Inversion of the Generalized Dunkl Intertwining Operator on \( \mathbb{R} \) and its Dual using Generalized Wavelets

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Abstract. We establish an inversion formula for a continuous wavelet transform associated with a class of singular differential-difference operators on \( \mathbb{R} \). We apply this result to derive new expressions for the inverse generalized Dunkl intertwining operator and its dual on \( \mathbb{R} \).

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1. Introduction

Consider the second-order singular differential operator on the real line

\[
\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx} \tag{1}
\]

where

\[
A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -\frac{1}{2},
\]

\( B \) being a positive \( C^\infty \) even function on \( \mathbb{R} \). In addition we suppose that

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A is increasing on $[0, \infty[$ and $\lim_{x \to \infty} A(x) = \infty$;

(ii) $A'/A$ is decreasing on $]0, \infty[$ and $\lim_{x \to \infty} A'(x)/A(x) = 0$;

(iii) There exists a constant $\delta > 0$ such that the function $e^{\delta x}B'(x)/B(x)$ is bounded for large $x \in ]0, \infty[$ together with its derivatives.

Lions [5] has constructed an automorphism $\mathcal{X}$ of the space $\mathcal{E}_e(\mathbb{R})$ of $C^\infty$ even functions on $\mathbb{R}$, which intertwines $\Delta$ and the second derivative operator $d^2/dx^2$; that is, satisfying the intertwining relation

$$\mathcal{X} \frac{d^2}{dx^2} f = \Delta \mathcal{X} f, \quad f \in \mathcal{E}_e(\mathbb{R}).$$

It is known [14] that the Lions operator $\mathcal{X}$ admits the integral representation

$$\mathcal{X} f(x) = \int_0^{\vert x \vert} G(x, y)f(y)dy, \quad x \neq 0,$$

where $G(x, \cdot)$ is an even positive function on $\mathbb{R}$, continuous on $]-\vert x \vert, \vert x \vert[$ and supported in $[-\vert x \vert, \vert x \vert]$. Furthermore, the dual Lions operator

$$\mathcal{X} f(y) = \int_{\vert y \vert}^{\infty} G(x, y)f(x)A(x)dx, \quad y \in \mathbb{R},$$

is an automorphism of the space $\mathcal{E}_e(\mathbb{R})$ of even Schwartz functions on $\mathbb{R}$, satisfying the intertwining relation

$$\frac{d^2}{dx^2} \mathcal{X} f = \mathcal{X} \Delta f, \quad f \in \mathcal{E}_e(\mathbb{R}).$$

In [8] the second author has introduced on the space $\mathcal{E}(\mathbb{R})$ of $C^\infty$ functions on $\mathbb{R}$, the following operator

$$V f = \mathcal{X}(f_e) + \frac{d}{dx} \mathcal{X} I(f_o), \quad (2)$$

where

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2}, \quad (3)$$

and $I$ is the map defined by $Ih(x) = \int_0^x h(t)dt$.

Mainly, he showed that $V$ is an automorphism of $\mathcal{E}(\mathbb{R})$ satisfying for all $f \in \mathcal{E}(\mathbb{R})$,

$$V \frac{d}{dx} f = \Lambda V f, \quad (4)$$

where $\Lambda$ is a first-order differential-difference operator on $\mathbb{R}$ given by

$$\Lambda f(x) = \frac{df}{dx} + A'(x) \left( \frac{f(x) - f(-x)}{2} \right) \quad (5).$$
For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, the intertwining operator $V$ reads

$$V(f)(x) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_{-1}^{1} f(tx)(1 - t^2)^{\alpha-1/2}(1 + t) \, dt,$$

and referred to as the Dunkl intertwining operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The differential-difference operator $\Lambda$ reduces to the one-dimensional Dunkl operator

$$D_\alpha f = \frac{df}{dx} + (\alpha + \frac{1}{2}) \frac{f(x) - f(-x)}{x}.$$

Such operators have been introduced by Dunkl in connection with a generalization of the classical theory of spherical harmonics (see [1, 11] and the references therein). During the last years, the theory of Dunkl operators has found a wide area of applications in mathematics and mathematical physics. In fact, Dunkl operators have been used in the study of multivariable orthogonality structures with certain reflection symmetries [12, 16]. Moreover, they have been successfully involved in the description and solution of Calogero-Moser-Sutherland type quantum many body systems [4].

Define the dual operator $^tV$ of $V$ on the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$, by the relation

$$^tV f = ^t\mathcal{X}(f_o) + \frac{d}{dx} ^t\mathcal{X}J(f_o), \quad (6)$$

where $J$ is the map defined by

$$J h(x) = \int_{-\infty}^{x} h(y)dy, \quad x \in \mathbb{R}. \quad (7)$$

In this paper, it is shown that the dual operator $^tV$ is an automorphism of $\mathcal{S}(\mathbb{R})$ which satisfies the intertwining relation

$$\frac{d}{dx} ^tV f = ^tV \Lambda f, \quad f \in \mathcal{S}(\mathbb{R}).$$

Moreover, the following inversion formulas for $V$ and $^tV$ on certain specific subspaces of $\mathcal{S}(\mathbb{R})$ are provided

$$f = V \mathcal{K} ^tV f,$$

$$f = \mathcal{M} V ^tV f,$$

$$f = ^tV \mathcal{K} V f,$$

$$f = ^tV \mathcal{M} V f.$$

$\mathcal{K}$ and $\mathcal{M}$ being pseudo-differential operators. But the main contribution of this work is the determination of the inverse operators $V^{-1}$ and $^tV^{-1}$ through a continuous wavelet transform on $\mathbb{R}$ associated with the differential-difference operator $\Lambda$. For examples of use of wavelet type transforms in inverse problems the reader is referred to [2, 6, 7, 10, 15] and the references therein. The content of this paper is as follows. In Section 2 we provide some harmonic
analysis results related to the differential-difference operator $\Lambda$. Next we list some basic properties of the generalized Dunkl intertwining operator $V$ and its dual $V^\prime$. In section 3 we introduce the generalized continuous wavelet transform associated with $\Lambda$, and we prove for this transform Plancherel and reconstruction formulas. Using generalized wavelets, we obtain in Section 4 formulas which give the inverse operators $V^{-1}$ and $V^\prime^{-1}$ on Schwartz type spaces.

2. Preliminaries

In this section we provide some facts about harmonic analysis related to the differential-difference operator $\Lambda$. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [8].

Notation. We denote by

- $\mathcal{S}(\mathbb{R})$ the space of $C^\infty$ functions $f$ on $\mathbb{R}$, which are rapidly decreasing together with their derivatives, i.e., such that for all $m, n = 0, 1, \ldots$,

$$P_{m,n}(f) = \sup_{x \in \mathbb{R}} (1 + x^2)^m \left| \frac{d^n}{dx^n} f(x) \right| < \infty.$$ 

The topology of $\mathcal{S}(\mathbb{R})$ is defined by the semi-norms $P_{m,n}$, $m, n = 0, 1, \ldots$.

