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On the Solution of Some Difference Equations

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Abstract. We obtain in this paper the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary nonzero real numbers.

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1. Introduction

In this paper we obtain the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots, \tag{1}$$

where the initial conditions are arbitrary nonzero real numbers.

The study of Difference Equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results

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in this area, for example: Agarwal et al. [2] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Aloqeili [4] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [6–8] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Cinar et al. [9] studied the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [11–12] investigated the global stability, periodicity character and gave the solution of some special cases of the following difference equations

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

In [19] Elsayed dealed with the dynamics and found the solution of the following rational recursive sequences

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-1} x_{n-3} x_{n-5}}.$$

Karatas et al. [34] obtained the solution of the difference equation

$$x_{n+1} = \frac{ax_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}.$$

Simsek et al. [38]-[39] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}, \quad x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1} x_{n-3}}.$$

In [40] Stevic solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}.$$

Yalçınkaya et al. [49] considered the dynamics of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed [52] considered the behavior of the following difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}.$$

Other related results on rational difference equations can be found in refs. [2-51].

The study of these equations is quite challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore we believe that these results about such equations over prototypes for the development of the basic theory of the global behavior of nonlinear rational difference equations.

Let us introduce some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 (Equilibrium Point). *A point $\overline{x} \in I$ is called an equilibrium point of Eq.* (2) *if*

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq. (2)*, or equivalently, $\overline{x}$ is a fixed point of f.*

Definition 2 (Periodicity). *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

2. On the Difference Equation $x_{n+1} = \dfrac{x_{n-3}}{1 + x_{n-1}x_{n-3}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots, \tag{3}$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.* (3)*. Then for $n = 0, 1, \ldots$*

$$x_{4n-3} = \frac{d\prod_{i=0}^{n-1}(1+2ibd)}{\prod_{i=0}^{n-1}(1+(2i+1)bd)}, \quad x_{4n-1} = \frac{b\prod_{i=0}^{n-1}(1+(2i+1)bd)}{\prod_{i=0}^{n-1}(1+(2i+2)bd)},$$

$$x_{4n-2} = \frac{c\prod_{i=0}^{n-1}(1+2iac)}{\prod_{i=0}^{n-1}(1+(2i+1)ac)}, \quad x_{4n} = \frac{a\prod_{i=0}^{n-1}(1+(2i+1)ac)}{\prod_{i=0}^{n-1}(1+(2i+2)ac)},$$

where $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_{-0} = a$, $\prod_{i=0}^{-1} A_i = 1$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$x_{4n-7} = \frac{d\prod_{i=0}^{n-2}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)}, \quad x_{4n-5} = \frac{b\prod_{i=0}^{n-2}(1+(2i+1)bd)}{\prod_{i=0}^{n-2}(1+(2i+2)bd)},$$

$$x_{4n-6} = \frac{c\prod_{i=0}^{n-2}(1+2iac)}{\prod_{i=0}^{n-2}(1+(2i+1)ac)}, \quad x_{4n-4} = \frac{a\prod_{i=0}^{n-2}(1+(2i+1)ac)}{\prod_{i=0}^{n-2}(1+(2i+2)ac)}.$$

Now, it follows from Eq. (3) that

$$x_{4n-3} = \frac{x_{4n-7}}{1+x_{4n-5}x_{4n-7}} = \frac{\dfrac{d\prod_{i=0}^{n-2}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)}}{1+\dfrac{b\prod_{i=0}^{n-2}(1+(2i+1)bd)}{\prod_{i=0}^{n-2}(1+(2i+2)bd)}\dfrac{d\prod_{i=0}^{n-2}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)}}$$

$$= \frac{d\prod_{i=0}^{n-2}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)\left(1+\dfrac{bd\prod_{i=0}^{n-2}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+2)bd)}\right)}$$

$$= \frac{d\prod_{i=0}^{n-2}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)\left(1+\frac{bd}{(1+(2n-2)bd)}\right)}$$

$$= \frac{d\prod_{i=0}^{n-2}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)\left(1+\frac{bd}{(1+(2n-2)bd)}\right)}\frac{(1+(2n-2)bd)}{(1+(2n-2)bd)}$$

$$= \frac{d\prod_{i=0}^{n-1}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)((1+(2n-2)bd)+bd)}$$

$$= \frac{d\prod_{i=0}^{n-1}(1+2ibd)}{\prod_{i=0}^{n-2}(1+(2i+1)bd)(1+(2n-1)bd)}.$$

