Generalised Hyers-Ulam Product-Sum Stability of a Cauchy Type Additive Functional Equation

Matina J. Rassias

Department of Statistics, University of Glasgow, Mathematics Building, Office No. 208, University Gardens, Glasgow G12 8QW, U.K.


2000 Mathematics Subject Classifications: Primary 39B. Secondary 26D.

Key Words and Phrases: Generalised “product-sum” Hyers-Ulam stability, Cauchy type additive functional equation.

1. Introduction and Preliminaries

In 1940 (and 1964) Stanislaw M. Ulam [9] proposed the following stability problem, well-known as Ulam stability problem:

“When is true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?”

In particular he stated the stability question:

“Let $G_1$ be a group and $G_2$ a metric group with the metric $\rho(\cdot,\cdot)$. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $\rho(f(x),f(y)) < c$ for all $x, y \in G_1$, then a unique homomorphism $h : G_1 \to G_2$ exists with $\rho(h(x),h(y)) < \delta$ for all $x \in G_1$?”

In 1941 D.H. Hyers [2] solved this problem for linear mappings as follows:

Email address: rassias.matina@gmail.com
Theorem 1 (D.H. Hyers, 1941: [2]). If a mapping \( f : E \to E' \) satisfies the approximately additive inequality
\[
||f(x + y) - f(x) - f(y)|| \leq \varepsilon,
\]
for some fixed \( \varepsilon > 0 \) and all \( x, y \in E \), where \( E \) and \( E' \) are Banach spaces, then there exists a unique additive mapping \( A : E \to E' \), satisfying the formula
\[
A(x) = \lim_{n \to \infty} 2^{-n}f(2^n x),
\]
and inequality
\[
||f(x) - A(x)|| \leq \varepsilon
\]
for some fixed \( \varepsilon > 0 \) and all \( x \in E \).

No continuity conditions are required for this result.

Theorem 2 (T. Aoki, 1950: [1]). Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality
\[
||f(x + y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p),
\]
for all \( x, y \in E \), where \( \varepsilon > 0 \) and \( p < 1 \) constants. Then the limit
\[
A(x) = \lim_{n \to \infty} 2^{-n}f(2^n x),
\]
exists for all \( x \in E \) and \( A : E \to E' \) is the unique additive mapping which satisfies
\[
||f(x) - A(x)|| \leq \frac{2\varepsilon}{2 - 2^p}||x||^p
\]
for all \( x \in E \). If \( p < 0 \) then the inequality (1) holds for \( x, y \neq 0 \) and (2) for \( x \neq 0 \).

Theorem 3 (Th. M. Rassias, 1978: [6]). Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality
\[
||f(x + y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p),
\]
for all \( x, y \in E \), where \( \varepsilon > 0 \) and \( p < 1 \) constants. Then the limit
\[
A(x) = \lim_{n \to \infty} 2^{-n}f(2^n x),
\]
exists for all \( x \in E \) and \( A : E \to E' \) is the unique additive mapping which satisfies
\[
||f(x) - A(x)|| \leq \frac{2\varepsilon}{2 - 2^p}||x||^p
\]
for all \( x \in E \). If \( p < 0 \) then the inequality (3) holds for \( x, y \neq 0 \) and (4) for \( x \neq 0 \).

If, moreover, \( f(tx) \) is continuous in \( t \in R \) for each fixed \( x \in E \), then \( A(tx) = tA(x) \) for all \( x \in E \) and \( t \in R \). \( A : E \to E' \) is a unique linear additive mapping satisfying equation
\[
A(x + y) = A(x) + A(y).
\]
Theorem 4 (J. M. Rassias, 1982-1989: [3, 4, 5]). Let $X$ be a real normed linear space and $Y$ a real Banach space. Assume that $f : X \to Y$ is a mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and $f$ satisfies the functional inequality
\[ ||f(x + y) - f(x) - f(y)|| \leq \theta ||x||^p ||y||^q, \]
for all $x, y \in X$. Then the limit
\[ A(x) = \lim_{n \to \infty} 2^{-n} f(2^n x), \]
exists for all $x \in X$ and $A : X \to Y$ is the unique additive mapping which satisfies
\[ ||f(x) - A(x)|| \leq \frac{\theta}{|2^r - 2|} ||x||^r \]
for all $x \in X$.