- $\mathcal{S}_e(\mathbb{R})$ (resp. $\mathcal{S}_o(\mathbb{R})$) the subspace of $\mathcal{S}(\mathbb{R})$ consisting of even (rep. odd) functions.

- $\mathcal{B}(\mathbb{R})$ the subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions $f$ such that for all $n = 0, 1, \ldots$,

$$\int_{\mathbb{R}} f(x)b_n(x)A(x)dx = 0,$$

with $b_n(x) = V \left( \frac{x^n}{n!} \right)(x)$, $V$ being the generalized Dunkl intertwining operator given by (2).

- $\mathcal{W}(\mathbb{R})$ the subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions $f$ such that for all $n = 0, 1, \ldots$,

$$\int_{\mathbb{R}} f(x)x^n dx = 0.$$

- $\mathcal{H}(\mathbb{R})$ the subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions $f$ such that for all $n = 0, 1, \ldots$,

$$\frac{d^n}{dx^n} f(0) = 0.$$

Put

$$\mathcal{B}_e(\mathbb{R}) = \mathcal{S}_e(\mathbb{R}) \cap \mathcal{B}(\mathbb{R}), \quad \mathcal{B}_o(\mathbb{R}) = \mathcal{S}_o(\mathbb{R}) \cap \mathcal{B}(\mathbb{R}),$$
$$\mathcal{W}_e(\mathbb{R}) = \mathcal{S}_e(\mathbb{R}) \cap \mathcal{W}(\mathbb{R}), \quad \mathcal{W}_o(\mathbb{R}) = \mathcal{S}_o(\mathbb{R}) \cap \mathcal{W}(\mathbb{R}),$$
$$\mathcal{H}_e(\mathbb{R}) = \mathcal{S}_e(\mathbb{R}) \cap \mathcal{H}(\mathbb{R}), \quad \mathcal{H}_o(\mathbb{R}) = \mathcal{S}_o(\mathbb{R}) \cap \mathcal{H}(\mathbb{R}).$$
Remark 1.

(i) Due to our assumptions on the function $A$ there is a positive constant $k$ such that
$$A(x) \sim k |x|^{2\alpha+1}, \quad \text{as } |x| \to \infty.$$ 

(ii) It follows from (4) that
$$\Lambda b_{n+1} = b_n$$
for all $n \in \mathbb{N}$. Further, by [9] we have for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$,
$$|b_n(x)| \leq k |x|^n,$$
k being a positive constant depending only on $n$.

(iii) It is easily checked that the space $\mathcal{S}(\mathbb{R})$ is invariant under the differential-difference operator $\Lambda$.

For each $\lambda \in \mathbb{C}$ the differential-difference equation
$$\Lambda u = i \lambda u, \quad u(0) = 1,$$
admits a unique $C^\infty$ solution on $\mathbb{R}$, denoted $\Psi_\lambda$ given by
$$\Psi_\lambda(x) = \left\{ \begin{array}{ll}
\varphi_\lambda(x) + \frac{1}{i \lambda} \frac{d}{dx} \varphi_\lambda(x) & \text{if } \lambda \neq 0, \\
1 & \text{if } \lambda = 0,
\end{array} \right.$$  

where $\varphi_\lambda$ designates the solution of the differential equation
$$\Delta u = -\lambda^2 u, \quad u(0) = 1, \quad u'(0) = 0,$$
$\Delta$ being the differential operator defined by (1).

Remark 2.

(i) If $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, then
$$\Psi_\lambda(x) = j_\alpha(\lambda x) + \frac{i \lambda x}{2(\alpha + 1)} j_{\alpha+1}(\lambda x),$$
where $j_\gamma$ $(\gamma > -1/2)$ stands for the normalized spherical Bessel function of index $\gamma$ given by
$$j_\gamma(z) = \frac{\Gamma(\gamma + 1)}{n! \Gamma(n + \gamma + 1)} \left( \frac{z}{2} \right)^{2n} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!}, \quad (z \in \mathbb{C}).$$
(ii) It follows by (4) and (9) that

\[ \Psi_\lambda(x) = V(e^{i\lambda \cdot x})(x) \]

(12)

for all \( x \in \mathbb{R} \) and \( \lambda \in \mathbb{C} \).

The next statement provides a new estimate for the eigenfunction \( \Psi_\lambda(x) \).

Lemma 1. For all \( \lambda, x \in \mathbb{R} \), we have

\[ |\Psi_\lambda(x)| \leq 1. \]

Proof. For \( \lambda = 0 \), the result is obvious. For \( \lambda \neq 0 \), set

\[ u_\lambda(x) = |\Psi_\lambda(x)|^2 = \left| \varphi_\lambda(x) + \frac{1}{i\lambda} \frac{d}{dx} \varphi_\lambda(x) \right|^2 = (\varphi_\lambda(x))^2 + \frac{1}{\lambda^2} \left( \frac{d}{dx} \varphi_\lambda(x) \right)^2. \]

Notice that \( u_\lambda(x) \) is even in \( x \). By (11),

\[ \frac{d}{dx} u_\lambda(x) = 2\varphi_\lambda(x) \frac{d}{dx} \varphi_\lambda(x) + \frac{2}{\lambda^2} \frac{d}{dx} \varphi_\lambda(x) \frac{d^2}{dx^2} \varphi_\lambda(x) = -\frac{2}{\lambda^2} A'(x) \left( \frac{d}{dx} \varphi_\lambda(x) \right)^2. \]

As the function \( A \) is increasing on \( [0, \infty[ \), it follows that \( u_\lambda \) is decreasing on \( ]0, \infty[ \). As \( u_\lambda(0) = 1 \), we deduce that \( u_\lambda(x) \leq 1 \) for all \( x \geq 0 \). This ends the proof.

Notation. For a positive Borel measure \( \mu \) on \( \mathbb{R} \), and \( p = 1 \) or 2, we write \( L^p(\mathbb{R}, d\mu) \) for the class of measurable functions \( f \) on \( \mathbb{R} \) for which

\[ \|f\|_{p,\mu} = \left( \int_{\mathbb{R}} |f(x)|^p d\mu(x) \right)^{1/p} < \infty. \]

Definition 1. The generalized Fourier transform of a function \( f \) in \( L^1(\mathbb{R}, A(x) dx) \) is defined by

\[ \mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{-\lambda}(x) A(x) dx. \]

(13)

Remark 3. Let \( f \in L^1(\mathbb{R}, A(x) dx) \). By Lemma 1, it follows that \( \mathcal{F}(f) \) is continuous on \( \mathbb{R} \) and \( \| \mathcal{F}(f) \|_\infty \leq \| f \|_{1,A} \).

An outstanding result about the generalized Fourier transform \( \mathcal{F} \) is as follows.

