Simultaneous Generalizations of Regularity and Normality

A. K. Das
School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India

Abstract. A generalization of regularity called $\theta$-regularity was earlier introduced to decompose normality and also utilised to factorize regularity. Every normal space need not be regular, but every normal space is $\theta$-regular. In this paper three variants of $\theta$-regular spaces is introduced and studied.

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1. Introduction and Preliminaries

Many generalizations of regularity that exists in the mathematical literature fails to be a generalization of normality. But in order to obtain a decomposition of normality, the notion of $\theta$-regularity was introduced in [6] which is a simultaneous generalization of regularity as well as normality. It is obvious from the definition that every regular space is $\theta$-regular as in a regular space every closed set is $\theta$-closed [14]. In general a normal space need not be regular, but in contrast every normal space is $\theta$-regular [6]. Also it is observed in [5] that the notion of $\theta$-regularity serves as a decomposition of regularity in terms of $R_0$ and $R_1$ spaces. In this paper we introduced three more variants of $\theta$-regular spaces and studied their properties.

Let $X$ be a topological space and let $A \subset X$. Throughout the present paper, the closure of a set $A$ will be denoted by $\overline{A}$ or $clA$ and the interior by $intA$. A set $U \subset X$ is said to be regularly open if $U = int\overline{U}$. The complement of a regularly open set is called regularly closed. A point $x \in X$ is called a $\theta$-limit point [14] of $A$ if every closed neighbourhood of $x$ intersects $A$. Let $cl_\theta A$ denotes the set of all $\theta$-limit point of $A$. The set $A$ is called $\theta$-closed if $A = cl_\theta A$. The complement of a $\theta$-closed set will be referred to as a $\theta$-open set. The family of $\theta$-open sets forms a topology on $X$. A space $X$ is said to be almost regular [9] if every regularly closed set and a point not in it are contained in disjoint open sets. A space is called almost normal [10] if every pair of disjoint closed sets, one of which is regularly closed, are contained in disjoint open sets and a space $X$ is said to be mildly normal [12] (or $\kappa$-normal [13]) if every pair of disjoint regularly closed sets are contained in disjoint open sets. A space $X$ is said to be

Email addresses: ak.das@smvd. ac.in, akdasdu@yahoo.co.in

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nearly compact [11] if every open covering of \( X \) admits a finite subcollection the interiors of the closures of whose members cover \( X \).

**Definition 1.** A topological space \( X \) is said to be

(i) \( \theta \)-normal [6] if every pair of disjoint closed sets one of which is \( \theta \)-closed are contained in disjoint open sets;

(ii) weakly \( \theta \)-normal [6] if every pair of disjoint \( \theta \)-closed sets are contained in disjoint open sets;

(iii) functionally \( \theta \)-normal ([4, 6]) if for every pair of disjoint closed sets \( A \) and \( B \) one of which is \( \theta \)-closed there exists a continuous function \( f : X \to [0,1] \) such that \( f(A) = 0 \) and \( f(B) = 1 \);

(iv) weakly functionally \( \theta \)-normal (wf \( \theta \)-normal) ([4, 6]) if for every pair of disjoint \( \theta \)-closed sets \( A \) and \( B \) there exists a continuous function \( f : X \to [0,1] \) such that \( f(A) = 0 \) and \( f(B) = 1 \); and

(v) \( \Sigma \)-normal [7] if for each closed set \( F \) and each open set \( U \) containing \( F \), there exists a regular \( F_{\sigma} \) set \( V \) such that \( F \subset V \subset U \).

### 2. Variants of \( \theta \)-regular Spaces

**Definition 2.** A topological space \( X \) is said to be

(i) \( \theta \)-regular [6] if for each closed set \( F \) and each open set \( U \) containing \( F \), there exists a \( \theta \)-open set \( V \) such that \( F \subset V \subset U \).

(ii) weakly \( \theta \)-regular if for each \( \theta \)-closed set \( F \) and each open set \( U \) containing \( F \), there exists a \( \theta \)-open set \( V \) such that \( F \subset V \subset U \).

(iii) point \( \theta \)-regular if for each closed singleton \( \{x\} \) and each open set \( U \) containing \( x \), there exists a \( \theta \)-open set \( V \) such that \( x \in V \subset U \).

(iv) point weakly \( \theta \)-regular if for each \( \theta \)-closed singleton \( \{x\} \) and each open set \( U \) containing \( x \), there exists a \( \theta \)-open set \( V \) such that \( x \in V \subset U \).