If, moreover, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$. $A : X \to Y$ is a unique linear additive mapping satisfying equation
\[ A(x + y) = A(x) + A(y). \]

For the theorem that follows, let $(E, \perp)$ denote an orthogonality normed space with norm $||.||_E$ and $(F, ||.||_F)$ is a Banach space.

Theorem 5 (Ravi, K., Arunkumar, M. and Rassias, J. M., 2008: [7]). Let $f : E \to F$ be a mapping which satisfies the inequality
\[ ||f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2(m^2 - 2)f(x) + 2f(y)||_F \leq \varepsilon \left( ||x||^p_E ||y||^p_E + (||x||^{2p}_E + ||y||^{2p}_E) \right) \]
for all $x, y \in E$ with $x \perp y$, where $\varepsilon$ and $p$ are constants with $\varepsilon, p > 0$ and either $m > 1; p < 1$ or $m < 1; p > 1$ with $m \neq 0; m \neq \pm 1; m \neq \pm \sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$.

Then the limit
\[ Q(x) = \lim_{n \to \infty} \frac{f(m^n x)}{m^{2n}} \]
exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that
\[ ||f(x) - Q(x)||_F \leq \frac{\varepsilon}{2|m^2 - m^{2p}|} ||x||^{2p}_E \]
for all $x \in E$.

Note that the mixed type product-sum function
\[ (x, y) \to \varepsilon \left( ||x||^p_E ||y||^p_E + (||x||^{2p}_E + ||y||^{2p}_E) \right) \]
was introduced by J. M. Rassias ([7, 8]).

In this paper we introduce a Cauchy type additive functional equation and investigate the generalised Hyers-Ulam “product-sum” stability of this equation.
2. Cauchy Type Additive Functional Equation

Let $X$ be a real normed linear space and $Y$ a real Banach space.

**Definition 1.** A mapping $f : X \to Y$ is called approximately Cauchy type additive, if the approximately Cauchy additive functional inequality

$$
||f(x + y) + f(x - y) + f(y - x) - f(x) - f(y)|| \leq \varepsilon (||x||^2 + ||y||^2 + ||x||^\alpha + ||y||^\alpha)
$$

(6)

holds for every $x, y \in X$ with $\varepsilon \geq 0$ and $\alpha \neq 1$.

**Lemma 1.** Mapping $A : X \to Y$ satisfies the Cauchy-type additive equation

$$
A(x + y) + A(x - y) + A(y - x) = A(x) + A(y)
$$

(7)

for all $x, y \in X$ if and only if there exists a mapping $C : X \to Y$ satisfying the Cauchy additive equation

$$
C(x + y) = C(x) + C(y)
$$

for all $x, y \in X$ such that $A(x) = C(x)$ for all $x \in X$.

**Proof.** ($\Rightarrow$) Let mapping $A : X \to Y$ satisfy the Cauchy-type additive equation

$$
A(x + y) + A(x - y) + A(y - x) = A(x) + A(y)
$$

(7)

for all $x, y \in X$. Assume that there exists a mapping $C : X \to Y$ such that $A(x) = C(x)$ for all $x \in X$. Observe that for $x = y = 0$ and $x = x, y = x$ from (7) we obtain respectively

$$
C(0) = A(0) = 0
$$

and

$$
C(-x) = A(-x) = -A(x) = -C(x), \text{ for } x \in X. \tag{8}
$$

From (7) and (8) it is obvious that

$$
C(x + y) + C(x - y) + C(y - x) = C(x) + C(y), \text{ or}
$$

$$
C(x + y) + C(x - y) + C(-x - y) = C(x) + C(y), \text{ or}
$$

$$
C(x + y) = C(x) + C(y).
$$

Hence, $C$ satisfies the Cauchy additive equation.

