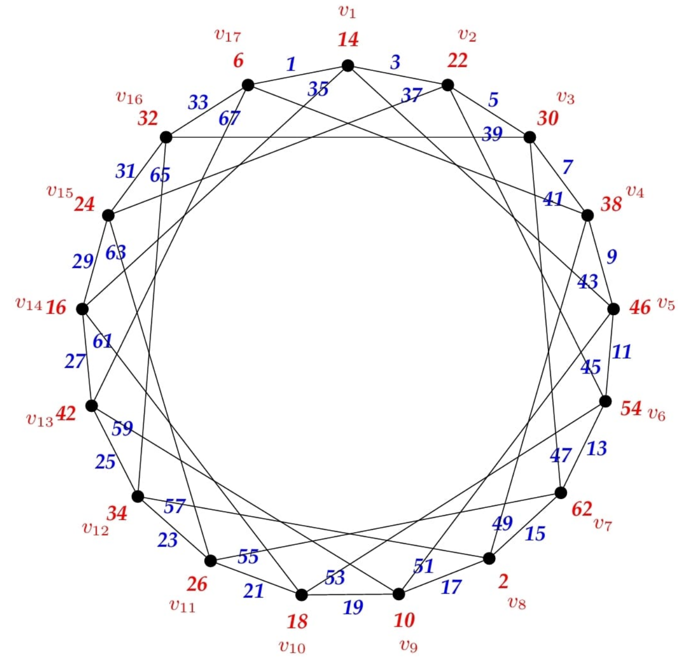

Figure 19: An Edge-Odd Graceful Labeling of $4 - \mathcal{P}_{17}$.

8. Paley Graph of prime power order

An edge-graceful, edge-even graceful, and edge-odd graceful labelling for Paley graph of order $q = p^n$, where $n > 1$, remains an open challenge.

The Paley graph $\mathcal{P}_q$ of order $q = p^n$, where $n > 1$, has the vertex set $V(\mathcal{P}_q) = \mathcal{F}_q$. Here, $\mathcal{F}_q \neq \{0, 1, 2, \ldots, q\}$, and so the algorithms introduced in this paper are not applicable for this case. For example, if $p = 5, n = 2$, we get the smallest case without edge-graceful, edge-even graceful, and edge-odd graceful labelling, where $V(\mathcal{P}_{25}) = \mathcal{F}_{25}$ $= \{0,\ 1,\ 2,\ 3,\ 4,\ a,\ 2a,\ 3a,\ 4a,\ a+1,\ a+2,\ a+3,\ a+4,\ 2a+1,\ 2a+2,\ 2a+3,\ 2a+4,\ 3a+1,\ 3a+2,\ 3a+3,\ 3a+4,\ 4a+1,\ 4a+2,\ 4a+3,\ 4a+4\} = \mathbb{Z}_5[x]/(x^2+2)$. And $(\mathcal{F}^*_{25})^2 = \{1, 2, 3, 4,\ a+2, a+3, 2a+1,\ 2a+4,\ 3a+1, 3a+4,\ 4a+2, 4a+3\}$, so $E(\mathcal{P}_{25}) = \{(x_i, x_i + x_j)\ \forall x_i \in \mathcal{F}_{25}\ \ and\ \ \forall x_j \in (\mathcal{F}^*_{25})^2\}$. (see Figure 20). The case of $\mathcal{P}_9$ is shown in Figure 3.

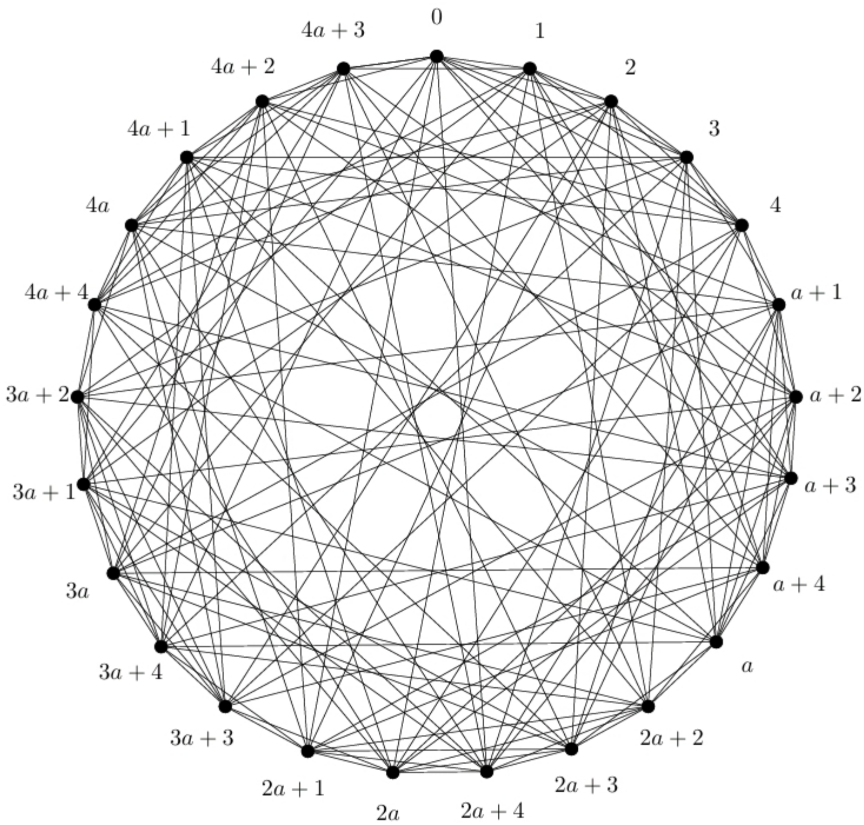

Figure 20: The Paley graph $\mathcal{P}_{25}$.

The structure of this field is more complex, and the simple modular arithmetic used in the proof for no longer holds directly. Therefore, we need to understand the structure of the field deeply to construct an edge-graceful, edge-even graceful, and edge-odd graceful labelling algorithm. For more information about the structure of the field, see [3]. Also, an other challenge is to find different types of labellings for Paley graph and its generalizations.

9. Conclusions

We introduced three algorithms which produce an edge-graceful, edge-even graceful, and edge-odd graceful labelling for Paley graphs and their generalizations. We proved that Paley graphs and their generalizations including cubic and quadruple Paley graphs of prime order are edge-graceful, edge-even graceful, and edge-odd graceful graphs.

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