Fredholmness of Combinations of Two Idempotents

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Abstract. If $P$ and $Q$ are two idempotents on a Hilbert space, in this paper, we prove that Fredholmness of $aP + bQ - cPQ$ is independent of the choice of $a, b, c$ with $ab \neq 0$.

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1. Introduction

Idempotents are important and have wide applications in the theory of linear algebra and operator theorem. It is shown in [17] that every $n \times n$ matrix over a field of characteristic zero is a linear combination of three idempotents and in [16] that every bounded linear operator on a complex infinite Hilbert space is a sum of at most five idempotents. See also [5],[18],[19].

Let $X$ be a Banach space, and $P, Q$ be two idempotent operators on $X$. Many researchers (see [1]-[15] and the references within) have addressed stability properties of the linear combination $aP + bQ$; it has been proved that some properties such as invertibility, nullity, Fredholmness, closeness of the range and complementarity of the Kernel of linear combinations of $P$ and $Q$ are independent of the choice of coefficients $a$ and $b$, provided $ab \neq 0$ and $a + b \neq 0$.

A natural question is whether the results above can be extended to more general situations. In this note we consider the Fredholmness of some special combinations $aP + bQ - cPQ$ and $aP + bQ - cPQ - dQP$ when $P, Q$ are idempotents. We prove that Fredholmness and index of any combinations $aP + bQ - cPQ$ are independent of the choice of $a, b, c$ with $ab \neq 0$. As an application, we obtain that the invertibility of combinations $aP + bQ - cPQ$ are equivalent to the invertibility of $P + Q$ for all $a, b, c \in \mathbb{C}$ with $ab \neq 0$, which generalizes the result of [4]. Moreover, counter examples are shown that the combination $aP + bQ - cPQ - dQP$ fails to retain any such properties.

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2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space, and let all bounded linear operators on $\mathcal{H}$ be denoted by $\mathcal{B}(\mathcal{H})$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $P^2 = P$. The set $\mathcal{P}$ of all idempotents in $\mathcal{B}(\mathcal{H})$ is invariant under similarity; that is, if $P \in \mathcal{P}$ and $S \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then $S^{-1}PS$ is still an idempotent since $(S^{-1}PS)^2 = S^{-1}PSS^{-1}PS = S^{-1}P^2S = S^{-1}PS$. An idempotent $P$ is called an orthogonal projection if $P^2 = P = P^*$, where $P^*$ is the adjoint of $P$. Moreover, for an idempotent $P \in \mathcal{P}$, there exists an invertible operator $U \in \mathcal{B}(\mathcal{H})$ such that $U^{-1}PU$ is an orthogonal projection. In fact, if $P \in \mathcal{P}$, then $P$ can be written in the form of

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp$, where $\mathcal{R}(M)$ denotes the range of the operator $M$. In this case, we have

$$\begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where $\bar{P} = \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix}$ is invertible and $\bar{P}^{-1} = \begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$. If $A$ is positive, then $A^{1/2}$ denotes the positive square root of $A$. An operator $T$ is Fredholm if the nullities of $T$ denoted by $\text{nul}(T)$ and $T^*$ are finite and the range of $T$ is closed. For a Fredholm operator $T$, its index, $\text{ind}T$, is by definition $\text{nul}(T) - \text{nul}(T^*)$. It is know that the Fredholmness of $T$ is preserved under compact perturbations and is equivalent to the existence of an operator $T'$ with $TT' - I$ and $T'T - I$ being compact. For details of Fredholmness, see[3], Chapter XI.

For the proof of the main theorem we need the following two lemmas which are well known, so the proofs are omitted.

**Lemma 1** ([3]). Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a bounded linear operator on $\mathcal{H} \oplus \mathcal{H}$. Then $A$ is a positive operator if and only if $A_{11} \geq 0, A_{22} \geq 0, A_{12} = A_{21}^*$ and there exists a contraction $D$ from $\mathcal{H}$ into $\mathcal{H}$ such that

$$A = \begin{pmatrix} A_{11} & \frac{1}{2}A_{12} \frac{1}{2}A_{11}^*DA_{22}^* \\ \frac{1}{2}A_{22}^*DA_{11} & A_{22} \end{pmatrix}.$$ 

**Lemma 2** ([3]). Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an operator on $\mathcal{H} \oplus \mathcal{H}$, where $A$ is Fredholm with $A'$ acting on $\mathcal{H}$ satisfying $AA' = I + K_1$ and $A'A = I + K_2$ for some compact operators $K_1$ and $K_2$. Then $T$ is Fredholm if and only if $D - CA'B$ is. In this case, $\text{ind}T = \text{ind}A + \text{ind}(D - CA'B)$. 