Hence, we have

$$x_{4n-3} = \frac{d\prod_{i=0}^{n-1}(1+2ibd)}{\prod_{i=0}^{n-1}(1+(2i+1)bd)}.$$

Similarly one can prove the other relations. The proof is complete.

Theorem 2. *Eq.* (3) *has a unique equilibrium point which is the number zero.*

Proof. For the equilibrium points of Eq. (3), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^2}.$$

Then

$$\overline{x} + \overline{x}^3 = \overline{x},$$

or,

$$\overline{x}^3 = 0.$$

Thus the equilibrium point of Eq. (3) is $\overline{x} = 0$.

Theorem 3. *Every positive solution of Eq.* (3) *is bounded and* $\lim_{n\to\infty} x_n = 0$.

Proof. It follows from Eq. (3) that

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}} \leq x_{n-3}.$$

Then the subsequences $\{x_{4n-3}\}_{n=0}^{\infty}$, $\{x_{4n-2}\}_{n=0}^{\infty}$, $\{x_{4n-1}\}_{n=0}^{\infty}$, $\{x_{4n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-3}, x_{-2}, x_{-1}, x_0\}$.

Lemma 1. *Eq.* (3) *has no prime period two solution.*

Numerical Examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (3).

Example 1. *Consider* $x_{-3} = 4$, $x_{-2} = 9$, $x_{-1} = 6$, $x_0 = 7$. *See Fig. 1.*

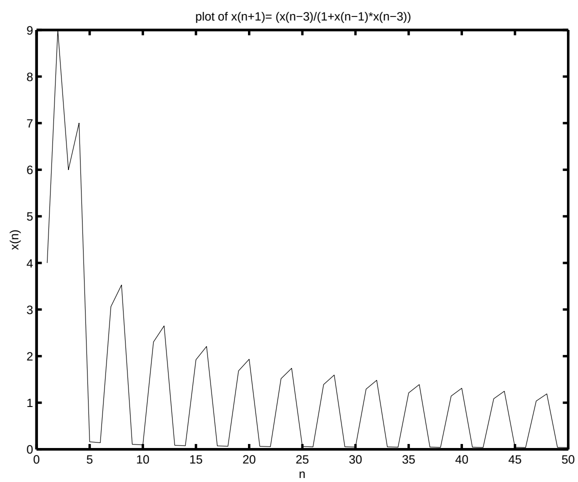

Figure 1

Example 2. *See Fig. 2, since* $x_{-3} = 1.4$, $x_{-2} = 0.9$, $x_{-1} = 0.6$, $x_0 = 0.7$.

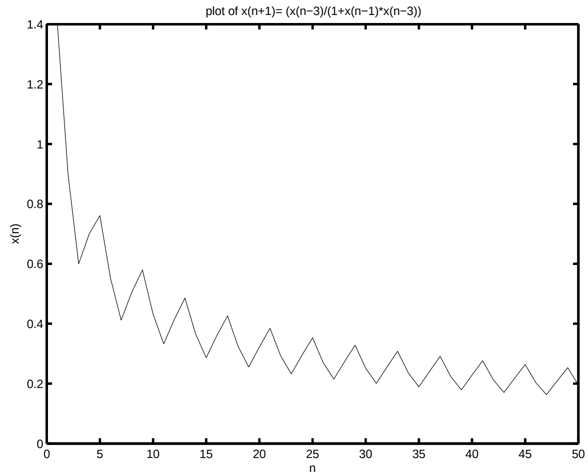

Figure 2

3. On the Difference Equation $x_{n+1}=\dfrac{x_{n-3}}{1-x_{n-1}x_{n-3}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1}=\frac{x_{n-3}}{1-x_{n-1}x_{n-3}}, \quad n=0,1,\ldots, \tag{4}$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 4. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.* (4)*. Then for $n=0,1,\ldots$*

