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On $\mathscr{I}$-Convergence in the Topology Induced by Probabilistic Norms

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Abstract. The concepts of $\mathscr{I}$ -convergence is a natural generalization of statistical convergence and it is dependent on the notion of the ideal of subsets of $\mathbb{N}$ of positive integer set. In this paper we study the $\mathscr{I}$ -convergence of sequences, $\mathscr{I}$ -convergence of sequences of functions and $\mathscr{I}$-Cauchy sequences in probabilistic normed spaces and prove some important results.

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1. Introduction

The concepts of statistical convergence was introduced (independently) by Fast [7] and Steinhause [25]. In their studies, the concept of ordinary convergence of sequence of real numbers was extended to statistical convergence in the following way: a sequence $\{x_n\} \subset \mathbb{R}$ is said to be statistically convergent to the real number $x_0 \in \mathbb{R}$ provided that each ϵ neighborhood $\mathcal{N}_\epsilon(x_0)$ of x_0, the set consisting of all elements not contained by $\mathcal{N}_\epsilon(x_0)$ has natural density zero for any $\epsilon > 0$. The notion of natural density here can be described as a function $\delta : 2^{\mathbb{N}} \to [0,1]$ and given by $\delta(K) := \lim_{n\to\infty} n^{-1}|\{k \in K : k \leq n\}|$ where $K \subset \mathbb{N}$, and $|A|$ denotes the cardinality of the set A. The concept of statistical convergence was further discussed and developed by many authors including [1,4,8–11,19,21]. Statistical convergence has also been discussed in more general abstract spaces such as the fuzzy number spaces [22], locally convex spaces [18], Banach spaces [15] and characterization of Banach spaces [5]. Recently, Karakus [13] has extended the concept of statistical convergence for sequences in probabilistic normed spaces (PN space) and proved several interesting results. In another paper, Karakus and Demirci [14] studied the concept of statistical convergence of double sequences on PN spaces. The idea of $\mathscr{I}$ -convergence for sequences, was inspired by the concept of statistical convergence introduced in [7], see Kostyrko et al. [16] for a comprehensive bibliography. It is a natural generalization of the concept of statistical convergence. The $\mathscr{I}$ -convergence is based on the notion of the ideal $\mathscr{I}$ of subsets of $\mathbb{N}$, the set of positive integers. Here, a sequence $\{x_n\} \subset \mathbb{R}$ is said to be $\mathscr{I}$ -convergent to the real number $x_0 \in \mathbb{R}$ provided that each ϵ neighborhood $\mathcal{N}_\epsilon(x_0)$ of x_0, the set consisting of all elements not contained by $\mathcal{N}_\epsilon(x_0)$ belongs to $\mathscr{I}$ for any $\epsilon > 0$. Further works on ideal convergence can be found in [2,3,6,12,17,20] The work of Karakus [13] inspired us to study the $\mathscr{I}$ -convergence and other related properties in PN spaces. In this context, we obtain some results that parallel to the one given in [3, 12, 20, 24].

Now we recall some notation and definitions used in this paper (see [26]).

Definition 1.1. *A function* $f:\mathbb{R}\to\mathbb{R}_0^+$ *is called a distribution function if it is non-decreasing and left-continuous with* $\inf_{t\in\mathbb{R}} f(t)=0$ *and* $\sup_{t\in\mathbb{R}} f(t)=1$. *We denote the set of all distribution function by* Δ^+.

Definition 1.2. *A* t*-norm* T *is a continuous mapping* $T:[0,1]\times[0,1]\to[0,1]$ *such that for all* $a,b,c,d\in[0,1]$
(i) $T(a,b)=T(b,a)$;
(ii)*T(a,T(b,c))=T(T(a,b),c);*
(iv) $T(a,b)\le T(c,d)$ *whenever* $a\le c$ *and* $b\le d$;
(v) $T(a,1)=a$.

Example 1.1. *The operation* $T(a,b)=ab$, $T(a,b)=max(a+b-1,0)$ *and* $T(a,b)=min(a,b)$ *on* $[0,1]$ *are* t*-norms.*

The following definition is due to A. N. Šerstnev [23].

Definition 1.3. *A probabilistic normed space (briefly, a PN space) is a triplet* (X,F,T)*, where* X *is a real linear space,* T *is a continuous* t*-norm, and* F *(called* probabilistic norm*) is a mapping from* X *into* Δ^+ *(writing* $F(x)$ *as* F_x*), the following conditions hold for every* $x,y\in X$ *and every* $s,t>0$*:*
(N1) $F_x(t)=1$ *if and only if* $x=\theta$*(the null vector of* X*);*
(N2) $F_{\alpha x}(t)=F_x(\frac{t}{|\alpha|})$ *for* $\alpha\neq 0$;
(N3) $F_{x+y}(s+t)\ge T(F_x(s),F_y(t))$;

Example 1.2. *Let* $(X,\|\cdot\|)$ *is a normed space and* $T(a,b)=ab$ *(or T(a,b)=min(a,b)). Define*

$$F_x(t)=\frac{t}{t+\|x\|}$$

where $x \in X$ *and* $t > 0$. *Then* (X, F, T) *is a PN space.*

Let (X, F, T) be a PN space. Since T is a continuous t-norm, the system of (ϵ, λ)-*neighborhoods* of θ (the null vector in X)

$$\{\mathcal{N}_\theta(\epsilon, \lambda) \colon \epsilon > 0, \lambda \in (0, 1)\}, \tag{1.1}$$

where

$$\mathcal{N}_\theta(\epsilon, \lambda) = \{x \in X : F_x(\epsilon) > 1 - \lambda\} \tag{1.2}$$

determines a first countable Hausdorff topology on X, called the F-*topology*. Thus, the F-topology can be completely specified by means of F-convergence of sequences.