The above notion of \( \theta \)-regularity is exclusively different from the concept of \( \theta \)-regularity introduced by Jankovic [3] which was utilized by Kovar [8] to study covering axioms including compactness and paracompactness. In [8], Kovar proved that Jankovic’s \( \theta \)-regularity coincides with the notion of point paracompactness introduced by Boyte [1]. From here onward the term “\( \theta \)-regularity” will always be meant in the sense of Definition 2.

The following implications are obvious, but none of them are reversible.
**Example 1** (A point $\theta$-regular space which is not $\theta$-regular.) Let $X = \{a, b, c, d, e\}$ and $T = \{\{a, b, c\}, \{c, d, e\}, \{c\}, \varnothing, X\}$. Here $X$ is vacuously point $\theta$-regular, but not $\theta$-regular as $\{a, b\} \subset \{a, b, c\}$ but there is no $\theta$-open set containing $\{a, b\}$ and contained in $\{a, b, c\}$.

**Example 2** (A point weakly $\theta$-regular space which is not point $\theta$-regular.) Co-finite topology is point weakly $\theta$-regular but not point $\theta$-regular.

**Example 3** (A point weakly $\theta$-regular space which is not point $\theta$-regular.) Let $X = \{a, b, c\}$ and $T = \{\{a, b\}, \{b, c\}, \{b\}, \varnothing, X\}$. Here $X$ is vacuously point weakly $\theta$-regular, but not point $\theta$-regular as $\{a\} \subset \{a, b\}$ but there is no $\theta$-open set containing $\{a\}$ and contained in $\{a, b\}$.

**Question 1.** Does there exists a point weakly $\theta$-regular space which is not weakly $\theta$-regular?

It is obvious from the definitions that, a $R_0$-space is regular if and only if it is $\theta$-regular and a $T_1$-space is $T_3$ if and only if it is point $\theta$-regular. Similarly, a Hausdorff space is $T_3$ if and only if it is point weakly $\theta$-regular.

**Theorem 1.** For a point $\theta$-regular space, the following statements are equivalent.

(i) For every pair of distinct points $x$ and $y$ in $X$, there exist $\theta$-open sets $P$ and $Q$ such that $x \in U, y \in V$ and $P \cap Q = \varnothing$.

(ii) $X$ is $\theta T_2$.

(iii) $X$ is Urysohn.

(iv) $X$ is $T_2$.

(v) $X$ is $T_1$.

**Proof.** Let $x$ and $y$ be two disjoint points in $X$. Since $X$ is $T_1$, the closed set $\{x\}$ is contained in an open set $X - \{y\}$. Thus by point $\theta$-regularity of $X$, there exists a $\theta$-open set $V$ such that $x \in V \subset X - \{y\}$. Since $V$ is $\theta$-open there exists a open set $U$ such that $x \in U \subset \overline{U} \subset V \subset X - \{y\}$. i.e., $x \in U$ and $y \in X - \overline{U}$. Again by point $\theta$-regularity, there exist $\theta$-open sets $P$ and $Q$ such that $x \in P$, $y \in Q$ and $P \cap Q = \varnothing$. 

Theorem 2. For a $T_1$ space, the following statements are equivalent.

(i) $X$ is $T_3$.
(ii) $X$ is regular.
(iii) $X$ is $\theta$-regular.
(iv) $X$ is point $\theta$-regular.

Proof. Let $X$ be a $T_1$ point $\theta$-regular space. Let $x \notin A$, where $A$ is a closed set in $X$. Since $X$ is a $T_1$ space, the singleton $\{x\}$ is closed and contained in $X - A$. By Point $\theta$-regularity of $X$, there exists a $\theta$-open set $V$ such that $x \in V \subset X - A$. Since $V$ is $\theta$-open there exists an open set $U$ such that $x \in U \subset \overline{U} \subset V \subset X - A$. Therefore $X$ is regular and thus $T_3$.

Theorem 3. Every $T_1$ point $\theta$-regular space is Hausdorff.

Proof. Let $X$ be a $T_1$ point $\theta$-regular space and let $x, y$ be two distinct points in $X$. Since $X$ is $T_1$, $\{x\}$ is a closed singleton contained in the open set $X - \{y\}$. By point $\theta$-regularity of $X$, there exists a $\theta$-open set $U$ such that $x \in U \subset X - \{y\}$. Thus there exists an open set $V$ such that $x \in V \subset \overline{V} \subset U \subset X - \{y\}$. So $V$ and $X - \overline{V}$ are two disjoint open sets containing $x$ and $y$ respectively.