3. Main results

**Theorem 1.** Let $P$ and $Q$ in $\mathcal{B}(\mathcal{H})$ be two idempotents, then the Fredholmness of $aP + bQ - cPQ$ is independent of the choice of $a$, $b$, $c$ with $ab \neq 0$ and $\text{ind}(aP + bQ - cPQ) = \text{ind}(P + Q)$.

**Proof.** Let $P$ and $Q$ be two idempotents. By the discussion above, since $aP + bQ - cPQ$ is Fredholm if and only if $aS^{-1}PS + bS^{-1}QS - c(S^{-1}PS)(S^{-1}PS)$ is Fredholm, to consider the Fredholmness of $aP + bQ - cPQ$, without loss of generality, we can assume that one of $P$ and $Q$ is an orthogonal projection. For example, assume that $Q$ is an orthogonal projection. Of course, $Q$ is a positive operator. In this case, by Lemma 1, $P$ and $Q$ have the following operator matrix forms:

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q_1 & \frac{1}{2}D\frac{1}{2} \\ Q_2D^*\frac{1}{2} & 2 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp$, where $Q_1$ and $Q_2$ are positive operators on $\mathcal{R}(P)$ and $\mathcal{R}(P)^\perp$, respectively, and $D$ is a contraction operator from $\mathcal{R}(P)^\perp$ into $\mathcal{R}(P)$. Furthermore, $Q_1$ and $Q_2$ have the following operator matrix forms:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{11} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} Q_{22} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respect to the space decomposition

$$\mathcal{R}(P) = \mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1) \oplus (\mathcal{R}(P) \ominus (\mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1)))$$

and the space decomposition

$$\mathcal{R}(P)^\perp = (\mathcal{R}(P)^\perp \ominus \mathcal{N}(I - Q_2)) \ominus \mathcal{N}(I - Q_2) \oplus \mathcal{N}(Q_2),$$

respectively. Then denote $\mathcal{H}_0 = \mathcal{N}(Q_1)$, $\mathcal{H}_1 = \mathcal{N}(I - Q_1)$, $\mathcal{H}_2 = \mathcal{R}(P) \ominus (\mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1))$, $\mathcal{H}_3 = \mathcal{R}(P)^\perp \ominus \mathcal{N}(I - Q_2)$ and $\mathcal{H}_4 = \mathcal{N}(I - Q_2)$, $\mathcal{H}_5 = \mathcal{N}(Q_2)$, therefore $P$ and $Q$ have the following matrix representations:

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & \frac{1}{2}D\frac{1}{2} & 0 & 0 \\ 0 & 0 & Q_{22}D^*\frac{1}{2} & Q_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} I & 0 & 0 & P_{11} & P_{12} & P_{13} \\ 0 & I & 0 & P_{21} & P_{22} & P_{23} \\ 0 & 0 & I & P_{31} & P_{32} & P_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=0}^{5} \mathcal{H}_{i}$ for some contraction $D_{1}$ from $\mathcal{H}_{3}$ to $\mathcal{H}_{2}$. If we let

$$Q_{0} = \begin{pmatrix} Q_{11} & Q_{22}^{\frac{1}{2}}D_{1}Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}}D_{1}^{*}Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix},$$

then $Q$ being an orthogonal projection implies that $Q_{0}$ is also an orthogonal projection on $\mathcal{H}_{2} \oplus \mathcal{H}_{3}$. That is, $Q_{0} = Q_{0}^{2}$. We obtain

$$\begin{aligned}
Q_{11} &= Q_{11}^{2} + Q_{11}^{\frac{1}{2}}D_{1}Q_{22}D_{1}^{*}Q_{11}^{\frac{1}{2}} , \\
Q_{11}^{\frac{1}{2}}D_{1}Q_{22}^{\frac{1}{2}} &= Q_{11}^{\frac{1}{2}}D_{1}Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}}D_{1}Q_{22}^{\frac{1}{2}}, \\
Q_{22}^{\frac{1}{2}}D_{1}^{*}Q_{11}^{\frac{1}{2}} &= Q_{22}^{\frac{1}{2}}D_{1}^{*}Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}}D_{1}^{*}Q_{11}^{\frac{1}{2}}, \\
Q_{22} &= Q_{22}^{2} + Q_{22}^{\frac{1}{2}}D_{1}Q_{11}D_{1}Q_{22}.
\end{aligned}$$