$$x_{4n-3} = \frac{d\prod_{i=0}^{n-1}(1-2ibd)}{\prod_{i=0}^{n-1}(1-(2i+1)bd)}, \quad x_{4n-1}=\frac{b\prod_{i=0}^{n-1}(1-(2i+1)bd)}{\prod_{i=0}^{n-1}(1-(2i+2)bd)},$$

$$x_{4n-2} = \frac{c\prod_{i=0}^{n-1}(1-2iac)}{\prod_{i=0}^{n-1}(1-(2i+1)ac)}, \quad x_{4n}=\frac{a\prod_{i=0}^{n-1}(1-(2i+1)ac)}{\prod_{i=0}^{n-1}(1-(2i+2)ac)},$$

where $x_{-3}=d$, $x_{-2}=c$, $x_{-1}=b$, $x_{-0}=a$, $\prod_{i=0}^{-1}A_i=1$ and $jbd\neq 1$ $jac\neq 1$ for $j=1,2,3,\ldots$.

Proof. As the proof of Theorem 1.

Theorem 5. *Eq.* (4) *has a unique equilibrium point which is the number zero.*

Proof. As the proof of Theorem 2.

Numerical Examples

Example 3. *Consider* $x_{-3} = 0.7$, $x_{-2} = 0.5$, $x_{-1} = 3$, $x_0 = 4$. *See Fig. 3.*

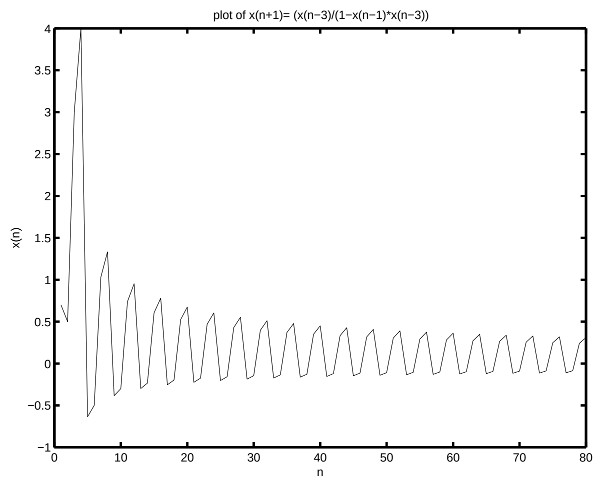

Figure 3

Example 4. *See Fig. 4, since* $x_{-3} = 7$, $x_{-2} = 11$, $x_{-1} = 0.3$, $x_0 = 4$.

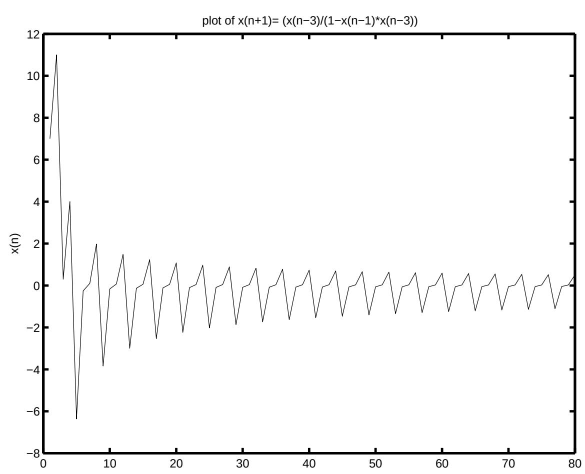

Figure 4

4. On the Difference Equation $x_{n+1} = \dfrac{x_{n-3}}{-1 + x_{n-1}x_{n-3}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots, \tag{5}$$

where the initial conditions are arbitrary non zero real numbers with $x_{-3}x_{-1} \neq 1$, $x_{-2}x_0 \neq 1$.