Theorem 1. [8]

(i) For every \( f \in L^1 \cap L^2(\mathbb{R}, A(x) dx) \) we have the Plancherel formula

\[ \int_{\mathbb{R}} |f(x)|^2 A(x) dx = \int_{\mathbb{R}} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda). \]
where
\[ d\sigma(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2}, \]
c(\zeta) being a continuous functions on \( ]0, \infty[ \) such that
\[ c(\zeta)^{-1} \sim k_1 \zeta^{\alpha + \frac{1}{2}}, \text{ as } \zeta \to \infty, \]
\[ c(\zeta)^{-1} \sim k_2 \zeta^{\alpha + \frac{1}{2}}, \text{ as } \zeta \to 0, \]
for some \( k_1, k_2 \in \mathbb{C} \).

(ii) The generalized Fourier transform \( \mathcal{F}_\Lambda \) extends uniquely to a unitary isomorphism from \( L^2(\mathbb{R}, A(x)dx) \) onto \( L^2(\mathbb{R}, d\sigma) \). The inverse transform is given by
\[ \mathcal{F}_\Lambda^{-1} g(x) = \int_{\mathbb{R}} g(\lambda) \Psi(\lambda, x) d\sigma(\lambda) \]
where the integral converges in \( L^2(\mathbb{R}, A(x)dx) \).

Remark 4.

(i) The tempered measure \( \sigma \) is called the spectral measure associated with the differential-difference operator \( \Lambda \).

(ii) For \( A(x) = |x|^{2\alpha+1}, \alpha > -1/2 \), we have
\[ c(s) = \frac{2^{\alpha+1} \Gamma(\alpha + 1)}{s^{\alpha+1/2}}. \]

The following lemma will play a key role in the remainder of this section.

Lemma 2. The map \( J \), given by (7), is a topological isomorphism
- from \( \mathcal{S}_o(\mathbb{R}) \) onto \( \mathcal{S}_e(\mathbb{R}) \);
- from \( \mathcal{B}_o(\mathbb{R}) \) onto \( \mathcal{B}_e(\mathbb{R}) \).

Proof.

(i) It is sufficient to show that \( J \) maps continuously \( \mathcal{S}_o(\mathbb{R}) \) into \( \mathcal{S}_e(\mathbb{R}) \). Let \( f \in \mathcal{S}_o(\mathbb{R}) \). Clearly \( Jf \) is a \( C^\infty \) even function on \( \mathbb{R} \). For \( n = 1, 2, \ldots, P_{m,n}(Jf) = P_{m,n-1}(f) \). Moreover,
\[ (1 + x^2)^m |f(x)| \leq (1 + x^2)^m \int_{|x|}^\infty |f(t)| dt \]
\[ \int_{|x|}^{\infty} (1 + t^2)^m |f(t)| dt \leq \int_{|x|}^{\infty} \frac{dt}{(1 + t^2)^m} P_{m+1,0}(f) \]

Hence \( P_{m,0}(Jf) \leq \frac{\pi}{2} P_{m+1,0}(f) \).

(ii) Let \( f \in \mathcal{B}_e(\mathbb{R}) \). By using (8) and by integrating by parts we have for any \( n = 0, 1, \ldots, \)

\[ \int_{\mathbb{R}} Jf(x) b_n(x) A(x) dx = \int_{\mathbb{R}} Jf(x) \Lambda b_{n+1}(x) A(x) dx \]
\[ = - \int_{\mathbb{R}} \Lambda Jf(x) b_{n+1}(x) A(x) dx \]
\[ = - \int_{\mathbb{R}} f(x) b_{n+1}(x) A(x) dx = 0, \]

which shows that \( Jf \in \mathcal{B}_e(\mathbb{R}) \). Conversely, let \( f \in \mathcal{B}_o(\mathbb{R}) \). Identity (8) together with an integration by parts yields for any \( n = 1, 2, \ldots, \)

\[ \int_{\mathbb{R}} f'(x) b_n(x) A(x) dx = \int_{\mathbb{R}} \Lambda f(x) b_n(x) A(x) dx \]
\[ = - \int_{\mathbb{R}} f(x) \Lambda b_n(x) A(x) dx \]
\[ = - \int_{\mathbb{R}} f(x) b_{n-1}(x) A(x) dx = 0, \]

which shows that \( f' \in \mathcal{B}_o(\mathbb{R}) \).

**Proposition 1.**

(i) For all \( f \) in \( \mathcal{S}(\mathbb{R}) \), we have

\[ \mathcal{F}_\Lambda (\Lambda f)(\lambda) = i\lambda \mathcal{F}_\Lambda (f)(\lambda). \tag{14} \]

(ii) For all \( f \) in \( \mathcal{S}(\mathbb{R}) \), we have

\[ \mathcal{F}_\Lambda (f)(\lambda) = \mathcal{F}_\Delta (f_e)(\lambda) + i\lambda \mathcal{F}_\Delta (Jf_o)(\lambda), \tag{15} \]

where \( \mathcal{F}_\Delta \) stands for the Fourier transform related to the differential operator \( \Delta \), defined on \( \mathcal{S}(\mathbb{R}) \) by

\[ \mathcal{F}_\Delta (h)(\lambda) = \int_{\mathbb{R}} h(x) \varphi_\lambda(x) A(x) dx, \quad \lambda \in \mathbb{R}, \]

\( f_e \) and \( f_o \) being respectively the even and odd parts of \( f \) given by (3).
Proof.

(i) Let $f \in \mathcal{S}(\mathbb{R})$. By (5), (10) and (13),

$$
\mathcal{F}_\lambda(Af)(\lambda) = \int_{\mathbb{R}} \left( f'_o(x) + \frac{A'(x)}{A(x)} f_o(x) \right) \varphi_\lambda(x) A(x) dx
$$

$$
= - \frac{1}{i\lambda} \int_{\mathbb{R}} f'_e(x) \varphi'_\lambda(x) A(x) dx
$$

$$
= \kappa_1 - \frac{\kappa_2}{i\lambda}.
$$

By integrating by parts we get

$$
\kappa_1 = \int_{\mathbb{R}} (A(x)f_o(x))' \varphi_\lambda(x) dx = - \int_{\mathbb{R}} f_o(x) \varphi'_\lambda(x) A(x) dx
$$

and

$$
\kappa_2 = \int_{\mathbb{R}} f'_e(x) \varphi'_\lambda(x) A(x) dx
$$

$$
= - \int_{\mathbb{R}} f_e(x)(A(x)\varphi'_\lambda(x))' dx
$$

$$
= - \int_{\mathbb{R}} f_e(x) \Delta \varphi_\lambda(x) A(x) dx
$$

$$
= \lambda^2 \int_{\mathbb{R}} f_e(x) \varphi_\lambda(x) A(x) dx
$$

by virtue of (11). Hence

$$
\kappa_1 - \frac{\kappa_2}{i\lambda} = i\lambda \int_{\mathbb{R}} \left( f_o(x) \varphi_\lambda(x) - f_o(x) \frac{\varphi'_\lambda(x)}{i\lambda} \right) A(x) dx
$$

$$
= i\lambda \int_{\mathbb{R}} f(x) \Phi_{-\lambda}(x) A(x) dx.
$$

This clearly yields (14).