It is clear that $x - y \in \mathcal{N}_\theta$ means $y \in \mathcal{N}_x$ and vice-versa.

A sequence (x_n) is said to be F-convergent to $\xi \in X$ if for every $\epsilon > 0$, and for every $\lambda \in (0, 1)$ there exists a number $N \in \mathbb{N}$ such that

$$x_n - \xi \in \mathcal{N}_\theta(\epsilon, \lambda) \quad \text{for all} \quad n \geq N.$$

or equivalently,

$$x_n \in \mathcal{N}_\xi(\epsilon, \lambda) \quad \text{for all} \quad n \geq N.$$

In this case we write $F - \lim x_n = \xi$.

Lemma 1.1. *Let* $(X, \|\cdot\|)$ *be a real normed space and* (X, F, T) *be a PN space induced by the probabilistic norm* $F_x(t) = \frac{t}{t + \|x\|}$, *where* $x \in X$ *and* $t > 0$. *Then for every* $\{x_n\}$ *in* X

$$\lim x_n = \xi \Rightarrow \mathfrak{I} - \lim x_n = \xi.$$

Proof. Let suppose that $\lim x_n = \xi$. Then for every $t > 0$ there exists a positive integer $N = N(t)$ such that

$$\|x_n - \xi\| < t \quad \text{for all} \quad n \geq N.$$

We observe that for any given $\epsilon > 0$,

$$\frac{\epsilon + \|x_n - \xi\|}{\epsilon} < \frac{\epsilon + t}{\epsilon}$$

which is equivalent to

$$\frac{\epsilon}{\epsilon + \|x_n - \xi\|} > \frac{\epsilon}{\epsilon + t} = 1 - \frac{t}{\epsilon + t}.$$

Therefore, by letting $\lambda = \frac{t}{\epsilon+t} \in (0,1)$ we have

$$F_{x_n - \xi}(\epsilon) > 1 - \lambda \quad \text{for all} \quad n \geq N.$$

This implies that $x_n \in \mathscr{N}_\xi(\epsilon, \lambda)$ for all $n \geq N$ as required.

We recall the definition and notations of ideal.

Definition 1.4. *A non-empty subset $\mathscr{I}$ of $2^{\mathbb{N}}$ is called an* ideal *on $\mathbb{N}$ if*
(i) $B \in \mathscr{I}$ whenever $B \subseteq A$ for some $A \in \mathscr{I}$ (closed under subsets),
(ii)$A \cup B \in \mathscr{I}$ whenever $A, B \in \mathscr{I}$ (closed under unions).

An ideal called *proper* if $\mathbb{N} \notin \mathscr{I}$. An ideal called *admissible* if its proper and contains all finite subsets.

Filter $\mathscr{F}$ is a dual notion to ideal $\mathscr{I}$-it is closed under supersets and intersections. It holds that $\{\mathbb{N} \backslash A \colon A \in \mathscr{I}\}$ is a filter if and only if $\mathscr{I}$ is ideal. The filter $\mathscr{F}(\mathscr{I})$ is called the filter associated with the ideal $\mathscr{I}$. Thus, one can write

$$A \in \mathscr{I} \Leftrightarrow A^c \in \mathscr{F}(\mathscr{I}).$$

where A^c denotes the complement of A.

Ideal can be viewed as a way to describe which sets will be considered "small", i.e., finite. Filter is collection of all "large" sets.

2. $\mathscr{I}$ -Convergence for Sequences in PN Spaces

In this section we define the ideal convergence of a sequence in (X, F, T) and prove some important results.

Definition 2.1. *Let $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and (X, F, T) be a PN space. The sequence (x_n) in X is said to be $\mathscr{I}^F - convergent$ to $x \in X$ ($\mathscr{I} - convergent$ to $x \in X$ with respect to F-topology) if for each $\epsilon > 0$, and $\lambda \in (0,1)$*

$$\{n \in \mathbb{N} : x_n \notin \mathscr{N}_x(\epsilon, \lambda)\} \in \mathscr{I}.$$

The vector x is called the $\mathscr{I}^F - limit$ of the sequence $\{x_n\}$ and we write $\mathscr{I}^F - \lim x_n = x$.

Definition 2.2. *Let (X, F, T) be a PN space and $\mathscr{I}$ be an admissible ideal in $\mathbb{N}$. The sequence $\{x_n\}$ in X is said to be $\mathscr{I}^{F*}$-convergent to $\xi \in X$ (i.e., $\mathscr{I}^{F*} - \lim x_n = \xi$) if and only if there exists a set $M = \{m_1 < m_2 < \cdots\} \in \mathscr{F}(\mathscr{I})$ such that $\mathfrak{I} - \lim x_{m_k} = \xi$.*

Lemma 2.1. *Let (X, F, T) be a PN space. $\mathscr{I}^F - limit$ of any sequence if exists is unique.*