Theorem 4. For a $T_2$ space, the following statements are equivalent.

(i) $X$ is $T_3$.
(ii) $X$ is regular.
(iii) $X$ is $\theta$-regular.
(iv) $X$ is weakly $\theta$-regular.
(v) $X$ is point $\theta$-regular.
(vi) $X$ is point weakly $\theta$-regular.

Proof. Obvious.

Theorem 5. Every functionally $\theta$-normal space is weakly $\theta$-regular.

Proof. Let $A$ be a $\theta$-closed set contained in an open set $U$. Let $B = X - U$. Then $A$ and $B$ are disjoint closed sets in $X$. By functional $\theta$-normality of $X$, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Let $V = f^{-1}[0, 1/2)$. Then $A \subset V \subset U$. We claim that $V$ is a $\theta$-open set. Let $x \in V$. Then $f(x) \in [0, 1/2)$. So there is a closed neighbourhood $N$ of $f(x)$ contained in $[0, 1/2) \subset [0, 1]$. Let $U_x = int f^{-1}(N)$. Then $x \in U_x \subset \overline{U_x} \subset f^{-1}(N) \subset V$. Hence $V$ is $\theta$-open. Therefore $X$ is $\theta$-regular.
Remark 1. Functionally $\theta$-normal spaces need not be $\theta$-regular. i.e.; Let $X = \{a, b, c\}$, $\tau = \{\{a, b\}, \{b\}, \{b, c\}, \phi, X\}$ is a functionally $\theta$-normal space which is not $\theta$-regular.

Theorem 6. Every nearly compact weakly $\theta$-regular space is $\theta$-normal.

Proof. Let $A$ and $B$ be two disjoint closed sets of $X$ where $A$ is $\theta$-closed. Then $A \subset X - B$. Thus by $\theta$-regularity of $X$ there exist an $\theta$-open set $V$ such that $A \subset V \subset X - B$. Since $\tau$ is $\theta$-open, for every $x \in A$ there exist an open set $U_x$ such that $x \in U_x \subset \overline{U_x} \subset V \subset X - B$. Then $\mathcal{U} = \{U_x : x \in A\}$ is an open cover of $A$. Since $A$ is $\theta$-closed, by [2, Proposition 2.1], $A$ is $N$-closed relative to $X$. Hence $\mathcal{U}$ has finite subcollection such that $A \subset \bigcup_{i=1}^{n} \text{int} \overline{U_{x_i}}$. Thus $B \subset \bigcap_{i=1}^{n} (X - \overline{U_{x_i}})$. therefore $X$ is $\theta$-normal.

Corollary 1. Every nearly compact $\theta$-regular space is normal.

Proof. The above result is obvious, since every $\theta$-regular $\theta$-normal space is normal.

Remark 2. The following example shows that the hypothesis of $\theta$-regularity in the above Corollary cannot be weakened to “weak $\theta$-regularity” as nearly compact weakly $\theta$-regular spaces need not be almost normal. e.g.; The set $X = \{a, b, c, d\}$ with topology $\tau = \{\{a, b\}, \{b\}, \{b, c\}, \{c\}, \{b, c, d\}, \{a, b, c\}, X, \emptyset\}$ is compact and weakly $\theta$-regular but not almost normal as the regularly closed set $\{c, d\}$ and closed set $\{a\}$ cannot be separated by disjoint open sets.

It is well known that every compact Hausdorff space is normal. However, in the absence of Hausdorffness or regularity a compact space may fail to be normal. Thus it is useful to know which topological property weaker than Hausdorffness with compactness implies normality. The property of being a $T_1$-space fails to do the job since the cofinite topology on an infinite set is a compact $T_1$ space which is not normal. However, it is well known that Every compact $R_1$-space is normal.

The following result of [6] is an improvement of well known results such as every compact Hausdorff space is normal and every compact (or Lindelöf) regular space is normal.

Theorem 7. Every paracompact $\theta$-regular space is normal.

Theorem 8. Every Lindelöf $\theta$-regular space is normal.

Remark 3. The condition of $\theta$-regularity in the above theorem cannot be weakened as the example cited in Remark 2 is a paracompact weakly $\theta$-regular space which fails to be almost normal.

Although every compact $\theta$-regular space is normal, but it is in the absence of $T_1$ property, as every $T_1$ $\theta$-regular space is regular. Thus it is very natural to ask the following Question.