(ii) If $f \in \mathcal{S}_e(\mathbb{R})$, identity (15) is obvious. Assume $f \in \mathcal{S}_e(\mathbb{R})$. By using (10), (11), (13) and by integrating by parts we obtain

$$
\mathcal{F}_\lambda(f)(\lambda) = \frac{1}{i\lambda} \int_{\mathbb{R}} f(x) \varphi'_\lambda(x) A(x) dx
$$

$$
= \frac{1}{i\lambda} \int_{\mathbb{R}} J f(x)(A(x)\varphi'_\lambda(x))' dx
$$

By integrating by parts we get
\[ = \frac{1}{i\lambda} \int_{\mathbb{R}} J f(x) \Delta \varphi_{\lambda}(x) A(x) dx \]
\[ = i\lambda \int_{\mathbb{R}} J f(x) \varphi_{\lambda}(x) A(x) dx \]
\[ = i\lambda \mathcal{F}_{\Delta} f(\lambda), \]

which completes the proof.

**Theorem 2.** The generalized Fourier transform \( \mathcal{F}_{\lambda} \) is a topological isomorphism

- from \( \mathcal{S}(\mathbb{R}) \) onto itself;
- from \( \mathcal{B}(\mathbb{R}) \) onto \( \mathcal{H}(\mathbb{R}) \).

**Proof.** By [13] we know that the transform \( \mathcal{F}_{\Delta} \) is a topological isomorphism

- from \( \mathcal{S}(\mathbb{R}) \) onto itself;
- from \( \mathcal{B}(\mathbb{R}) \) onto \( \mathcal{H}(\mathbb{R}) \).

The result follows then from (15), Lemma 2 and the fact that the operator \( \lambda \mapsto \lambda f \) is a topological isomorphism

- from \( \mathcal{S}(\mathbb{R}) \) onto \( \mathcal{S}(\mathbb{R}) \);
- from \( \mathcal{H}(\mathbb{R}) \) onto \( \mathcal{H}(\mathbb{R}) \).

**Proposition 2.**

(i) For all \( f \in \mathcal{S}(\mathbb{R}) \),
\[
\mathcal{F}_{\lambda}(f) = \mathcal{F}_{u} \circ \mathcal{V}(f),
\]

where \( \mathcal{F}_{u} \) denotes the usual Fourier transform on \( \mathbb{R} \) given by
\[
\mathcal{F}_{u}(f)(\lambda) = \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx.
\]

(ii) For all \( f \in \mathcal{S}(\mathbb{R}) \),
\[
\frac{d}{dx} \mathcal{V} f = \mathcal{V} \Delta f.
\]

**Proof.** Assertion (i) follows by applying the usual Fourier transform \( \mathcal{F}_{u} \) to both sides of (6) and by using the identity
\[
\mathcal{F}_{\Delta} h(\lambda) = \mathcal{F}_{u} \left( \mathcal{V} h \right)(\lambda),
\]
(see [13]). The intertwining relation (17) follows by applying the usual Fourier transform \( \mathcal{F}_{u} \) to both its sides and by using (14) and (16).
Theorem 3. The intertwining operator \( T^V \) is a topological isomorphism

- from \( \mathcal{S}(\mathbb{R}) \) onto itself;
- from \( \mathcal{B}(\mathbb{R}) \) onto \( \mathcal{W}(\mathbb{R}) \).

Proof. We deduce the result from (16), Theorem 2 and the fact that the usual Fourier transform \( \mathcal{F}_u \) is a topological isomorphism

- from \( \mathcal{S}(\mathbb{R}) \) onto itself;
- from \( \mathcal{W}(\mathbb{R}) \) onto \( \mathcal{H}(\mathbb{R}) \).

Definition 2.

(i) The generalized translation operators \( T^x \), \( x \in \mathbb{R} \), are defined on \( L^2(\mathbb{R}, A(x)dx) \) by the relation

\[ \mathcal{F}_\lambda(T^x f)(\lambda) = \Psi_\lambda(x)\mathcal{F}_\lambda(f)(\lambda). \] (18)

(ii) The generalized convolution product of two functions \( f \) and \( g \) in \( L^2(\mathbb{R}, A(x)dx) \) is defined by

\[ f \# g(x) = \int_{\mathbb{R}} T^x f(-y)g(y)A(y)dy. \] (19)

Remark 5. Let \( f \) and \( g \) be in \( L^2(\mathbb{R}, A(x)dx) \). Then

(i) By (18), Lemma 1 and Theorem 1, we deduce that

\[ \| T^x f \|_{2,\Lambda} \leq \| f \|_{2,\Lambda} \] (20)

for any \( x \in \mathbb{R} \).

(ii) It follows from (19), (20) and Schwarz inequality that \( f \# g \in L^\infty(\mathbb{R}) \) and

\[ \| f \# g \|_{\infty} \leq \| f \|_{2,\Lambda} \| g \|_{2,\Lambda}. \] (21)

(iii) By virtue of (18), (19) and Theorem 1, \( f \# g \) may be rewritten as

\[ f \# g(x) = \int_{\mathbb{R}} \mathcal{F}_\lambda(f)(\lambda)\mathcal{F}_\lambda(g)(\lambda)\Psi_\lambda(x)d\sigma(\lambda). \] (22)

Proposition 3. Let \( f \in L^2(\mathbb{R}, A(x)dx) \) and \( g \in L^1 \cap L^2(\mathbb{R}, A(x)dx) \). Then \( f \# g \in L^2(\mathbb{R}, A(x)dx) \),

\[ \| f \# g \|_{2,\Lambda} \leq \| f \|_{2,\Lambda} \| g \|_{1,\Lambda}, \] (23)

and

\[ \mathcal{F}_\lambda(f \# g) = \mathcal{F}_\lambda(f) \mathcal{F}_\lambda(g). \] (24)
Proof. By Schwarz inequality, $\mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(g) \in L^1(\mathbb{R}, d\sigma)$. Moreover, by Remark 3, $\mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(g) \in L^2(\mathbb{R}, d\sigma)$ and $\|\mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(g)\|_{L^2} \leq \|\mathcal{F}_\Lambda(f)\|_{L^2} \|g\|_{L^1}$. The result follows then by combining (22) and Theorem 1.

Proposition 4. If $f, g \in \mathcal{S}(\mathbb{R})$, then $f \# g \in \mathcal{S}(\mathbb{R})$ and

$$\mathcal{F} f \# g = \mathcal{F} f \ast \mathcal{F} g,$$

where $\ast$ denotes the usual convolution on $\mathbb{R}$.

Proof. The fact that $f \# g \in \mathcal{S}(\mathbb{R})$ follows from (24) and Theorem 2. Identity (25) follows by applying the usual Fourier transform to both its sides and by using (16) and (24).

Remark 6. Notice by (24) and Theorem 2 that $\mathcal{B}(\mathbb{R}) \# \mathcal{S}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$.