Theorem 6. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.* (5). *Then for $n = 0, 1, \ldots$*

$$\begin{aligned} x_{4n-3} &= \frac{d}{(-1+bd)^n}, \quad x_{4n-1} = b\,(-1+bd)^n\,, \\ x_{4n-2} &= \frac{c}{(-1+ac)^n}, \quad x_{4n} = a\,(-1+ac)^n\,, \end{aligned}$$

where $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_{-0} = a$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned} x_{4n-7} &= \frac{d}{(-1+bd)^{n-1}}, \quad x_{4n-5} = b\,(-1+bd)^{n-1}\,, \\ x_{4n-6} &= \frac{c}{(-1+ac)^{n-1}}, \quad x_{4n-4} = a\,(-1+ac)^{n-1}\,. \end{aligned}$$

Now, it follows from Eq.(5) that

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-7}}{-1 + x_{4n-5}x_{4n-7}} = \frac{\dfrac{d}{(-1+bd)^{n-1}}}{-1 + b\,(-1+bd)^{n-1}\,\dfrac{d}{(-1+bd)^{n-1}}} \\ &= \frac{d}{(-1+bd)^{n-1}\,(-1+bd)}. \end{aligned}$$

Hence, we have

$$x_{4n-3} = \frac{d}{(-1+bd)^n}.$$

Similarly

$$\begin{aligned} x_{4n-2} &= \frac{x_{4n-6}}{-1 + x_{4n-4}x_{4n-6}} = \frac{\dfrac{c}{(-1+ac)^{n-1}}}{-1 + a\,(-1+ac)^{n-1}\,\dfrac{c}{(-1+ac)^{n-1}}} \\ &= \frac{c}{(-1+ac)^{n-1}\,(-1+ac)}. \end{aligned}$$

Hence, we have

$$x_{4n-3} = \frac{c}{(-1+ac)^n}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 7. *Eq.* (5) *has three equilibrium points which are* 0, $\sqrt{2}$, $-\sqrt{2}$.

Proof. For the equilibrium points of Eq. (5), we can write

$$\overline{x} = \frac{\overline{x}}{-1+\overline{x}^2}.$$

Thus we have

$$-\overline{x} + \overline{x}^3 = \overline{x},$$

or,

$$\overline{x}(\overline{x}^2 - 2) = 0.$$

Thus the equilibrium points of Eq. (5) are 0, $\sqrt{2}$, $-\sqrt{2}$.

Theorem 8. *Eq.* (5) *has a periodic solutions of period four iff* $ac = bd = 2$ *and will be take the form* $\{d, c, b, a, d, c, b, a, \ldots\}$.

Proof. First suppose that there exists a prime period four solution

$$d, c, b, a, d, c, b, a, \ldots,$$

of Eq. (5), we see from Eq. (5) that

$$\begin{aligned} d &= \frac{d}{(-1+bd)^n}, \quad b = b(-1+bd)^n, \\ c &= \frac{c}{(-1+ac)^n}, \quad a = a(-1+ac)^n, \end{aligned}$$

or,

$$(-1+bd)^n = 1, \quad (-1+ac)^n = 1.$$

Then

$$bd = 2, \ \ ac = 2.$$

Second suppose $ac = 2$, $bd = 2$. Then we see from Eq. (5) that

$$\begin{aligned} x_{4n-3} &= d, \quad x_{4n-2} = c, \\ x_{4n-1} &= b, \quad x_{4n} = a. \end{aligned}$$

Thus we have a period four solution and the proof is complete.

Lemma 2. *Eq.* (5) *has no prime period two solution.*

Lemma 3. *Assume that* ac, $bd \neq 1 \pm 1$. *Then Eq.* (5) *has unbounded solutions.*

Numerical Examples

Example 5. *We consider* $x_{-3} = 0.4$*,* $x_{-2} = 0.9$*,* $x_{-1} = 0.16$*,* $x_0 = 1.7$*. See Fig. 5.*