It can be derived by using the injectivity of $Q_{11}$, $I - Q_{11}$, $Q_{22}$ and $I - Q_{22}$ that

$$\begin{aligned}
D_{1}D_{1}^{*} &= I, \\
D_{1}^{*}D_{1} &= I, \\
Q_{22} &= D_{1}^{*}(I - Q_{11})D_{1}.
\end{aligned}$$

Note that

$$aP + bQ - cPQ = \begin{pmatrix} U_{11} & U_{13} & U_{14} & U_{15} & U_{16} \\ 0 & U_{22} & U_{23} & U_{24} & U_{25} & U_{26} \\ 0 & 0 & V_{11} & V_{12} & U_{35} & U_{36} \\ 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & U_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=0}^{5} \mathcal{H}_{i}$, where

$$\begin{aligned}
U_{11} &= aI, & U_{13} &= -cP_{11}Q_{22}^{\frac{1}{2}}D_{1}^{*}Q_{11}^{\frac{1}{2}}, & U_{15} &= aP_{12} - cP_{12}, \\
U_{14} &= aP_{11} - cP_{11}Q_{22}, & U_{16} &= aP_{13}, & U_{22} &= (a + b - c)I, \\
U_{23} &= -cP_{21}Q_{22}^{\frac{1}{2}}D_{1}^{*}Q_{11}^{\frac{1}{2}}, & U_{24} &= aP_{21} - cP_{21}Q_{22}, & U_{25} &= aP_{22} - cP_{22}, \\
U_{25} &= aP_{22} - cP_{22}, & U_{35} &= aP_{32} - cP_{32}, & U_{36} &= aP_{33}, \\
U_{55} &= bI.
\end{aligned}$$
and

\[
V_{11} = aI + bQ_{11} - c(Q_{11} + P_{31}Q_{22}D_1^* Q_{11}^{\frac{1}{2}})
\]

\[
= aI + bQ_{11} - c(Q_{11} + P_{31}D_1^* Q_{11}^{\frac{1}{2}}(I - Q_{11}^{\frac{1}{2}})),
\]

\[
V_{12} = aP_{31} + bQ_{11}^\frac{1}{2}D_1 Q_{22}^\frac{1}{2} - c(Q_{11}^\frac{1}{2}D_1 Q_{22}^\frac{1}{2} + P_{31}Q_{22}),
\]

\[
= aP_{31} + bQ_{11}^\frac{1}{2}(I - Q_{11})^\frac{1}{2} D_1 - c(Q_{11}^\frac{1}{2}(I - Q_{11})^\frac{1}{2} D_1
\]

\[
+ P_{31}D_1^* (I - Q_{11})^\frac{1}{2} D_1),
\]

\[
V_{21} = bQ_{22}^\frac{1}{2}D_1^* Q_{11}^\frac{1}{2} = bD_1^* Q_{11}^\frac{1}{2}(I - Q_{11})^\frac{1}{2},
\]

\[
V_{22} = bQ_{22} = bD_1^*(I - Q_{11}) D_1.
\]

We claim that \( aP + bQ - cPQ \) is Fredholm if and only if \( I - Q_{11} \) is invertible and \( I - P_{31}D_1^*(I - P_{11})^{-\frac{1}{2}} P_{11}^\frac{1}{2} \) is Fredholm. Indeed, if \( aP + bQ - cPQ \) is Fredholm, then, letting \( A \) be an operator on \( \mathcal{H} \) such that \( K = (aP + bQ - cPQ)A - I \) is compact, we have, with

\[
A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}
\] and \( K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \) on \( \mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp \),

\[
\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}
\]

\[
\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}
\] = \( \begin{pmatrix} I + K_1 & K_2 \\ K_3 & I + K_4 \end{pmatrix} \).