($\Leftarrow$) Let mapping $C : X \to Y$ satisfy the Cauchy additive equation

$$
C(x + y) = C(x) + C(y)
$$

(9)

for all $x, y \in X$. Assume that there exists a mapping $A : X \to Y$ such that $A(x) = C(x)$ for all $x \in X$. Observe that for $x = y = 0$, from (9) we obtain

$$
A(0) = C(0) = 0. \tag{10}
$$
Thus, from (9) and (10) one gets
\[ A(x) + A(y) = C(x) + C(y) = C(x + y) = A(x + y) = A(x + y) + A(x - y) + (y - x) = A(x + y) + A(x - y) + A(y - x). \]

Hence, \( A \) satisfies the Cauchy type additive equation.

Thus the proof of Lemma 1 is complete.

**Theorem 6.** Assume that \( f : X \to Y \) is an approximately Cauchy type additive mapping satisfying \((6)\). Then, there exists a unique Cauchy type additive mapping \( A : X \to Y \) which satisfies the formula
\[ A(x) = \lim_{n \to \infty} f_n(x), \]
where
\[ f_n(x) = \begin{cases} 2^{-n}f(2^n x), & -\infty < \alpha < 1 \\ 2^n f(2^{-n} x), & \alpha > 1 \end{cases} \]
for all \( x \in X \) and \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \), which is the set of natural numbers and
\[ ||f(x) - A(x)|| \leq \frac{3\varepsilon}{2 - 2^a} ||x||^a \]
for some fixed \( \varepsilon > 0 \), \( \alpha \neq 1 \) and all \( x \in X \).

If, moreover, \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then \( A(tx) = tA(x) \) for all \( t \in \mathbb{R} \) and \( x \in X \). \( A : X \to Y \) is a unique linear Cauchy type additive mapping satisfying equation
\[ A(x + y) + A(x - y) + A(y - x) = A(x) + A(y). \] (11)

**Proof.** We start our proof considering: \(-\infty < \alpha < 1\).

Step 1 By substituting \( x = y = 0 \) and \( x = y \) in \((6)\), respectively, we can observe that
\[ f(0) = 0 \]
and
\[ ||f(x) - 2^{-1}f(2x)|| \leq \frac{3}{2}\varepsilon||x||^a. \]

Hence, for \( n \in \mathbb{N} - \{0\} \)
\[ ||f(x) - 2^{-n}f(2^n x)|| \leq ||f(x) - 2^{-1}f(2x)|| + ||2^{-1}f(2x) - 2^{-2}f(2^2 x)|| + \ldots + ||2^{-(n-1)}f(2^{n-1} x) - 2^{-n}f(2^n x)|| \]
\[ \leq \frac{3}{2}(1 + 2^{a-1} + \ldots + 2^{(n-1)(a-1)})\varepsilon||x||^a \]
\[ = \frac{3}{2 - 2^a}(1 - 2^{n(a-1)})\varepsilon||x||^a. \]
Thus,
\[ ||f(x) - 2^{-n}f(2^n x)|| \leq \frac{3}{2 - 2^a}(1 - 2^n(\alpha - 1))\epsilon||x||^a,\]
for \( n \in \mathbb{N} - \{0\} \) and \(-\infty < \alpha < 1\).

**Step 2** Following, we need to show that if there is a sequence \( \{f_n\} : f_n(x) = 2^{-n}f(2^n x) \), then \( \{f_n\} \) converges.

For every \( n > m > 0 \), we can obtain
\[
||f_n(x) - f_m(x)|| = ||2^{-n}f(2^n x) - 2^{-m}f(2^m x)||
\leq 2^{-m} \frac{3\epsilon}{2 - 2^a}(1 - 2^{n-m}(\alpha - 1))||x||^a
\leq 2^{-m} \frac{3\epsilon}{2 - 2^a}||x||^a \rightarrow 0,
\]
for \( m \rightarrow \infty \), as \( \alpha < 1 \). Therefore, \( \{f_n\} \) is a Cauchy sequence. Since \( Y \) is complete we can conclude that \( \{f_n\} \) is convergent. Thus, there is a well-defined \( A : X \rightarrow Y \) such that \( A(x) = \lim_{n \rightarrow \infty} 2^{-n}f(2^n x) \), for \(-\infty < \alpha < 1\).