Proof. Let $\{x_n\}$ be any sequence and suppose that $\mathscr{I}^F - \lim x_n = \xi$, $\mathscr{I}^F - \lim x_n = \eta$ where $\xi \neq \eta$. Since $\xi \neq \eta$, select $\epsilon > 0$ and $\lambda \in (0,1)$ such that $\mathscr{N}_\xi(\epsilon, \lambda)$ and $\mathscr{N}_\eta(\epsilon, \lambda)$ are disjoint neighborhoods of ξ and η. Since ξ and η both are $\mathscr{I}^F - limit$ of the sequence $\{x_n\}$, we have $A = \{n \in \mathbb{N} : x_n \notin \mathscr{N}_\xi(\epsilon, \lambda)\}$ and $B = \{n \in \mathbb{N} : x_n \notin \mathscr{N}_\eta(\epsilon, \lambda)\}$ are both belongs to $\mathscr{I}$. This implies that the sets $A^c = \{n \in \mathbb{N} : x_n \in \mathscr{N}_\xi(\epsilon, \lambda)\}$ and $B^c = \{n \in \mathbb{N} : x_n \in \mathscr{N}_\eta(\epsilon, \lambda)\}$ belongs to $\mathscr{F}(\mathscr{I})$. Since $\mathscr{F}(\mathscr{I})$ is a filter in $\mathbb{N}$, we have $A^c \cap B^c$ is a nonempty in $\mathscr{F}(\mathscr{I})$. In this way we obtain a contradiction to the fact that the neighborhoods $\mathscr{N}_\xi(\epsilon, \lambda)$ and $\mathscr{N}_\eta(\epsilon, \lambda)$ of ξ and η are disjoints. Hence we have $\xi = \eta$. This completes the proof.

Lemma 2.2. *Let (X, F, T) be a PN space and $\mathscr{I}_{fin}$ be Fréchet ideal (finite subsets on $\mathbb{N}$). Then $F - convergence$ implies $\mathscr{I}_{fin}^F - convergence$.*

Proof. Let $\epsilon > 0$ and $\lambda \in (0,1)$. Suppose that $\{x_n\}$ is $F - convergent$ to ξ. Then, there exists a number $N \in \mathbb{N}$ such that $x_n \in \mathcal{N}_\xi(\epsilon, \lambda)$ for every $n \geq N$. This implies that the set $A = \{n \in \mathbb{N} : x_n \notin \mathcal{N}_\xi(\epsilon, \lambda)\} \subseteq \{1, 2, \cdots, N-1\}$. Since the right hand side belongs to $\mathscr{I}_f$, we have $A \in \mathscr{I}_{\text{fin}}$. This shows that $\{x_n\}$ is $\mathscr{I}_{\text{fin}}^F - convergent$ to ξ.

The following example shows that the converse of above theorem is not valid.

Example 2.1. *By letting $X = \mathbb{R}$ in Example 1, we have $(\mathbb{R}, F, T)$ is a PN space induced by the probabilistic norm $F_x(\epsilon) = \frac{\epsilon}{\epsilon + \|x\|}$. Let us suppose that $A \in \mathscr{I}_{fin}$. Define a sequence $\{x_n\}$ in $\mathbb{R}$ via*

$$x_n = \begin{cases} 1, & \text{if } n \in A \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every $\epsilon > 0$ and $\lambda \in (0,1)$, let $K = \{n \in \mathbb{N} : x_n \notin \mathcal{N}_\theta(\epsilon, \lambda)\}$. We observe that

$$\begin{aligned} x_n \notin \mathcal{N}_\theta(\epsilon, \lambda) &\Rightarrow F_{x_n}(\epsilon) \leq 1 - \lambda \\ &\Rightarrow \frac{\epsilon}{\epsilon + \|x_n\|} \leq 1 - \lambda \\ &\Rightarrow \|x_n\| \geq \frac{\epsilon\lambda}{1-\lambda} > 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} K &= \{n \in \mathbb{N} : \|x_n\| > 0\} \\ &= \{n \in \mathbb{N} : x_n = 1\} \\ &= A \in \mathscr{I}_f. \end{aligned}$$

Therefore $\mathscr{I}_{fin}^F - \lim x_n = \theta$. But the sequence $\{x_n\}$ is not convergent to θ in $(\mathbb{R}, \|\cdot\|)$. By Lemma 1, this implies that $F - \lim x_n \neq \theta$.

Lemma 2.3. *Let (X, F, T) be a PN space and $\mathscr{I}$ is an admissible ideal on X. Then $\mathscr{I}^F_{fin}$ -convergence implies $\mathscr{I}^F$ -convergence.*

Proof. For $\mathscr{I}$ be an admissible ideal, we have $\bigcup \mathscr{I} = \mathbb{N}$. This implies that $\mathscr{I}_{\text{fin}} \subset \mathscr{I}$. So, $\mathscr{I}^F_{\text{fin}}$ -convergence implies $\mathscr{I}^F$ -convergence.

The following lemma is an immediate consequence of definition of statistical convergence sequence.

Lemma 2.4. *Let (X, F, T) be a PN space. If $\mathscr{I}_\delta = \{A \subseteq \mathbb{N} \colon \delta(A) = 0\}$ where $\delta(A)$ be the density of A, then $\mathscr{I}^F_\delta$-convergence coincide with statistical convergence.*

Lemma 2.5. *If $\{x_n\}$ and $\{y_n\}$ are two sequences in (X, F, T) with $T(a, a) > a$ for every $a \in (0, 1)$, then*
(i) If $\mathscr{I}^F - \lim x_n = \xi$ and $\mathscr{I}^F - \lim y_n = \eta$, then $\mathscr{I}^F - \lim(x_n + y_n) = \xi + \eta$.
(ii)If $\mathscr{I}^F - \lim x_n = \xi$ and $\alpha \in \mathbb{R}$, then $\mathscr{I}^F - \lim \alpha x_n = \alpha\xi$.
(iii) If $\mathscr{I}^F - \lim x_n = \xi$ and $\mathscr{I}^F - \lim y_n = \eta$, then $\mathscr{I}^F - \lim(x_n - y_n) = \xi - \eta$.