3. Generalized Wavelets

Definition 3. We say that a function $g \in L^2(\mathbb{R}, A(x)dx)$ is a generalized wavelet if it satisfies the admissibility condition :

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Lambda g(a\lambda)|^2 \frac{da}{a} < \infty,$$

for almost all $\lambda \in \mathbb{R}$.

Remark 7.

(i) The admissibility condition (26) can also be written as

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Lambda g(\lambda)|^2 \frac{d\lambda}{\lambda} = \int_0^\infty |\mathcal{F}_\Lambda g(-\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

(ii) If $g$ is real-valued we have $\mathcal{F}_\Lambda g(-\lambda) = \overline{\mathcal{F}_\Lambda g(\lambda)}$, so (26) reduces to

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Lambda g(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$ 

(iii) If $0 \neq g \in L^2(\mathbb{R}, A(x)dx)$ is real-valued and satisfies

$$\exists \eta > 0 \text{ such that } \mathcal{F}_\Lambda g(\lambda) - \mathcal{F}_\Lambda g(0) = o(\lambda^\eta), \text{ as } \lambda \to 0^+,$$

then (26) is equivalent to $\mathcal{F}_\Lambda g(0) = 0$.

(iv) According to (iii) and Theorem 2, each real-valued function $g$ in $\mathcal{B}(\mathbb{R})$ is a generalized wavelet.
Proposition 5.

(i) Let \( h \in L^2(\mathbb{R}, d\sigma) \) and \( a > 0 \). Then the function \( \lambda \mapsto h(a\lambda) \) belongs to \( L^2(\mathbb{R}, d\sigma) \) and we have

\[
\|h(a\cdot)\|_{2,\sigma} \leq \frac{k(a)}{\sqrt{a}} \|h\|_{2,\sigma},
\]

where

\[
k(a) = \sup_{\lambda > 0} \frac{|c(\lambda)|}{|c(\lambda/a)|}.
\]

(ii) For every \( a > 0 \), the dilatation operator

\[
H_a(f)(x) = \frac{1}{\sqrt{a}} f \left( \frac{x}{a} \right), \quad x \in \mathbb{R},
\]

is a topological automorphism of \( L^2(\mathbb{R}, d\sigma) \).

Proof.

(i) Notice first that according to the properties of the function \( c(\lambda) \) given in Theorem 1, there exist two positive constants \( m_1 \) and \( m_2 \) such that

\[
\frac{m_1}{a^{\alpha+1/2}} \leq k(a) \leq \frac{m_2}{a^{\alpha+1/2}} \quad \text{for all } a > 0.
\]

We have

\[
\|h(a\cdot)\|_{2,\sigma}^2 = \int_{\mathbb{R}} |h(a\lambda)|^2 \frac{d\lambda}{|c(\lambda)|^2} = \frac{1}{a} \int_{\mathbb{R}} |h(s)|^2 \frac{|c(|s|)|^2}{|c(|s|/a)|^2 |c(|s|)|^2} ds \leq \frac{k^2(a)}{a} \|h\|_{2,\sigma}^2.
\]

(ii) We deduce the result from (i).

Proposition 6. Let \( g \in L^2(\mathbb{R}, A(x)dx) \) and \( a > 0 \). Then there exists a function \( g_a \in L^2(\mathbb{R}, A(x)dx) \) (and only one) such that

\[
\mathcal{F}_\Lambda(g_a)(\lambda) = \mathcal{F}_\Lambda(g)(a\lambda)
\]

for almost every \( \lambda \in \mathbb{R} \). This function is given by the relation

\[
g_a = \frac{1}{\sqrt{a}} \mathcal{F}_\Lambda^{-1} \circ H_a^{-1} \circ \mathcal{F}_\Lambda(g)
\]

and satisfies

\[
\|g_a\|_{2,\sigma} \leq \frac{k(a)}{\sqrt{a}} \|g\|_{2,A}.
\]
Proof. The result follows by combining Theorem 1 and Proposition 5.

Remark 8. For \( A(x) = |x|^{2a+1}, \ a > -1/2 \), the function \( g_{a} \), \( a > 0 \), is given by

\[
g_{a}(x) = \frac{1}{a^{2a+2}} g \left( \frac{x}{a} \right), \quad x \in \mathbb{R}.
\]

Proposition 7. Let \( g \) be in \( \mathcal{S}(\mathbb{R}) \). Then for all \( a > 0 \), the function \( g_{a} \) belongs to \( \mathcal{S}(\mathbb{R}) \) and we have the relation

\[
g_{a} = \frac{1}{\sqrt{a}} V^{-1} \circ H_{a} \circ V(g).
\]

Proof. The result follows from (16), (28), Theorem 2, and the fact that \( \mathcal{F}_{a} \circ H_{a} = H_{a^{-1}} \circ \mathcal{F}_{a} \).

Notation. For a function \( g \) in \( L^{2}(\mathbb{R}, A(x)dx) \) and for \( (a, b) \in ]0, \infty[ \times \mathbb{R} \) we write

\[
g_{a,b}(x) := \sqrt{a} T^{-b} g_{a}(x),
\]

where \( T^{-b} \) are the generalized translation operators given by (18).

Definition 4. Let \( g \in L^{2}(\mathbb{R}, A(x)dx) \) be a generalized wavelet. The generalized continuous wavelet transform \( \Phi_{g} \) is defined for regular functions \( f \) on \( \mathbb{R} \) by:

\[
\Phi_{g}(f)(a, b) = \int_{\mathbb{R}} f(x) g_{a,b}(x) A(x) dx.
\]

This transform can also be written in the form

\[
\Phi_{g}(f)(a, b) = \sqrt{a} f \# \tilde{g}_{a}(b),
\]

where \# is the generalized convolution product given by (19), and \( \tilde{g}_{a}(x) = g_{a}(-x), \ x \in \mathbb{R} \).

Lemma 3. For all \( f, g \in L^{2}(\mathbb{R}, A(x)dx) \) and all \( h \in \mathcal{S}(\mathbb{R}) \) we have the identity

\[
\int_{\mathbb{R}} f \# g(x) \mathcal{F}_{A}^{-1}(h)(x) A(x) dx = \int_{\mathbb{R}} \mathcal{F}_{A}(f)(\lambda) \mathcal{F}_{A}(g)(\lambda) h^{-}(\lambda) d\sigma(\lambda)
\]

where \( h^{-}(\lambda) = h(-\lambda), \ \lambda \in \mathbb{R} \).

