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A Generalization of Durbin-Watson Statistic

A. K. Gupta[1*], D. G. Kabe[2], and S. NiwitPong[3]

[1] *Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, USA*
[2] *5971 Greensboro Dr., Mississauga, Ontario, Canada*
[3] *Department of Applied Statistics, King Mongkut's University of Technology North Bangkok, Thailand*

Abstract. Two generalizations of the Durbin-Watson Statistic d, for testing that the serial correlation, in a given univariate normal regression model, is zero, to its multivariate counter part, are proposed. In the univariate case the moments of d are obtained in terms of generalized gamma functions. Our methodology is based on the generalized quadratic form of the central Wishart distribution.

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1. Introduction

For the univariate normal linear regression model

$$Y = X\beta + e, e \sim N(0, \sigma^2 I) \tag{1}$$

where Y is an n component (column) vector, β has q components, X is $n \times q$ and of rank $q < n$, σ^2 is unknown, the Durbin-Watson statistic d is defined as follows,

$$\begin{aligned}(Y - X\beta)'(Y - X\beta) &= (\beta - \hat{\beta})'X'X(\beta - \hat{\beta}) + Y'(I - X(X'X)X')Y \\ &= (\beta - \hat{\beta})'X'X(\beta - \hat{\beta}) + Y'QQ'Y,\end{aligned}$$

$\hat{\beta} = (X'X)^{-1}X'Y$ and $Q'Q = I$, Q is $n \times m$, m=(n-q) matric of rank $m < n$. It follows that

$$Y'QQ'Y = f'f, \tag{2}$$

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*Corresponding author.

Email address: gupta@bgsu.edu (A. Gupta)

f is $m \times n$, and the density of f is

$$g(f) = K \exp\left\{-\frac{1}{2\sigma^2} f'f\right\}, -\infty < f < \infty, \tag{3}$$

where K, as a generic letter, denotes the normalizing constants of density functions in this paper.

Now [6, p. 200] define d to be

$$d = f'Af/f'f, A = Q'A_1Q, \tag{4}$$

$$A_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & 1 & 0 & \dots & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & \dots & 1 & 2 & 1 \end{pmatrix}$$

where $n \times n$ A_1 is of rank $(n-1)$.

Next setting $f = ht$, $f'f = 1$,(4) reduces to

$$d = h'Ah, h'h = 1, \tag{5}$$

and hence

$$E(d^g) = K \int_{h'h=1} (h'Ah)^g dh = \frac{(\frac{1}{2})_g C_{(g)}(A)}{(\frac{m}{2})_g}, \tag{6}$$

where $C_{(\theta)}$ is the zonal polynomial [1, p. 29].

The integrals of the type (6) are known in the literature as the generalized quadratic form of the central Wishart distribution (GQFCWD) integrals. The power function integrals of the type (6) may be called the generalized quadraic form of the noncentral Wishart distribution (GQFNCWD) integrals.

Mathai et al. [6, Chapter 5] list a number of integrals of the type (6) and their generalizations, however, none of them are suitable for the moments problem of d in the present context. We formulate some suitable integrals in our context for the moments problem.

The model (1) generalizes to the model

$$Y = X\beta + E, E \sim N(0, I \otimes \Sigma), \tag{7}$$

where Y is $p \times n$, β is $p \times q$, $n > (p+q)$, X is $q \times n$ and of rank $q < n$, Σ is $p \times p$ unknown.

We now write

$$\begin{aligned}(Y - \beta X)(Y - \beta X)' &= (\beta - \hat{\beta})XX'(\beta - \hat{\beta})' + Y(I - X'(X'X)^{-1}X)Y' \\ &= (\hat{\beta} - \beta)XX'(\hat{\beta} - \beta)' + YQQ'Y',\end{aligned}$$

$\hat{\beta} = (X'X)^{-1}X'Y$ and the first generalized d to be

$$d = tr(YAY')/tr(YQQ'Y') = tr(FAF')/tr(FF') = tr(HAH') \tag{8}$$

where $HH' = I$ and H is $p \times (n-q)$ or $p \times m$.

The second generalized d is, where m is assumed to be n,

$$d = |HAH'| = |H\Lambda H'|, HH' = I, \tag{9}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix of the roots of A.