Proof. (i) Let $\epsilon > 0$ and $\lambda \in (0, 1)$. Since $\mathscr{I}^F - \lim x_n = \xi$ and $\mathscr{I}^F - \lim y_n = \eta$, the sets $A = \{n \in \mathbb{N} \colon x_n \notin \mathscr{N}_\xi(\frac{\epsilon}{2}, \lambda)\}$ and $B = \{n \in \mathbb{N} \colon x_n \notin \mathscr{N}_\eta(\frac{\epsilon}{2}, \lambda)\}$ are belongs to $\mathscr{I}$. Let $C = \{n \in \mathbb{N} \colon x_n + y_n \notin \mathscr{N}_{\xi+\eta}(\epsilon, \lambda)\}$. Since $\mathscr{I}$ is an ideal it is sufficient to show that $C \subset A \cup B$. This is equivalent to show that $C^c \supset A^c \cap B^c$ where A^c and B^c are belongs to $\mathscr{F}(\mathscr{I})$. Let $n \in A^c \cap B^c$, i.e., $n \in A^c$ and $n \in B^c$ then by (N4) we have

$$\begin{aligned} F_{(x_n+y_n)-(\xi+\eta)}(\epsilon) &\geq \tau_T(F_{x_n-\xi}, F_{y_n-\eta})(\epsilon) \\ &\geq T\left(F_{x_n-\xi}(\frac{\epsilon}{2}), F_{y_n-\eta}(\frac{\epsilon}{2})\right) \\ &> T(1-\lambda, 1-\lambda) \\ &> 1-\lambda. \end{aligned}$$

Hence, $n \in C^c \supset A^c \cap B^c \in \mathscr{F}(\mathscr{I})$ which implies $C \subset A \cup B \in \mathscr{I}$ and the result follows.

(ii)Let $\epsilon > 0$ and $\lambda \in (0,1)$. Since $\mathscr{I}^F - \lim x_n = \xi$, we have $A = \{n \in \mathbb{N}: x_n \notin \mathscr{N}_\xi(\epsilon,\lambda)\} \in \mathscr{I}$. This implies that $A^c = \{n \in \mathbb{N}: x_n \in \mathscr{N}_\xi(\epsilon,\lambda)\} \in \mathscr{F}(\mathscr{I})$. Let $n \in A^c$. For the case $\alpha = 0$, We have

$$F_{0x_n - 0\xi}(\epsilon) = F_0\epsilon = 1 > 1 - \lambda$$

and for the case $\alpha \neq 0$, we have

$$\begin{aligned} F_\alpha x_n - \alpha\xi(\epsilon) &= F_{x_n-\xi}(\frac{\epsilon}{|\alpha|}) \\ &\geq T\left(F_{x_n-\xi}(\epsilon), F_0(\frac{\epsilon}{|\alpha|} - \epsilon)\right) \\ &> T(1-\lambda,,1) \\ &= 1-\lambda. \end{aligned}$$

This shows that $\{n \in \mathbb{N}: \alpha x_n \notin \mathscr{N}_{\alpha\xi}(\epsilon,\lambda)\} \in \mathscr{I}$ and consequently we have $\mathscr{I}^{\mathfrak{F}} - \lim \alpha x_n = \alpha\xi$.

(iii) The proof is obvious from (i) and (ii).

Definition 2.3. *Let (X,F,T) be a PN space. A subset $A = \{x_n\}$ of X is said to be $\mathscr{I}^F$-bounded on PN spaces if for every $\lambda \in (0,1)$, there exists $\epsilon > 0$ such that*

$$\{n \in \mathbb{N}: x_n \notin \mathscr{N}_\theta(\epsilon,\lambda)\} \in \mathscr{I}.$$

Let (X,F,T). We denote $\mathscr{I}_b^F(X)$ the set of all $\mathscr{I}^F$-bounded $I^F - convergent$ sequences on X and $l_\infty^F(X)$ the set of all $\mathscr{I}^F$ -bounded sequences on X.

Theorem 2.1. *Let (X,F,T) be a PN space such that $T(a,a) > a$ for every $a \in (0,1)$. Let $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$. Then $\mathscr{I}_b^F(X)$ is a closed linear subspace of the set $l_\infty^F(X)$.*

Proof. In view of Lemma (), it is clear that the set $\mathscr{I}_b^F(X)$ is a linear subspace of the set $l_\infty^F(X)$. So to prove the result it is sufficient to prove that $\mathscr{I}_b^F(X) = \overline{\mathscr{I}_b^F(X)}$. It is clear that $\mathscr{I}_b^F(X) \subset \overline{\mathscr{I}_b^F(X)}$. Now we show that $\overline{\mathscr{I}_b^{Fm}(X)} \subset \mathscr{I}_b^F(X)$. Let $y \in \overline{\mathscr{I}_b^F(X)}$. We notice that since $\mathscr{N}_y(\epsilon,\lambda) \cap \mathscr{I}_b^F(X) \neq \emptyset$ for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an $x \in \mathscr{N}_y(\epsilon,\lambda) \cap \mathscr{I}_b^F(X)$ such that the set $K = \{n \in \mathbb{N} \colon x \notin \mathscr{N}_y(\frac{\epsilon}{2},\lambda)\}$ belongs to $\mathscr{I}$. This implies that $K^c = \{n \in \mathbb{N} \colon x \in \mathscr{N}_y(\frac{\epsilon}{2},\lambda)\} \in \mathscr{F}(\mathscr{I})$. Now let $n \in K^c$, then by (N4), we have

$$\begin{aligned}
F_{y_n}(\epsilon) &= F_{y_n - x_n + x_n}(\epsilon) \\
&\geq T\left(F_{y_n - x_n}(\frac{\epsilon}{2}), F_{x_n}(\frac{\epsilon}{2})\right) \\
&> T(1-\lambda, 1-\lambda) \\
&> 1-\lambda.
\end{aligned}$$

Thus, we have $\{n \in K^c \colon y_n \in \mathscr{N}_\theta(\epsilon) > 1-\lambda\} \in \mathscr{F}(\mathscr{I})$ which implies that $\{n \in \mathbb{N} \colon y_n \notin \mathscr{N}_\theta(\epsilon) > 1-\lambda\} \in \mathscr{I}$. Thus $y \in \mathscr{I}_b^F(X)$ and this completes the proof.