Carrying out the multiplication here yields

\[
bQ_{22}^\frac{1}{2}D_1^* Q_{11}^\frac{1}{2}A_2 + bQ_{22}A_4 = I + K_4
\]

or

\[
bQ_{22}^\frac{1}{2}(D_1^* Q_{11}^\frac{1}{2}A_2 + Q_{22}^\frac{1}{2}A_4) = I + K_4.
\]

This shows that \( Q_{22}^\frac{1}{2} \) is Fredholm and hence so is \( Q_{22} \). Therefore, \( Q_{22} \) is invertible and thus so is \( I - Q_{11} \) by (1). The Fredholmness of \( aP + bQ - cPQ \) is equivalent to that of

\[
\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}
\]

by (3), which is in turn equivalent to that of

\[
V_{11} - V_{12}V_{22}V_{21} = aI + bQ_{11} - (aP_{31} + bQ_{11}^\frac{1}{2}D_1 Q_{22}^\frac{1}{2})(bQ_{22}^\frac{1}{2}D_1^* Q_{11}^\frac{1}{2})
\]

by Lemma 2. But this letter operator is equal to

\[
aI + bQ_{11} - (aP_{31} + bQ_{11}^\frac{1}{2}D_1 D_1^*(I - Q_{11})^\frac{1}{2} D_1)D_1^*(I - Q_{11})^{-\frac{1}{2}}D_1 D_1^* Q_{11}^\frac{1}{2},
\]
which can be further simplified to
\[ a(I - P_{31}D_1^*(I - Q_{11})^{-\frac{1}{2}}Q_{11}^{\frac{1}{2}}) \]
by (1). This proves one direction. For the other, if \( I - Q_{11} \) is invertible and \( I - P_{31}D_1^*(I - Q_{11})^{-\frac{1}{2}}Q_{11}^{\frac{1}{2}} \) is Fredholm then we can reverse the above arguments to show that \( aP + bQ - cPQ \) is Fredholm. The equivalence of Fredholmness of \( aP + bQ - cPQ \) and \( P + Q \) follows easily.

Finally, we also have
\[ \text{ind}(aP + bQ - cPQ) = \text{ind}(I - P_{31}D_1^*(I - Q_{11})^{-\frac{1}{2}}Q_{11}^{\frac{1}{2}}) = \text{ind}(P + Q), \]
which complete the proof.

As an application, we immediately have the following corollary.

**Corollary 1.** Let \( P, Q \) be two idempotents in \( \mathcal{B}(X) \). Then

(i) the invertibility of \( aP + bQ - cPQ \) is independent of the choice of \( a, b, c \in \mathbb{C} \) and \( ab \neq 0 \).

(ii) the invertibility of \( aP + bQ - cPQ \) is equivalent to the invertibility of \( aP + bQ \) for all choice of \( a, b, c \in \mathbb{C} \) and \( ab \neq 0 \).

**Proof.**

(i) Let \( a_0P + b_0Q - c_0QP \) be invertible for some \( a_0, b_0, c_0 \in \mathbb{C} \) with \( a_0b_0 \neq 0 \). Then \( a_0P + b_0Q - c_0QP \) is Fredholm with the nullity and defect equal to zero. By the above Theorem, \( aP + bQ - cPQ \) is invertible for all \( a, b, c \in \mathbb{C} \) with \( ab \neq 0 \).

(ii) Let \( c = 0 \), then the (ii) follows from (i).

**Remark 1.** Let \( c = 0 \), we obtain the Theorems of [4] and [7].

As to the invertibility of \( aP + bQ - cPQ \), there is a natural question that does the combination \( aP + bQ - cPQ - dQP \) retain the invertibility for any \( ab \neq 0 \) and \( a + b = c + d \). However, there is an counterexample to note that this is impossible. Let \( P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( Q = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \), then \( P, Q \) are idempotent and the determinant of \( aP + bQ - cPQ - dQP \) is 0 when \( a = 12, b = -5, c = 10, d = -3 \) with \( a + b = c + d \), and is \(-3\) when \( a = 1, b = 1, c = -1, d = -1 \) with \( a + b = c + d \). So the invertibility of \( aP + bQ - cPQ - dQP \) depending on the choice of scalars \( a, b, c, d \) with \( a + b = c + d \). Therefore the idea of generalize the invertibility of \( aP + bQ - cPQ \) or \( aP + bQ - cQP \) to the invertibility of \( aP + bQ - cPQ - dQP \) or more generally \( aP + bQ - cPQ - dQP - ePQP - fQPQ - \cdots \) can not be achieved.
References