Proof. Fix \( g \in L^{2}(\mathbb{R}, A(x)dx) \) and \( h \in \mathcal{S}(\mathbb{R}) \). For \( f \in L^{2}(\mathbb{R}, A(x)dx) \) put

\[
S_{1}(f) = \int_{\mathbb{R}} f \# g(x) \mathcal{F}_{A}^{-1}(h)(x) A(x) dx
\]

and

\[
S_{2}(f) = \int_{\mathbb{R}} \mathcal{F}_{A}(f)(\lambda) \mathcal{F}_{A}(g)(\lambda) h^{-}(\lambda) d\sigma(\lambda).
\]
In view of Proposition 3 and Theorem 1, we see that $S_1(f) = S_2(f)$ for each $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$. Moreover, by using (21), Schwarz inequality and Theorem 1 we get

$$|S_1(f)| \leq \|f \otimes g\|_\infty \|\mathcal{F}_0^{-1}(h)\|_{1,A} \leq \|f\|_{2,A} \|g\|_{2,A} \|\mathcal{F}_0^{-1}(h)\|_{1,A}$$

and

$$|S_2(f)| \leq \|F_0(f)F_0(g)\|_{1,\sigma} \|h\|_\infty \leq \|F_0(f)\|_{2,\sigma} \|F_0(g)\|_{2,\sigma} \|h\|_\infty \leq \|f\|_{2,A} \|g\|_{2,A} \|h\|_\infty,$$

which shows that the linear functionals $S_1$ and $S_2$ are bounded on $L^2(\mathbb{R}, A(x)dx)$. Therefore $S_1 \equiv S_2$, and the lemma is proved.

**Lemma 4.** Let $f_1, f_2 \in L^2(\mathbb{R}, A(x)dx)$. Then $f_1 \# f_2 \in L^2(\mathbb{R}, A(x)dx)$ if and only if $\mathcal{F}_0(f_1) \mathcal{F}_0(f_2) \in L^2(\mathbb{R}, d\sigma)$ and we have

$$\mathcal{F}_0(f_1 \# f_2) = \mathcal{F}_0(f_1) \mathcal{F}_0(f_2)$$

in the $L^2$–case.

**Proof.** Suppose $f_1 \# f_2 \in L^2(\mathbb{R}, A(x)dx)$. By Lemma 3 and Theorem 1, we have for any $h \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \mathcal{F}_0(f_1)(\lambda) \mathcal{F}_0(f_2)(\lambda) h(\lambda) d\sigma(\lambda) = \int_{\mathbb{R}} f_1 \# f_2(x) \mathcal{F}_0^{-1}(h)(x) A(x) dx = \int_{\mathbb{R}} f_1 \# f_2(x) \mathcal{F}_0^{-1}(h)(x) A(x) dx = \int_{\mathbb{R}} \mathcal{F}_0(f_1) \mathcal{F}_0(f_2)(\lambda) h(\lambda) d\sigma(\lambda),$$

which shows that $\mathcal{F}_0(f_1) \mathcal{F}_0(f_2) = \mathcal{F}_0(f_1 \# f_2)$. Conversely, if $\mathcal{F}_0(f_1) \mathcal{F}_0(f_2) \in L^2(\mathbb{R}, d\sigma)$, then by Lemma 3 and Theorem 1, we have for any $h \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} f_1 \# f_2(x) \mathcal{F}_0^{-1}(h)(x) A(x) dx = \int_{\mathbb{R}} \mathcal{F}_0(f_1)(\lambda) \mathcal{F}_0(f_2)(\lambda) h(\lambda) d\sigma(\lambda) = \int_{\mathbb{R}} \mathcal{F}_0^{-1}[\mathcal{F}_0(f_1) \mathcal{F}_0(f_2)](\lambda) h(\lambda) A(x) dx,$$

which shows, in view of Theorem 2, that $f_1 \# f_2 = \mathcal{F}_0^{-1}[\mathcal{F}_0(f_1) \mathcal{F}_0(f_2)]$. This achieves the proof.

A combination of Lemma 4 and Theorem 1 gives us the following.
Lemma 5. Let $f_1, f_2 \in L^2(\mathbb{R}, A(x)dx)$. Then

$$\int_{\mathbb{R}} |f_1 \# f_2(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}_A(f_1)(\lambda)|^2 |\mathcal{F}_A(f_2)(\lambda)|^2 d\sigma(\lambda)$$

where both sides are finite or infinite.

Theorem 4. Let $g \in L^2(\mathbb{R}, A(x)dx)$ be a generalized wavelet. Then for all $f \in L^2(\mathbb{R}, A(x)dx)$, we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \frac{1}{C_g} \int_0^\infty \int_{\mathbb{R}} |\Phi_g(f)(a, b)|^2 A(b) db \frac{da}{a^2}.$$

Proof. Using (26), (27), (31), Fubini's Theorem and Lemma 5, we have

$$\frac{1}{C_g} \int_0^\infty \int_{\mathbb{R}} |\Phi_g(f)(a, b)|^2 A(b) db \frac{da}{a^2} =$$

$$= \frac{1}{C_g} \int_0^\infty \left( \int_{\mathbb{R}} |f \# \widetilde{g}_a(b)|^2 A(b) db \right) \frac{da}{a}$$

$$= \frac{1}{C_g} \int_0^\infty \left( \int_{\mathbb{R}} |\mathcal{F}_A(f)(\lambda)|^2 |\mathcal{F}_A(g)(a\lambda)|^2 d\sigma(\lambda) \right) \frac{da}{a}$$

$$= \int_{\mathbb{R}} |\mathcal{F}_A(f)(\lambda)|^2 d\sigma(\lambda)$$

The result is now a direct consequence of Theorem 1.

Theorem 5. Let $g \in L^2(\mathbb{R}, A(x)dx)$ be a generalized wavelet. Then for $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$ such that $\mathcal{F}_A(f) \in L^1(\mathbb{R}, d\sigma)$, we have

$$f(x) = \frac{1}{C_g} \int_0^\infty \left( \int_{\mathbb{R}} \Phi_g(f)(a, b) g_{a, b}(x) A(b) db \right) \frac{da}{a^2}, \text{ a.e.},$$

where, for each $x \in \mathbb{R}$, both the inner integral and the outer integral are absolutely convergent, but possibly not the double integral.

Proof. Put

$$\mathcal{S}(a, x) = \int_{\mathbb{R}} \Phi_g(f)(a, b) g_{a, b}(x) A(b) db$$

and

$$\mathcal{J}(x) = \frac{1}{C_g} \int_0^\infty \mathcal{S}(a, x) \frac{da}{a^2}.$$
By (30) and (31) we have
\[ \mathcal{J}(a, x) = a \int_{\mathbb{R}} \mathcal{F}_\lambda(f \# \tilde{g}_a)(b) T^{-x} \tilde{g}_a(b) \mathcal{F}_\lambda(g)(a \lambda) \, db. \]

From (20), (23) and Schwarz inequality we deduce that the integral \( \mathcal{J}(a, x) \) is absolutely convergent. On the other hand, by (18), (24) and (27),
\[ \mathcal{F}_\lambda(f \# \tilde{g}_a)(\lambda) = \mathcal{F}_\lambda(f)(\lambda) \mathcal{F}_\lambda(g)(a \lambda) \]
and
\[ \mathcal{F}_\lambda(T^{-x} \tilde{g}_a)(\lambda) = \Psi_\lambda(-x) \mathcal{F}_\lambda(g)(a \lambda). \]