Lemma 2.6. *If a sequence in a PN space (X,F,T) is $\mathscr{I}^{F*}$ -convergent, then it is $\mathscr{I}_f^F$ -convergent to the same limit.*

Proof. Let $\mathscr{I}^{F*} - \lim x_n = \xi$, then by definition, there exists $M = \{m_1 < m_2 < \cdots\} \in \mathscr{F}(\mathscr{I})$ such that $F - \lim x_{m_k} = \xi$. Let $\epsilon > 0$ and $\lambda \in (0,1)$ be given. Since $F - \lim x_{m_k} = \xi$, there exists $N \in \mathbb{N}$ such that $x_{m_k} \in \mathscr{N}_\xi(\epsilon,\lambda)$ for every $k \geq N$. Let $A = \{k \in \mathbb{N} \colon x_{m_k} \notin \mathscr{N}_\xi(\epsilon,\lambda)\}$. Then it is clear that $A \subset \{1,2,\cdots,N-1\} \in \mathscr{I}_f$. Therefore, the sequence $\{x_n\}$ is $\mathscr{I}^F - \lim x_n = \xi$.

3. $\mathscr{I}$ -Convergence for Continuous Functions in PN Spaces

In this short section, we extend the study of ideal convergence to a sequence of function f_n in (X,F,T) and prove a theorem about ideal convergence. We begin with the following definition.

Definition 3.1. *Let (X, F, T) be a PN spaces and $\mathscr{I}$ be an arbitrary admissible ideal in $\mathbb{N}$. We say that a sequence of functions $f_n : X \to X$ is $\mathscr{I}^F$*-convergent *to a function $f : X \to X$) denoted $\mathscr{I}^F - \lim f_n = f$, if for every $x \in X$, $\epsilon > 0$ and $\lambda \in (0, 1)$ the set*

$$\{n \in \mathbb{N} : f_n(x) - f(x) \notin \mathcal{N}_\theta(\epsilon, \lambda)\} \quad \textit{belongs to} \quad \mathscr{I}.$$

Theorem 3.1. *Let (X, F, T) be a PN spaces such that $\sup_{a<1} T(a, a) = 1$ and let $\mathscr{I}$ be an arbitrary admissible ideal in $\mathbb{N}$. Let $\mathscr{I}^F - \lim f_n = f$ (on X) where $f_n : X \to X$, $n \in \mathbb{N}$, are equi-continuous (on X) and $f : X \to X$. Then f is F -continuous (on X).*

Proof. Let $x_0 \in X$ and $x - x_0 \in \mathcal{N}_\theta(\epsilon, \lambda)$ be fixed. By equi-continuity of f_n's, for every $\epsilon > 0$, there exists a $\gamma \in (0, 1)$ with $\gamma < \lambda$ such that

$$f_n(x) - f_n(x_0) \in \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)$$

for every $n \in \mathbb{N}$. Since $\mathscr{I}^F - \lim f_n = f$, the set

$$K = \{n \in \mathbb{N} : f_n(x_0) - f(x_0) \notin \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)\} \bigcup \{n \in \mathbb{N} : f_n(x) - f(x)) \notin \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)\}$$

is in $\mathscr{I}$ and different from $\mathbb{N}$. Hence, there exists $n \in \mathscr{F}(K)$ such that

$$f_n(x_0) - f(x_0) \in \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)) \quad \text{and} \quad f_n(x) - f(x) \in \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma).$$

It follows that

$$\begin{aligned} F_{f(x_0)-f(x)}(\epsilon) &\geq T\left(F_{f(x_0)-f_n(x_0)}(\frac{\epsilon}{3}), T(F_{f_n(x_0)-f_n(x)}(\frac{\epsilon}{3}), F_{f_n(x)-f(x)}(\frac{\epsilon}{3})\right) \\ &> T(1-\gamma, T(1-\gamma, 1-\gamma)) \\ &> T(1-\gamma, 1-\gamma) \\ &> 1-\gamma \\ &> 1-\lambda. \end{aligned}$$

This implies that f is F-continuous (on X).

4. $\mathscr{I}$ -Continuity of a Function in PN Spaces

We begin with the definition of continuity an important type of sequential continuity in PN space.

Definition 4.1. *Let $\mathscr{I}$ be an ideal and (X, F, T) be a PN space. A map $f : X \to X$ is called $F - continuous$ at a point $\xi \in X$, if*

$$F - \lim x_n = \xi \quad \Longrightarrow \quad F - \lim f(x_n) = f(\xi).$$

This means for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a number $N \in \mathbb{N}$ such that for $n \geq N$, we have $x_n - \xi \in \mathscr{N}_\theta(\epsilon, \lambda)$ implies $f(x_n) - f(\xi) \in \mathscr{N}_\theta(\epsilon, \lambda)$.