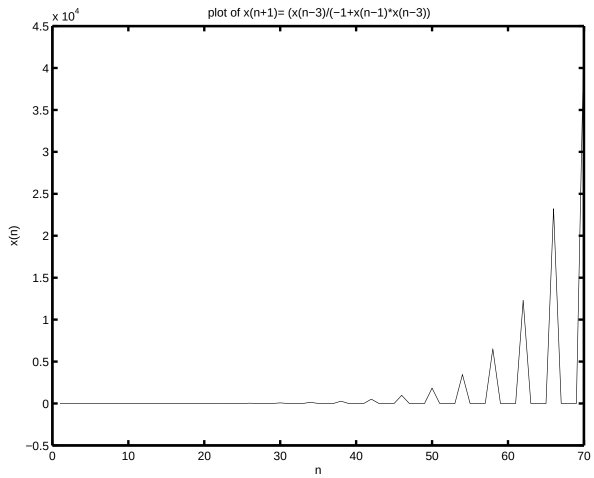

Figure 5

Example 6. *See Fig. 6, since* $x_{-3} = 0.7$*,* $x_{-2} = 0.5$*,* $x_{-1} = 20/7$*,* $x_0 = 4$*.*

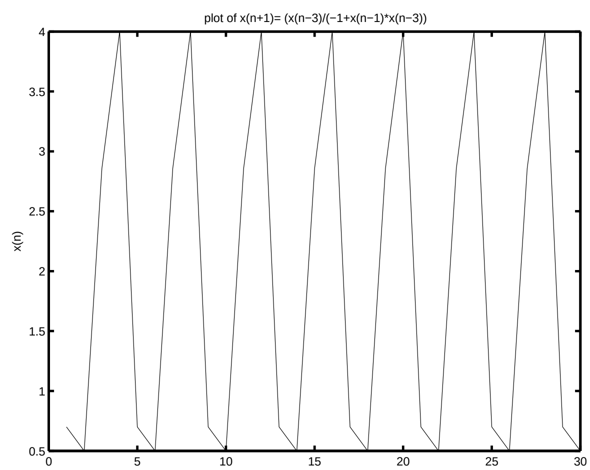

Figure 6

Example 7. *In Fig. 7, we assume* $x_{-3} = 0.7$*,* $x_{-2} = 0.5$*,* $x_{-1} = 3$*,* $x_0 = 4$*.*

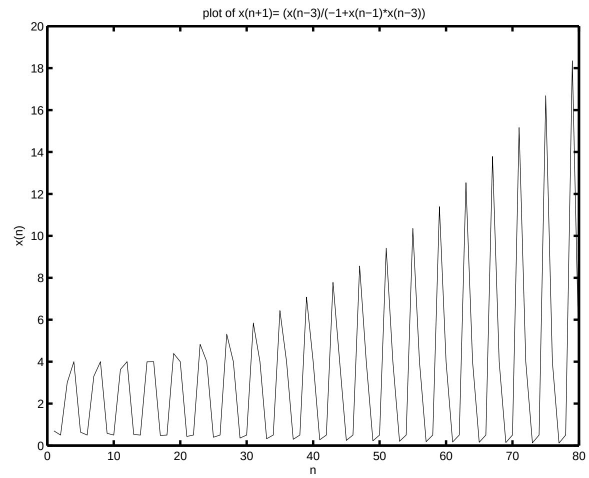

Figure 7

5. On the Difference Equation $x_{n+1}=\dfrac{x_{n-3}}{-1-x_{n-1}x_{n-3}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1}=\frac{x_{n-3}}{-1-x_{n-1}x_{n-3}},\quad n=0,1,\ldots, \tag{6}$$

where the initial conditions are arbitrary nonzero real numbers with
$x_{-3}x_{-1}\neq -1$, $x_{-2}x_0\neq -1$.

Theorem 9. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.* (6)*. Then for $n=0,1,\ldots$*

$$\begin{aligned} x_{4n-3} &= \frac{(-1)^n d}{(1+bd)^n}, \quad x_{4n-1}=(-1)^n b\,(1+bd)^n, \\ x_{4n-2} &= \frac{(-1)^n c}{(1+ac)^n}, \quad x_{4n}=(-1)^n a\,(1+ac)^n, \end{aligned}$$

where $x_{-3}=d$, $x_{-2}=c$, $x_{-1}=b$, $x_{-0}=a$.

Proof. As the proof of Theorem 6.