From (8), [6, p.270, equation 5.5.6] show that

$$E(d^g) = \int_{HH'=1} (tr(H\Lambda H'))^K dH = \left(\frac{p}{2}\right)_g C_{(g)}(\Lambda) / \left(\frac{pm}{2}\right)_g, \tag{10}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is the diagonal matrix of the roots of A_1.

However, it does not appear that the integral

$$E(d^g) = \int_{HH'=1} |H\Lambda H'|^g dH, \tag{11}$$

has been suitably evaluated in the literature, and the evaluation of the integral (11) is the main result of the present paper. The methodology for integrating (11) is based on [2, 3, 4] and [5, p. 352, equation 5.5.29].

We present our methodology in the next section, and section 3 evaluates the integral (11). The moments of d can be calculated in terms of gamma functions; however the density of d is not, as yet, available in the literature, except in trivial cases. Sometimes the same symbol denotes different quantities; however, its meaning is made explicit in the paper.

2. Methodology

Given the joint density of n gamma variates to be

$$g(y_1, y_2, \ldots, y_n) = K \exp\{-(\lambda_1 y_1 + \cdots + \lambda_n y_n)\} y_1^{g_1 - 1} \ldots y_n^{g_n - 1}, \tag{12}$$

the density of $t = (y_1 + \cdots + y_n)$ is desired. The moment generating function $\psi(\theta)$ of t is

$$\begin{aligned} \psi(\theta) &= (\alpha_1 - \theta)^{-g_1} \ldots (\alpha_n - \theta)^{-g_n} \\ &= (\alpha_1 - \theta)^{-g_1} ((\alpha_1 - \theta) - (\alpha_1 - \alpha_2))^{-g_2} \ldots ((\alpha_1 - \theta) - (\alpha_1 - \alpha_n))^{-g_n}, \end{aligned}$$

where α_1 is the largest parameter amongst the n positive parameters $\alpha_1, \ldots, \alpha_n$.

Now [2] expands $\psi(\theta)$ as

$$\begin{aligned} \psi(\theta) &= (\alpha_1 - \theta)^{(-g_1 + \cdots + g_n + r_2 + \cdots + r_n)} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \binom{g_2 + r_2 - 1}{r_2} \ldots \\ & \binom{g_n + r_n - 1}{r_n} (\alpha_1 - \alpha_2)^{r_2} \ldots (\alpha_1 - \alpha_n)^{r_n}, \end{aligned} \tag{13}$$

and inverting (13) finds the density of t to be

$$g(t) = K\exp\{-t\}t^{(g_1+\ldots+g_n-1)}\phi(g_2,\ldots,g_n;g_1+\cdots+g_n;(\alpha_1-\alpha_2)t,\ldots,(\alpha_1-\alpha_n)t)$$

where

$$\begin{aligned}\phi &= \sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\Gamma(g_2+r_2)\ldots\Gamma(g_n+r_n)(\alpha_1-\alpha_2)^{r_2}\ldots(\alpha_1-\alpha_2)^{r_n}}{\Gamma(g_1+\cdots+g_n+r_2+\cdots+r_n)r_2!\ldots r_n!}\\ &= {}_1F_1(g_2+\cdots+g_n;g_1+\cdots+g_n);((n-1)\alpha_1-\alpha_2-\ldots\alpha_n)t). \qquad (14)\end{aligned}$$

To prove (14), we observe that the sum of two noncentral Wishart $p\times p$ matrices A and B, with noncentrality parameter $p\times p$ matrices Δ and Ω, and n and q degrees of freedom respectively is again noncentral Wishart with $(n+q)$ degrees of freedom, and noncentrality parameter matrix $(\Delta+\Omega)$. With $2g=(p+1)$, we write this result as

$$\begin{aligned}&\int_{A+B=D}\exp\{-tr(A+B)\}|A|^{n-g}|B|^{q-g}oF_1(n;\Delta A)oF_1(q;\Omega B)dAdB\\ =\ & K\exp\{-trD\}|D|^{n+q-g}oF_1(q;(\Delta+\Omega)D) \qquad (15)\end{aligned}$$

or formally that

$$oF_1(n;\Delta A)oF_1(q;\Omega B) = oF_1(n+q;(\Delta+\Omega)(A+B)). \qquad (16)$$