Definition 4.2. *Let $\mathscr{I}$ be an ideal and (X, F, T) be a PN space. A map $f : X \to X$ is called $\mathscr{I}^F - continuous$ at a point $\xi \in X$, if*

$$\mathscr{I}^F - \lim x_n = \xi \quad \Longrightarrow \quad \mathscr{I}^F - \lim f(x_n) = f(\xi).$$

Theorem 4.1. *Let (X, F, T) be a PN space and $\mathscr{I}$ be an arbitrary ideal in $\mathbb{N}$. If $f : X \to X$ is F -continuous then it is $\mathscr{I}^F$-continuous.*

Proof. Let $\{x_n\} \in X$ and $\mathscr{I}^F - \lim x_n = \xi$. Then by F -continuity of f at $\xi \in X$ we means for every $\epsilon > 0$ and $\lambda \in (0, 1)$, we have $x_n - \xi \in \mathscr{N}_\theta(\epsilon, \lambda)$ implies $f(x_n) - f(\xi) \in \mathscr{N}_\theta(\epsilon, \lambda)$. Thus $\{n \in \mathbb{N} : f(x_n) - f(\xi) \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \subset \{n \in \mathbb{N} : x_n - \xi \notin \mathscr{N}_\theta(\epsilon, \lambda)\}$. Since $\mathscr{I}^F - \lim x_n = \xi$, we have $\{n \in \mathbb{N} : x_n - \xi \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \in \mathscr{I}$. This implies that $\{n \in \mathbb{N} : f(x_n) - f(\xi) \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \in \mathscr{I}$ which means $\mathscr{I}^F - \lim f(x_n) = f(\xi)$. Hence, f is an $\mathscr{I}^F$-continuous.

Theorem 4.2. *Let (X, F, T) be a PN space and $\mathscr{I}$ be an arbitrary admissible ideal in $\mathbb{N}$. If $f : X \to X$ is $\mathscr{I}^F$ -continuous then f is $\mathscr{I}^F_{fin}$ -continuous.*

Proof. Let f is $\mathscr{I}^F$ -continuous at $\xi \in X$. Suppose that f is not $\mathscr{I}_f^F$ -continuous, then the set $A = \{n \in \mathbb{N} \colon f(x_n) - f(\xi) \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \notin \mathbb{I}_f$, i.e., A is infinite set whenever $\{n \in \mathbb{N} \colon x_n - \xi \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \in \mathscr{I}_f$. Let $\{y_n\}$ be the subsequence of $\{x_n\}$ given by the subset A of $\mathbb{N}$. Then $\{n \in \mathbb{N} \colon f(y_n) - f(\xi) \notin \mathscr{N}_\theta(\epsilon, \lambda)\} = \mathbb{N}$. Also, the subsequence $\{y_n\}$ holds $\mathscr{I}_f^F - \lim y_n = \xi$. By Lemma 4, this implies $\mathscr{I}^F - \lim y_n = \xi$. Thus, by $\mathscr{I}^F$ continuity of f, we have $\mathscr{I}^F - \lim f(y_n) = f(\xi)$. Hence $\{n \in \mathbb{N} \colon f(y_n) - f(\xi) \notin \mathscr{N}_\theta(\epsilon, \lambda)\} = \mathbb{N} \in \mathscr{I}$, a contradiction. Therefore f is $\mathscr{I}_f^F$ -continuous.

From theorem 4.3 and 4.4, we can easily prove the following lemma.

Lemma 4.1. *Let (X, F, T) be a PN space and $\mathscr{I}$ be an arbitrary admissible ideal in $\mathbb{N}$. If $f : X \to X$ is a map, then the following implication hold:*

$$F - continuous \Rightarrow \mathscr{I}^F - continuous \Rightarrow \mathscr{I}_{fin} - continuous$$

5. $\mathscr{I}$-Cauchy Sequences in PN Spaces

Definition 5.1. *Let (X, F, T) be a PN space. A sequence $\{x_n\}$ in X is said to be F -Cauchy, if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a number $N = N(\epsilon, \lambda) \in \mathbb{N}$ such that*

$$x_n - x_m \in \mathscr{N}_\theta(\epsilon, \lambda) \quad \textit{for every} \quad n, m \geq N.$$

Definition 5.2. *Let (X, F, T) be a PN space and $\mathscr{I}$ be an admissible ideal. Then a sequence (x_n) in X is called $\mathscr{I}^F - Cauchy$ sequence in X if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $M = M(\epsilon, \lambda) \in \mathbb{N}$ such that*

$$\{n \in \mathbb{N} \colon x_n - x_M \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \in \mathscr{I}.$$

Definition 5.3. *Let (X, F, T) be a PN space and $\mathscr{I}$ be an admissible ideal. Then a sequence (x_n) in X is called $\mathscr{I}^{F*} - Cauchy$ sequence in X if for every $\epsilon > 0$ and $\lambda \in (0, 1)$,* *there exists a set* $M = \{m_1 < m_2 < \cdots < m_k, \cdots\} \in \mathscr{F}(\mathscr{I})$ *such that the subsequence* $x_M = (x_{m_k})$ *is* $F - Cauchy$ *in* X*, i.e. there exists a number* $k_0 \in \mathbb{N}$ *such that*

$$x_{m_k} - x_{m_p} \in \mathscr{N}_\theta(\epsilon, \lambda) \quad \textit{for every} \quad k, p \geq k_0.$$

Theorem 5.1. *Let* (X, F, T) *be a PN space and* $\mathscr{I}$ *in* $\mathbb{N}$ *is an admissible ideal. If* $\{x_n\}$ *in* X *is* $\mathscr{I}^{F*} - Cauchy$ *then it is* $\mathscr{I}^F - Cauchy$.