**Step 3** Observe that
\[
||f(x) - f_n(x)|| = ||f(x) - 2^{-n}f(2^n x)|| \leq \frac{3\epsilon}{2 - 2^a}(1 - 2^n(\alpha - 1))||x||^a,
\]
from which by letting \( n \rightarrow \infty \) we obtain
\[ ||f(x) - A(x)|| \leq \frac{3\epsilon}{2 - 2^a}||x||^a. \quad (12) \]

**Step 4** Claim that mapping \( A : X \rightarrow Y \) satisfies (11). In fact, by letting \( x \rightarrow 2^n x \) and \( y \rightarrow 2^n y \), from (6), we have:
\[
||f(2^n(x + y)) + f(2^n(x - y)) + f(2^n(y - x)) - f(2^n x) - f(2^n y)||
\leq \epsilon(||2^n x||^\alpha ||2^n y||^\alpha + ||2^n x||^\alpha + ||2^n y||^\alpha).
\]
Next, by multiplying with \( 2^{-n} \) we obtain
\[
0 \leq ||2^{-n}f(2^n(x + y)) + 2^{-n}f(2^n(x - y)) + 2^{-n}f(2^n(y - x)) - 2^{-n}f(2^n x) - 2^{-n}f(2^n y)||
\leq 2^{n(\alpha - 1)}\epsilon(||x||^\alpha ||y||^\alpha + ||x||^\alpha + ||y||^\alpha)
\]
and by letting \( n \rightarrow \infty \), for \(-\infty < \alpha < 1\) we can conclude that an \( A : X \rightarrow Y \) truly exists such that: \( A(x) = \lim_{n \rightarrow \infty} 2^{-n}f(2^n x) \) satisfies the **Cauchy-type additivity property**
\[
A(x + y) + A(x - y) + A(y - x) = A(x) + A(y).
\quad (13)
\]
Therefore, existence of Theorem holds.
Step 5 We need to prove that A is unique.

Observe, from (13), that
\[ A(0) = 0 \quad \text{and} \quad A(2x) = 2A(x). \]

Therefore, by induction we can show that
\[ A(2^n x) = 2^n A(x) \]
or equivalently
\[ A(x) = 2^{-n} A(2^n x). \] \hspace{1cm} (14)

Assume, now, the existence of another \( A' : X \to Y \), such that \( A'(x) = 2^{-n} A'(2^n x) \). With the aid of the (12)-(14) and the triangular inequality, one gets
\[
0 \leq ||A(x) - A'(x)|| = ||2^{-n} A(2^n x) - 2^{-n} A'(2^n x)|| \\
\leq ||2^{-n} A(2^n x) - 2^{-n} f(2^n x)|| + ||2^{-n} f(2^n x) - 2^{-n} A'(2^n x)|| \\
\leq 2^n (a-1) \frac{3\varepsilon}{2-2^a} ||x||^a \\
\to 0,
\]
as \( n \to \infty, \ (-\infty < \alpha < 1) \). Thus, the uniqueness of A is proved and the stability of Cauchy-type additive mapping \( A : X \to Y \) is established.

Step 6 To complete the proof of Theorem 6, we only need to examine whether \( A : X \to Y \) is a linear Cauchy-type mapping. To be more precise, we need to show that:

(1) \( A(x + y) + A(x - y) + A(y - x) = A(x) + A(y) \), and

(2) \( A(r x) = r A(x) \), \quad \forall r \in R. \)

Recall that we have shown already that (1) holds.

Therefore, we only need to show that (2) is valid \( \forall r \in R. \)

For that we will study four cases.

Case 1: Let \( r = k \in N = \{0, 1, 2, \ldots \} \).