So using Theorem 1 we obtain
\[ \mathcal{J}(a, x) = a \int_{\mathbb{R}} \mathcal{F}_\lambda(f)(\lambda) \mathcal{F}_\lambda(g)(a \lambda) \Psi_\lambda(x) \mathcal{F}_\lambda(g)(a \lambda) d\sigma(\lambda). \]

In particular, this implies that
\[
\frac{1}{C_{g}} \int_{0}^{\infty} |\mathcal{J}(a, x)| \frac{da}{a^2} \leq \int_{\mathbb{R}} |\mathcal{F}_\lambda(f)(\lambda)| \left( \frac{1}{C_{g}} \int_{0}^{\infty} |\mathcal{F}_\lambda(g)(a \lambda)|^2 \frac{da}{a} \right) d\sigma(\lambda) = \left\| \mathcal{F}_\lambda(f) \right\|_{1, \sigma} < \infty,
\]
that is, the integral \( \mathcal{J}(x) \) is absolutely convergent. Finally, using Fubini’s theorem we get
\[
\mathcal{J}(x) = \frac{1}{C_{g}} \int_{0}^{\infty} \left( \int_{\mathbb{R}} \mathcal{F}_\lambda(f)(\lambda) |\mathcal{F}_\lambda(g)(a \lambda)|^2 \Psi_\lambda(x) d\sigma(\lambda) \right) d\sigma(\lambda)
\]
\[
= \int_{\mathbb{R}} \mathcal{F}_\lambda(f)(\lambda) \left( \int_{0}^{\infty} |\mathcal{F}_\lambda(g)(a \lambda)|^2 \frac{da}{a} \right) \Psi_\lambda(x) d\sigma(\lambda)
\]
\[
= \int_{\mathbb{R}} \mathcal{F}_\lambda(f)(\lambda) \Psi_\lambda(x) d\sigma(\lambda),
\]
which ends the proof in view of Theorem 1.

4. Inversion of the Intertwining Operators Using Generalized Wavelets

In this section we suppose that the function \( |c(\lambda)|^{-2} \) is \( C^\infty \) on \( ]0, \infty[ \), and for all \( n \in \mathbb{N} \):

(i) \( \frac{d^n}{d\lambda^n} |c(\lambda)|^{-2} \neq 0 \) on \( ]0, \infty[ \);

(ii) \( \exists p_n \in \mathbb{N} \) and \( k_n > 0 \) such that \( \frac{d^n}{d\lambda^n} |c(\lambda)|^{-2} \leq k_n \lambda^{p_n} \) for \( \lambda \geq 1 \);

(iii) \( \frac{d^n}{d\lambda^n} |c(\lambda)|^{-2} \sim_{0^+} a_n \lambda^{q_n} \), where \( a_n \in \mathbb{R} \) and \( q_n \in \mathbb{Z} \).
Remark 9. These conditions are satisfied in the Dunkl operator case.

Proposition 8. The operator \( \mathcal{K} \) (resp. \( \mathcal{M} \)) defined by
\[
\mathcal{K}(f) = \mathcal{F}_u^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_u(f) \right]
\]
(32)
\[
\text{(resp. } \mathcal{M}(f) = \mathcal{F}_\Lambda^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_\Lambda(f) \right] \text{)}
\]
(33)
is a topological automorphism of \( \mathcal{W}(\mathbb{R}) \) (resp. \( \mathcal{B}(\mathbb{R}) \)).

Proof. Clearly, the mapping \( f \mapsto 2\pi |c(|\lambda|)|^{-2} f \) is a topological automorphism of \( \mathcal{K}(\mathbb{R}) \), and its inverse is given by \( f \mapsto \frac{1}{2\pi} c(|\lambda|) f \). We deduce the result from Theorem 2 and the fact that the usual Fourier transform \( \mathcal{F}_u \) is a topological isomorphism from \( \mathcal{W}(\mathbb{R}) \) onto \( \mathcal{H}(\mathbb{R}) \).

Proposition 9. For \( f \) in \( \mathcal{B}(\mathbb{R}) \), we have
\[
\mathcal{M}(f) = 1V^{-1} \circ \mathcal{K} \circ 1V(f).
\]
(34)

Proof. By (16), (32) and (33),
\[
\mathcal{M}(f) = \mathcal{F}_\Lambda^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_\Lambda(f) \right]
\]
\[
= 1V^{-1} \circ \mathcal{F}_u^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_u \circ 1V(f) \right]
\]
\[
= 1V^{-1} \circ \mathcal{K} \circ 1V(f).
\]

Proposition 10.

(i) For all \( f \) in \( \mathcal{W}(\mathbb{R}) \) and \( g \) in \( \mathcal{H}(\mathbb{R}) \), we have
\[
\mathcal{K}(f \ast g) = \mathcal{K}(f) \ast g.
\]

(ii) For all \( f \) in \( \mathcal{B}(\mathbb{R}) \) and \( g \) in \( \mathcal{H}(\mathbb{R}) \), we have
\[
\mathcal{M}(f \# g) = \mathcal{M}(f) \# g.
\]

Proof. We have
\[
\mathcal{K}(f \ast g) = \mathcal{F}_u^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_u(f \ast g) \right]
\]
\[
= \mathcal{F}_u^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_u(f) \mathcal{F}_u(g) \right]
\]
\[
= \left\{ \mathcal{F}_u^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_u(f) \right] \right\} \ast g
\]
\[
= \mathcal{K}(f) \ast g
\]
and
\[
\mathcal{M}(f \# g) = \mathcal{F}_\Lambda^{-1} \left[ 2\pi |c(|\lambda|)|^{-2} \mathcal{F}_\Lambda(f \# g) \right]
\]
\[ F_\Lambda^{-1} \left[ 2\pi |c(\lambda)|^{-2} \mathcal{F}_\Lambda(f) \mathcal{F}_\Lambda(g) \right] = \{ F_\Lambda^{-1} \left[ 2\pi |c(\lambda)|^{-2} \mathcal{F}_\Lambda(f) \right] \} \# g = \mathcal{M}(f) \# g, \]

which ends the proof.

**Theorem 6.** 1. The intertwining operator \( V \) is a topological isomorphism from \( \mathcal{W}(\mathbb{R}) \) onto \( \mathcal{B}(\mathbb{R}) \).

2. We have the following inverse formulas for \( V \) and \( ^t V \):

(a) For \( f \in \mathcal{B}(\mathbb{R}) \),
\[ f = V \mathcal{K}^t V(f); \quad f = \mathcal{M} V^t V(f). \]

(b) For \( f \in \mathcal{W}(\mathbb{R}) \),
\[ f = \mathcal{K}^t V V(f); \quad f = V^t V V(f). \]

**Proof.** Let \( f \in \mathcal{B}(\mathbb{R}) \). From (12), (16) and Theorem 1 we have for all \( x \in \mathbb{R} \),
\[ f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda) \Psi_\lambda(x) \frac{d\lambda}{|c(|\lambda|)|^2} \]
\[ = V \left( \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda) e^{i\lambda \cdot x} \frac{d\lambda}{|c(|\lambda|)|^2} \right)(x) \]
\[ = V \left( \frac{1}{2\pi} \int_{\mathbb{R}} \left[ 2\pi |c(\lambda)|^{-2} \mathcal{F}_u \circ ^t V(f) \right] e^{i\lambda \cdot x} d\lambda \right)(x) \]
\[ = V \mathcal{K}^t V V(f)(x). \]

This when combined with (34) yields formula (36). By replacing \( f \) respectively by \( Vf \) and \( ^t V^{-1} f \) into formulas (35) and (36), we regain identities (37) and (38). From (35), Proposition 8 and Theorem 3, we deduce that \( V \) is a topological isomorphism from \( \mathcal{W}(\mathbb{R}) \) onto \( \mathcal{B}(\mathbb{R}) \).