Proof. Let $\{x_n\}$ be a $\mathscr{I}^{F*} - Cauchy$ sequence. Then for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists a set $M = \{m_1 < m_2 < \cdots < m_k, \cdots\} \in \mathscr{F}(\mathscr{I})$ and a number $k_0 \in \mathbb{N}$ such that $x_{m_k} - x_{m_p} \in \mathscr{N}_\theta(\epsilon, \lambda)$ for every $k, p \geq k_0$. Now, fix $N = m_{k_0+1}$. Then for every $\epsilon > 0$ and $\lambda \in (0,1)$, we have $x_{m_k} - x_N \in \mathscr{N}_\theta(\epsilon, \lambda)$ for every $k \geq k_0$. Let $H = \mathbb{N} \backslash M$. It is obvious that $H \in \mathscr{I}$ and $A(\epsilon, \lambda) = \{n \in \mathbb{N} \colon x_n - x_N \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\}$. Clearly, the right hand side of the last argument is belongs to $\mathscr{I}$. Therefore, for every $\epsilon > 0$ and $\lambda \in (0,1)$ we can find $N = N(\epsilon, \lambda) \in \mathbb{N}$ such that $A(\epsilon, \lambda) \in \mathscr{I}$, i.e., $\{x_n\}$ is $\mathscr{I}^F - Cauchy$ sequence in X.

Theorem 5.2. *Let* (X, F, T) *be a PN space such that* $T(a, a) > a$ *for every* $a \in (0,1)$ *and* $\mathscr{I}$ *be an admissible ideal. A sequence* $\{x_n\}$ *in* X *is* $\mathscr{I}^F$ *-convergent if and only if it is* $\mathscr{I}^F$ *-Cauchy.*

Proof: *Necessity: Suppose that* $\{x_n\}$ *is* $\mathscr{I}^F$ *-convergent to* $\xi \in X$*. Let* $\epsilon > 0$ *and* $\lambda \in (0,1)$ *be given. Since* $\mathscr{I}^F - \lim x_n = \xi$*, we have* $A = \{n \in \mathbb{N} \colon x_n \notin \mathscr{N}_\xi(\frac{\epsilon}{2}, \lambda)\} \in \mathscr{I}$*. This implies that* $A^c = \{n \in \mathbb{N} \colon x_n \in \mathscr{N}_\xi(\frac{\epsilon}{2}, \lambda)\} \in \mathscr{F}(\mathscr{I})$*. Now, by (N4), for every* $n, m \in A^c$*,*

$$\begin{aligned} \nu_{x_n - x_m}(\epsilon) &\geq T\left(\nu_{x_n - \xi}(\frac{\epsilon}{2}), \nu_{x_m - \xi}(\frac{\epsilon}{2})\right) \\ &> T(1-\lambda, 1-\lambda) \\ &> 1 - \lambda. \end{aligned}$$

Hence, $\{n \in \mathbb{N} \colon x_n - x_m \in \mathscr{N}_\theta(\epsilon, \lambda)\} \in \mathscr{F}(\mathscr{I})$*. This implies that* $\{n \in \mathbb{N} \colon x_n - x_m \notin \mathscr{N}_\theta(\epsilon, \lambda)\} \in \mathscr{I}$*, i.e.,* $\{x_n\}$ *is a* $\mathscr{I}^F$ *-Cauchy sequence.*

Proof. Sufficiency: Assume that $\{x_n\}$ is a $\mathscr{I}^F$ -Cauchy sequence. We shall prove that $\{x_n\}$ is $\mathscr{I}^F$ -convergent sequence. For this, let $\{\epsilon_p\}$ be a strictly decreasing sequence of positive real numbers such that $\epsilon_p \to 0$ as $p \to \infty$. Since $\{x_n\}$ is a $\mathscr{I}^F$ -Cauchy sequence, there exists a strictly increasing sequence $\{m_p\}$ of positive integers such that

$$A_p = \{n \in \mathbb{N} \colon x_n - x_{m_p} \notin \mathscr{N}_\theta(\epsilon_p, \lambda)\} \in \mathscr{I} \quad p = 1, 2, 3, \cdots .$$

This implies that

$$\emptyset \neq \{n \in \mathbb{N} \colon x_n - x_{m_p} \in \mathscr{N}_\theta(\epsilon_p, \lambda)\} \in \mathscr{F}(\mathscr{I}) \quad p = 1, 2, 3, \cdots . \tag{5.1}$$

Let p and q be two positive integers such that $p \neq q$. Then by (3), both the sets $\{n \in \mathbb{N} \colon x_n - x_{m_p} \in \mathscr{N}_\theta(\epsilon_p, \lambda)\}$ and $\{n \in \mathbb{N} \colon x_n - x_{m_q} \in \mathscr{N}_\theta(\epsilon_q, \lambda)\}$ are nonempty elements of $\mathscr{F}(\mathscr{I})$. Since $\mathscr{F}(\mathscr{I})$ is a filter on $\mathbb{N}$, therefore

$$\emptyset \neq \{n \in \mathbb{N} \colon x_n - x_{m_p} \in \mathscr{N}_\theta(\epsilon_p, \lambda)\} \cap \{n \in \mathbb{N} \colon x_n - x_{m_q} \in \mathscr{N}_\theta(\epsilon_q, \lambda)\} \in \mathscr{F}(\mathscr{I}).$$

Thus, for each p and q with $p \neq q$, we can select $n_p, n_q \in \mathbb{N}$ such that $x_{n_p} - x_{m_p} \in \mathscr{N}_\theta(\epsilon_p, \lambda)$ and $x_{n_q} - x_{m_q} \in \mathscr{N}_\theta(\epsilon_q, \lambda)$. Let $\epsilon = \epsilon_p + \epsilon_q$. Then by (N4), we have

$$\begin{aligned} \nu_{x_{m_p} - x_{m_q}}(\epsilon) &\geq T(\nu_{x_{n_p} - x_{m_p}}(\epsilon_p), \nu_{x_{n_p} - x_{m_q}}(\epsilon_q)) \\ &> T(1 - \lambda, 1 - \lambda) \\ &> 1 - \lambda. \end{aligned}$$