Mathai [5, p. 339, Theorem 5.5] defines

$$\begin{aligned}\phi(b_1,b_2;c;X_1,X_2) &= \int|U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}\\ &\quad {}_1F_1(b_1;d_1;X_1U_1)_1F_1(b_2;d_2;X_2U_2)dU_1dU_2\\ &= \int|U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}\\ &\quad \exp\{-tr(Z_1+Z_2)\}|Z_1|^{b_1-g}|Z_2|^{b_2-g}oF_1(d_1;X_1U_1Z_1)\\ &\quad oF_1(d_2;X_2U_2Z_2)dZ_1dZ_2dU_1dU_2\\ &= \int|U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}\\ &\quad \exp\{-tr(Z_1+Z_2)\}|Z_1|^{b_1-g}|Z_2|^{b_2-g}oF_1(d_1+d_2;\\ &\quad (X_1+X_2)(U_1+U_2)(Z_1+Z_2)dZ_1dZ_2dU_1dU_2\\ &= \int|U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}\\ &\quad \exp\{-tr(Z_1+Z_2)\}|Z_1|^{b_1-g}|Z_2|^{b_2-g}\\ &\quad {}_1F_1(b_1+b_2;d_1+d_2;(X_1+X_2)(U_1+U_2))dU_1dU_2\\ &= (K)_2F_2(d_1+d_2;b_1+b_2;d_1+d_2;c;X_1+X_2)\end{aligned}$$

$$= (K)_1F_1(b_1+b_2;c;X_1+X_2), \tag{17}$$

and hence (16) follows. In (17) all matrices are $p \times p$ positive symmetric matrices.

Obviously now we have the integral

$$\begin{aligned} g(T) &= K \int_{A_1+\cdots+A_n=T} \exp\{tr(\Sigma_1 A_1 + \cdots + \Sigma_n A_n)\}|A_1|^{g_1-g} \ldots |A_n|^{g_n-g} \\ &\quad dA_1 \ldots dA_n \\ &= K \exp\{-trT\}|T|^{g_1+\cdots+g_n-g}{}_1F_1(g_2+\cdots+g_n; g_1+\cdots+g_n; ((n-1) \\ &\quad \Sigma_1 - \Sigma_2 - \cdots - \Sigma_n)T) \end{aligned} \tag{18}$$

where all matrices in (18) are $p \times p$ positive definite symmetric matrices.

The moment generating function $\phi(\theta)$ of T is

$$\phi(\theta) = |\Sigma_1 - \theta|^{-g_1} \ldots |\Sigma_n - \theta|^{-g_n}, \tag{19}$$

and (18) is obtained by inverting (19), see e.g., [4], [5, p. 352, equation 5.5.29].

If now X $p \times p$ has the density

$$g(X) = K \exp\{-\frac{1}{2} tr(X\Lambda X')\}, -\infty < X < \infty, \tag{20}$$

then the moment generating function $M(\theta)$ of $T = XX'$ is

$$M(\theta) = |\lambda_1 I - \theta|^{-1/2} \ldots |\lambda_n I - \theta|^{-1/2}, \tag{21}$$

and hence from (18), the density function of the GQFCWD of T is

$$\begin{aligned} g(T) &= K \exp\{-\frac{1}{2} tr(T)\}|T|^{\frac{1}{2}(n-p-1)} \\ &\quad {}_1F_1(\frac{1}{2}(n-1); \frac{1}{2}n; ((n-1)\lambda_1 - \lambda_2 - \cdots - \lambda_n)T). \end{aligned} \tag{22}$$

Now [3] proves the following results. Let $p \times n$ Y, $-\infty < X < \infty$ and $q \times n$ D of rank d be given, then we have that

$$\begin{aligned} &\int_{YY'=T, DY'=V'} f(YY', DY')dY \\ &= K|DD'|^{\frac{-1}{2}p} f(T,V)|T - V(DD')^{-1}V'|^{\frac{1}{2}(n-q-p-1)}. \end{aligned} \tag{23}$$