For \( k = 0 \), from (2), we have \( A(0) = 0 \). This is verified if we substitute \( x = y = 0 \) in (13).

Assume, that \( A((k-1)x) = (k-1)A(x) \) is true \( \forall k \).

Then, we need to prove that \( A(kx) = kA(x) \).

Note that for \( x = x \), and \( y = 0 \) from (13), we can easily obtain \( A(-x) = (-1)A(x) \).

Let \( x = x \) and \( y = (k-1)x \) in (13). Then,
\[
A(kx) + A(- (k-2)x) + A((k-2)x) = A(x) + A((k-1)x),
\]
or
\[ A(kx) = kA(x), \quad \forall k \in N = \{0, 1, 2, \ldots \}. \]
Case 2: Let \( r = k \in \mathbb{Z} \).
We only need to observe that \( A \) is odd. Since, we have already proved that (2) is valid \( \forall k \in \mathbb{N} = \{0, 1, 2, \ldots\} \) we can then conclude that
\[
A(kx) = kA(x), \quad \forall k \in \mathbb{Z}.
\]

Case 3: Let \( r = \frac{k}{l} \in \mathbb{Q} \), for \( k \in \mathbb{Z}, \ l \in \mathbb{Z} - \{0\} \).
Then, \( A(x) = A\left(\frac{1}{l}x\right) = lA\left(\frac{1}{l}x\right) \), for \( l \in \mathbb{Z} - \{0\} \). Hence, \( A\left(\frac{1}{l}x\right) = \frac{1}{l}A(x) \).
Besides, for \( k \in \mathbb{Z}, A\left(\frac{k}{l}x\right) = A\left(\frac{k}{l}x\right) = kA\left(\frac{1}{l}x\right) \), from Case 2.
Thus, \( A\left(\frac{k}{l}x\right) = \frac{k}{l}A(x) \), or \( A(\frac{r}{l}x) = \frac{r}{l}A(x) \) for \( r \in \mathbb{Q} \).

Case 4: Let \( r \in \mathbb{R} \), where \( r = \frac{q}{n} \) : rational numbers.
Since \( \mathbb{R} \) is a complete space, every sequence \( \{\frac{q}{n}\} \) converges in \( \mathbb{R} \), i.e. \( \lim_{n \to \infty} \frac{q}{n} = q \in \mathbb{R} \).
Recall that \( A(x) = \lim_{n \to \infty} 2^{-n}f(2^{n}x) \) and \( f(tx) \) is continuous in \( t \) for each fixed \( x \) in \( X \). Therefore, \( A(tx) \) is continuous in \( t \) for each fixed \( x \) in \( X \). Besides,
\[
\lim_{n \to \infty} A(q_{n}x) = A\left(\lim_{n \to \infty} q_{n}x\right) = A(qx) \quad (15)
\]
and
\[
\lim_{n \to \infty} A(q_{n}x) = \lim_{n \to \infty} q_{n}A(x) = qA(x) \quad (16)
\]
From (15) and (16) Case 4. is now proved, which completes Step 6. and thus the proof of our Theorem 6 for the case of \( -\infty < \alpha < 1 \).

The proof for the case of \( \alpha > 1 \) is similar to the proof for \( -\infty < \alpha < 1 \).
In fact, we can find the general inequality
\[
||f(x) - 2^{n}f(2^{-n}x)|| \leq \frac{3\varepsilon}{2^{a} - 2} \left(1 - 2^{n(1-a)}\right)||x||^{a}, \quad (17)
\]
for all \( n \in \mathbb{N} - \{0\} \). Thus from this inequality (17) and the formula
\[
A(x) = \lim_{n \to \infty} 2^{n}f(2^{-n}x),
\]
for \( n \to \infty \), we get the inequality
\[
||f(x) - A(x)|| \leq \frac{3\varepsilon}{2^{a} - 2}||x||^{a}, \text{ for } \alpha > 1.
\]
The rest of the proof for \( \alpha > 1 \) is omitted as similar to the above mentioned proof for \( -\infty < \alpha < 1 \).
References