This implies that $\{x_{m_p}\}$ is a $F-Cauchy$ sequence and satisfies the Cauchy criterion. Say $\lim x_{m_p} = \xi$. Also we have $\epsilon \to 0$ as $p \to \infty$, so for each $\epsilon > 0$ we can choose $p_0 \in \mathbb{N}$ such that $\epsilon_{p_0} < \frac{\epsilon}{2}$ and

$$x_{m_p} \in \mathscr{N}_\xi(\frac{\epsilon}{2}, \lambda) \quad \text{for} \quad p \geq p_0.$$

Next we prove that $A = \{n \in \mathbb{N} \colon x_n \notin \mathscr{N}_\xi(\epsilon, \lambda)\} \subset A_{p_0} = \{n \in \mathbb{N} \colon x_n - x_{m_{p_0}} \notin \mathscr{N}_\theta(\epsilon_{p_0}, \lambda)\}$. Since A and A_{p_0} are both in $\mathscr{I}$, it is sufficient to show that $A^c \supset A^c_{p_0}$.

Let $n \in A^c_{p_0}$, then we have

$$\begin{aligned} \nu_{x_n-\xi}(\epsilon) &\geq T\left(\nu_{x_n-x_{m_{p_0}}}(\frac{\epsilon}{2}), \nu_{x_{m_{p_0}}-\xi}(\frac{\epsilon}{2})\right) \\ &> T(1-\lambda, 1-\lambda) \\ &> 1-\lambda. \end{aligned}$$

This implies that $n \in A^c$. Therefore $A \subset A_{p_0}$. Since $A_{p_0} \subset \mathscr{I}$, we conclude that $A \subset \mathscr{I}$. This proves that the sequence (x_n) is $\mathscr{I}^F$ -convergent to ξ.

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References

[1] S. Aytar and S. Pehlivan, Statistically monotonic and statistically bounded sequences of fuzzy numbers. Information Sciences, 176: 734-744 (2006).

[2] V. Balaz, J. Cervennansky, T. Kostyrko, T. Salat, $\mathscr{I}$-convergence and $\mathscr{I}$-continuity of real functions. Acta Mathematica, Faculty of Natural Sciences, Constantine the Philosopher University Nitra 5: 43-50 (2002).

[3] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions. J. Math. Anal. Appl. 328: 715-729 (2007).

[4] J. Connor, The statistical and strong p-Cesaro convergence of sequences. Analysis 8: 47-63 (1988).

[5] J. Connor, M. Ganichev, V. Kadets, A characterization of Banach spaces with separable duals via weak statistical convergence. J. Math. Anal. Appl. 244: 251-261 (2000).

[6] K. Dems, On $\mathscr{I}$-Cauchy sequences. Real Anal. Exch. 30: 123-128 (2004/2005).

[7] H. Fast, Sur la convergence statistique. Colloq. Math. 2: 241-244 (1951).

[8] A. R. Freedman and J. J. Sember, Densities and Summability. Pasific J. Math. 95: 293-305 (1981).

[9] J. A. Fridy, On statistical convergence. Analysis 5: 301-313 (1985).

[10] J. A. Fridy, Statistical limit points. Proc. Amer. Math. Soc. 118: 1187-1192 (1993).

[11] J. A. Fridy and C. Orhan, Statistical limit superior and limit inferior. Proc. Amer. Math. Soc. 125: 3625-3631 (1997).

[12] J. Jasinski, I. Reclaw, Ideal convergence of continuous functions. Topology and its Applications, 153: 3511-3518 (2006).

[13] S. Karakus, Statistical convergence on probabilistic normed spaces. Math. Comm. 12: 11-23 (2007).

[14] S. Karakus, K. Demirci, Statisitcval convergence of double sequences on probabilistic normed spaces. Int. J. Math. Math. Sci. 2007: 11 pages (2007)

[15] E. Kolk, The statistical convergence in Banach spaces. Acta Comment. Univ. Tartu. 928: 41-52 (1991).

[16] P. Kostyrko, T. Salat, W. Wilczynski, $\mathscr{I}$-convergence. Real Anal. Exch. 26, 2: 669-686 (2000/2001).

[17] P. Kostyrko, M. Macaj, T. Salat, M. Sleziak. $\mathscr{I}$-convergence and extremal $\mathscr{I}$-limit point. Math. Slovaca 55: 443-464 (2005).

[18] I. J. Maddox, Statistical convergence in a locally convex space. Math. Proc. Cambridge Phil. Soc. 104: 141-145 (1988).

[19] Mursaleen and O. H. H. Edely, Statistical convergence of double sequences. J. Math. Anal. Appl. 288: 223-231 (2003).

[20] A. Nabiev, S. Pehlivan, M. Gurdal, On $\mathscr{I}$-Cauchy sequence. Taiwanese J. Math. 11, 2: 569-576 (2007).

[21] T. Šalát, On statistical convergent sequences of real numbers. math. Slovaca 30: 139-150 (1980).

[22] E. Savaş, On statistically convergent sequences of fuzzy numbers. Information Sciences, 137: 277-282 (2001).

[23] A. N. Šerstnev, On the notion of a random normed spaces. Dokl. Akad. nauk SSSR 149: 280-283 (1963).

[24] M. Sleziak, $\mathscr{I}$-continuity in topological spaces. Preprint.

[25] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique. Collog. Math. 2: 73-74 (1951).

[26] B. Schweizer, A. Sklar, Probabilistic metric spaces, Elsevier/North-Holand. New York, 1983.