Question 2. Which non-regular, non-Hausdorff, $T_1$-compact spaces are normal?
Let us recall that a space $X$ is seminormal if for every closed set $F$ contained in an open set $U$ there exists a regularly open set $V$ such that $F \subset V \subset U$. A space is said to be $\theta$-seminormal [15] if for every $\theta$-closed set $F$ contained in an open set $U$ there exists a regularly open set $V$ such that $F \subset V \subset U$.

**Example 4.** A seminormal space which is not $\theta$-regular. Let $X$ be the set of positive integers. Define a topology on $X$ by taking every odd integer to be open and a set $U \subset X$ is open if for every even integer $p \in U$, the predecessor and the successor of $p$ are also in $U$. Since every open set is regularly open in this topology, the space is seminormal but the space is not $\theta$-regular.

**Theorem 9.** Every almost regular seminormal space is $\theta$-regular.

**Proof.** Let $F$ be a closed set contained in an open set $U$. Since $X$ is seminormal there exists a regularly open set $V$ such that $F \subset V \subset U$. Since in an almost regular space every regularly open set is $\theta$-open, the space is $\theta$-regular.

**Corollary 2.** An almost regular space is normal if and only if it is seminormal and weakly $\theta$-normal.

**Proof.** Proof is obvious, since every $\theta$-regular weakly $\theta$-normal space is normal.

**Theorem 10.** Every almost regular $\theta$-seminormal space is weakly $\theta$-regular.

### 3. Subspaces

**Lemma 1.** If $Y \subset X$ and $A$ is any $\theta$-open set in $X$ then $A \cap Y$ is $\theta$-open in $Y$.

**Theorem 11.** If $Y$ is a closed subspace of $X$ and $X$ is $\theta$-regular then $Y$ is $\theta$-regular.

**Proof.** Let $X$ be a $\theta$-regular space and $Y \subset X$. Let $F$ be a closed set in $Y$ which is contained in an open set $U$ of $Y$. Since $F$ is closed in $Y$ and $Y$ is a closed subspace of $X$, $F$ is closed in $X$. Since $U$ is open in $Y$, there exists an open set $V$ in $X$ such that $U = V \cap Y$. Thus $F \subset V$. By $\theta$-regularity of $X$, there exists a $\theta$-open set $W$ in $X$ such that $F \subset W \subset V$, i.e.,

$$F \cap Y \subset W \cap Y \subset V \cap Y \Rightarrow F \subset W \cap Y \subset U.$$

By the previous lemma $W \cap Y$ is $\theta$-open in $Y$. Hence $Y$ is $\theta$-regular.

**Theorem 12.** If $Y$ is a closed subspace of $X$ and $X$ is point $\theta$-regular, then $Y$ is point $\theta$-regular.

**Lemma 2.** If $Y$ is $\theta$-open in $X$ and $A$ is $\theta$-open in $Y$, then $A$ is $\theta$-open in $X$.

**Lemma 3.** If $Y$ is $\theta$-open in $X$ and $A$ is $\theta$-closed in $Y$ then $A$ is $\theta$-closed in $X$.

**Proof.** Let $Y$ be a $\theta$-open set in $X$ and let $A$ be $\theta$-closed in $Y$. Then $(Y - A)$ is $\theta$-open in $Y$. Thus by previous lemma $(Y - A)$ is $\theta$-open in $X$. Therefore $X - (Y - A)$ is $\theta$-closed in $X$. Hence $A$ is $\theta$-closed in $X$. 

Theorem 13. If $Y$ is a $\theta$-open subspace of $X$ and $X$ is weakly $\theta$-regular, then $Y$ is weakly $\theta$-regular.

Proof. Let $Y$ be a $\theta$-open subspace of $X$ and $X$ is weakly $\theta$-regular. Let $F$ be a $\theta$-closed set in $Y$ and contained in an open set $U$ of $Y$. Since $Y$ is $\theta$-open in $X$, $F$ is $\theta$-closed in $X$. Since $U$ is open in $Y$, there exists an open set $V$ in $X$ such that $U = V \cap Y$. So $F \subseteq V$. By weak $\theta$-regularity of $X$, there exists a $\theta$-open set $W$ in $X$ such that $F \subseteq W \subseteq V$. Thus $F \subseteq W \cap Y \subseteq V$, where $W \cap Y$ is $\theta$-open in $Y$. Hence $Y$ is weakly $\theta$-regular.

Theorem 14. If $Y$ is a $\theta$-open subspace of $X$ and $X$ is point weakly $\theta$-regular, then $Y$ is point weakly $\theta$-regular.

References