In order to invert the intertwining operators \( V \) and \( ^t V \) we shall need some technical lemmas.

**Lemma 6.** For all \( f \) in \( \mathcal{W}(\mathbb{R}) \) and \( g \) in \( \mathcal{S}(\mathbb{R}) \), we have
\[ V(f \ast g) = V(f) \# ^t V^{-1}(g). \]

**Proof.** By using relations (25), (35), (37) and Proposition 10(i) we have
\[ V^{-1} \left[ V(f) \# ^t V^{-1}(g) \right] = \mathcal{K}^t V \left[ V(f) \# ^t V^{-1}(g) \right] \]
\[ = \mathcal{K}^t \left[ ^t V V(f) \ast g \right] \]
\[ = \left[ \mathcal{K}^t V V(f) \right] \ast g \]
\[ = f \ast g. \]
Definition 5. The classical continuous wavelet transform on $\mathbb{R}$ is defined for regular functions by

$$S_g(f)(a, b) = \int_{\mathbb{R}} f(x) \overline{g_{a,b}^0(x)} \, dx, \quad a > 0, \ b \in \mathbb{R},$$

where

$$g_{a,b}^0(x) := \frac{1}{\sqrt{a}} g \left( \frac{x-b}{a} \right).$$

The function $g$ is a classical wavelet on $\mathbb{R}$, i.e., a function in $L^2(\mathbb{R}, dx)$ satisfying the admissibility condition:

$$0 < C_g^0 = \int_0^\infty |\mathcal{F}_a(g)(a\lambda)|^2 \frac{da}{a} < \infty,$$

for almost all $\lambda \in \mathbb{R}$.

A more complete and detailed discussion of the properties of the classical wavelet transform on $\mathbb{R}$ can be found in [3], from which we have the following inversion formula.

Theorem 7. Let $g \in L^2(\mathbb{R}, dx)$ be a classical wavelet. If both $f$ and $\mathcal{F}_a(f)$ are in $L^1(\mathbb{R}, dx)$ then we have

$$f(x) = \frac{1}{C_g} \int_0^\infty \left( \int_{\mathbb{R}} S_g(f)(a, b) g_{a,b}^0(x) \, db \right) \frac{da}{a^2}$$

for almost every $x \in \mathbb{R}$.

Remark 10. According to (16) and Definitions 3, 5, $g \in \mathcal{S}(\mathbb{R})$ is a generalized wavelet, if and only if, $\mathcal{V}(g)$ is a classical wavelet and we have:

$$C_{\mathcal{V}(g)} = C_g. \quad (40)$$

Lemma 7. Let $g \in \mathcal{W}(\mathbb{R})$ be real-valued. Then for all $f \in \mathcal{S}(\mathbb{R})$ we have

$$\Phi_{\mathcal{V}(g)}(f)(a, b) = \mathcal{M} \left[ S_g(\mathcal{V}(f) (a, \cdot)) \right] (b).$$

Proof. Notice that $\mathcal{V}(\mathcal{V}g) = \mathcal{V}^{-1}g$ by virtue of (35). Further, $g$ is a classical wavelet according to [3]. So it follows from Remark 10 that $\mathcal{V}(g) \in \mathcal{B}(\mathbb{R})$ is a generalized wavelet and

$$C_{\mathcal{V}(g)} = C_g^0. \quad (41)$$

Due to (25), (29), (31), (35), (38) and Definition 5 we have

$$\Phi_{\mathcal{V}(g)}(f)(a, b) = \sqrt{a} f \#(\mathcal{V}(g))_a(b)$$

$$= \sqrt{a} \mathcal{V}^{-1} \left[ \mathcal{V}f * \mathcal{V}(\mathcal{V}(g)_a) \right] (b)$$

$$= \mathcal{V}^{-1} \left[ \mathcal{V}f * \mathcal{H}_a \left( \mathcal{V}(g) \right) \right] (b)$$

$$= \mathcal{M} \left[ \mathcal{V}(f) * \mathcal{H}_a(g) \right] (b)$$

$$= \mathcal{M} \left[ \mathcal{V}(f) (a, \cdot) \right] (b).$$
Lemma 8. Let \( g \in \mathcal{B}(\mathbb{R}) \) be real-valued. Then for all \( f \in \mathcal{W}(\mathbb{R}) \), we have
\[
S_{tVg}(f)(a,b) = \mathcal{X}^{-1}V \left[ \Phi_g(Vf)(a,\cdot) \right](b).
\]

Proof. Observe that by Remarks 7(iv) and 10, \( \mathcal{V}g \) is a classical wavelet. Using (29), (31), (39) and Definition 5 we have
\[
V \left( S_{tVg}(f)(a,\cdot) \right)(b) = V(f * H_a (\mathcal{V}g))(b) = \sqrt{a} V(f * \mathcal{V}g(a))(b) = \sqrt{a} V(f# \varpi_a)(b) = \Phi_g(Vf)(a,b).
\]

Thus
\[
S_{tVg}(f)(a,b) = V^{-1} [\Phi_g(Vf)(a,\cdot)](b) = \mathcal{X}^{-1}V [\Phi_g(Vf)(a,\cdot)](b)
\]
by virtue of (35).

We can now state our main result.

Theorem 8.

(i) Let \( g \in \mathcal{W}(\mathbb{R}) \) be real-valued. Then for all \( f \in \mathcal{S}(\mathbb{R}) \) we have
\[
\mathcal{V}^{-1}f(x) = \frac{1}{C_g} \int_{0}^{\infty} \left( \int_{\mathbb{R}} \mathcal{M}V[S_g(f)(a,\cdot)](b)(V\mathcal{X}g)_{a,b}(x)A(b)db \right) \frac{da}{a^2}.
\]

(ii) Let \( g \in \mathcal{B}(\mathbb{R}) \) be real-valued. Then for all \( f \in \mathcal{B}(\mathbb{R}) \) we have
\[
V^{-1}f(x) = \frac{1}{C_g} \int_{0}^{\infty} \left( \int_{\mathbb{R}} \mathcal{X}^{-1}V[\Phi_g(Vf)(a,\cdot)](b) (\mathcal{V}g)_{a,b}(x)db \right) \frac{da}{a^2}.
\]

Proof. The result follows by combining Theorems 5, 7, Lemmas 7, 8 and identities (40), (41).

References