Theorem 10. *Eq.* (6) *has three equilibrium points which are* 0, $\sqrt{2}$, $-\sqrt{2}$.

Proof. As the proof of Theorem 7.

Theorem 11. *Eq.* (6) *has a periodic solutions of period four iff $ac=bd=-2$ and will be take the form $\{d,c,b,a,d,c,b,a,\ldots\}$.*

Proof. As the proof of Theorem 8.

Lemma 4. *Eq.* (6) *has no prime period two solution.*

Lemma 5. *Assume that ac, $bd\neq -1\pm 1$. Then Eq.* (6) *has unbounded solutions.*

Numerical Examples

Example 8. *We consider* $x_{-3} = 0.7$, $x_{-2} = 0.6$, $x_{-1} = 0.3$, $x_0 = 0.4$. *See Fig. 8.*

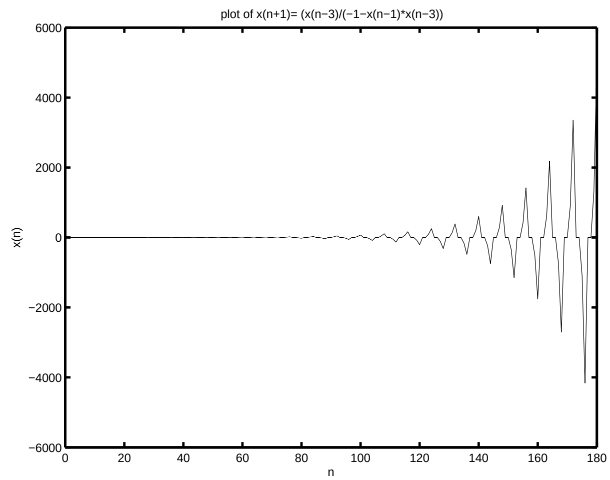

Figure 8

Example 9. *See Fig. 9, since* $x_{-3} = 0.7$, $x_{-2} = 6$, $x_{-1} = -3$, $x_0 = -0.4$.

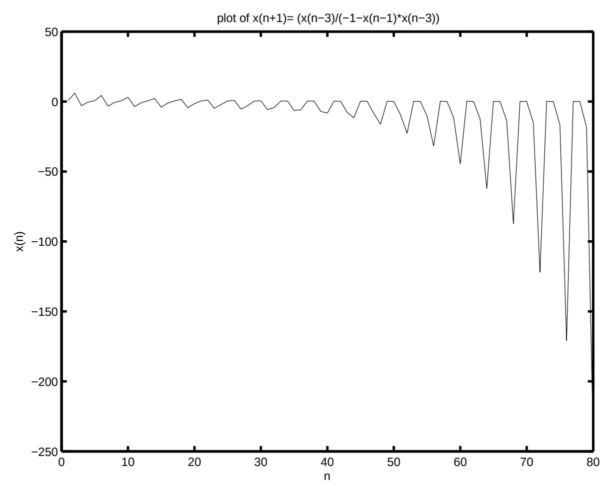

Figure 9

Example 10. *In Fig. 10, we assume* $x_{-3} = -2.5$, $x_{-2} = -6$, $x_{-1} = 0.8$, $x_0 = 1/3$.

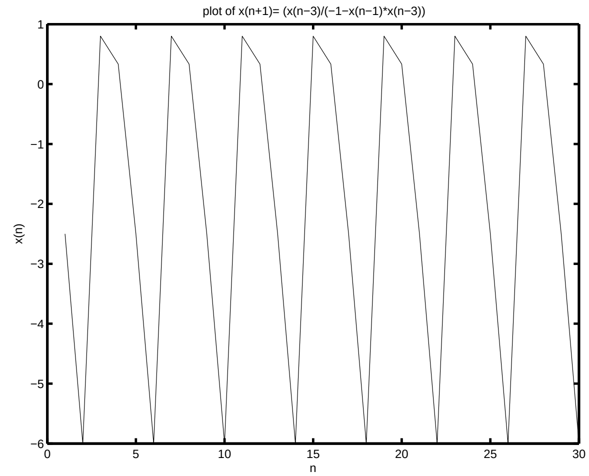

Figure 10

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