Next from (23) it follows that

$$\begin{aligned} &\int_{XX'=T=FF'+V(\mu\Lambda^2\mu')^{-1}V', \mu\Lambda X'=V'} \exp\{-\frac{1}{2} tr(X\Lambda X') + tr(\mu\Lambda X')\}dX \\ &= K \int \exp\{-\frac{1}{2} tr(T) + tr(V)\}|T - V(\mu\Lambda^2\mu')^{-1}V'|^{\frac{1}{2}(n-2p-1)} dV \end{aligned}$$

$$
\begin{aligned}
& {}_1F_1(\frac{1}{2}(n-1);\frac{1}{2}n;\frac{1}{2}((n-1)\lambda_1-\lambda_2-\cdots-\lambda_n)T)) \\
= \; & K\exp\{-\frac{1}{2}tr(T)\}|T|^{\frac{1}{2}(n-p-1)} \\
& {}_1F_1(\frac{1}{2}(n-1);\frac{1}{2}n;\frac{1}{2}((n-1)\lambda_1-\lambda_2-\cdots-\lambda_n)T)) \\
& oF_1(\frac{1}{2}n;\frac{1}{4}\mu\Lambda^2\mu' T), \qquad (24)
\end{aligned}
$$

which is known as (GQNCWD). The integration with respect to V is a known integral in the theory of noncentral Wishart distribution. Here F is $p\times(n-p)$ matrix of rank (n-p), and the integral is first evaluated with respect to F and then with respect to V.

We now proceed with d statistic generalizations. All other results given by [6, Chapter 5] relating to the GQFNCWD can be simply and elegantly rewritten by our methodology.

3. d Statistic Generalizations

We observe from (10) that

$$
\begin{aligned}
E(d^g) &= K\Big(\frac{d}{d\theta}\Big)^g_{\theta=0}\int_{HH'=I}\exp\{tr(\theta H\Lambda H')\}dH \\
&= K\Big(\frac{d}{d\theta}\Big)^g_{\theta=0}{}_1F_1\frac{1}{2}(n-1);\frac{1}{2}n;((n-1)\lambda_1-\cdots-\lambda_n)\theta)). \qquad (25)
\end{aligned}
$$

Mathai et al. [6, p. 302] show that

$$
\Big(\frac{d}{d\theta}\Big)^g{}_1F_1(\alpha;\beta;\theta) = \frac{\Gamma_p(\alpha+g)}{\Gamma_p(\beta+g)}{}_1F_1(\alpha+g;\beta+g;\theta), \qquad (26)
$$

and hence (25) yields

$$
E(d)^g = K((n-1)\lambda_1-\lambda_2-\cdots-\lambda_n)^{pg}\frac{\Gamma_p(\frac{1}{2}(n-1)+g)}{\Gamma_p(\frac{1}{2}n+g)}. \qquad (27)
$$

It follows from (27) that (5) may be written as

$$
\begin{aligned}
E(d)^g &= \Big(\frac{1}{2}\Big)_g C_{(g)}(\Lambda)\Big/\Big(\frac{m}{2}\Big)_g \\
&= \frac{\Gamma(\frac{1}{2}(m-1)g)((m-1)\lambda_1-\cdots-\lambda_m)^g}{\Gamma(\frac{1}{2}m+g)} \qquad (28)
\end{aligned}
$$

Further it follows from (9) that

$$
E(d)^g = \Big(\frac{p}{2}\Big)_g C_{(g)}(\Lambda)\Big/\Big(\frac{pm}{2}\Big)_g
$$

$$= \frac{\Gamma(\frac{1}{2}(m-1)p+g)((m-1)\lambda_1-\cdots-\lambda_m)^g}{\Gamma(\frac{1}{2}mp+g)} \quad (29)$$

Once again we write Kabe's [2] result as

$$\begin{aligned} g(t) &= \int_{y_1+\cdots+y_n=t} \exp\{-(\Lambda_1 y_1+\cdots+\Lambda_n y_n)\} y_1^{g_1-1}\ldots y_n^{g_n-1} dy_1\ldots dy_n \\ &= K\exp\{-t\}t^{g_1+\cdots+g_n-1}{}_1F_1(g_2+\cdots+g_n; g_1+\cdots+g_n; \\ &\quad ((n-1)\lambda_1-\cdots-\lambda_n)t) \end{aligned} \quad (30)$$

and hence

$$\begin{aligned} &\int_{y_1+\cdots+y_n=1} (\lambda_1 y_1+\cdots+\lambda_n y_n)^g y_1^{g_1-1}\ldots y_n^{g_n-1} dy_1\ldots dy_n \\ &= K\frac{\Gamma(g_2+\cdots+g_n+g}{\Gamma(g_1+\cdots+g_n+g)}((n-1)\lambda_1-\cdots-\lambda_n))^g. \end{aligned}$$

Similar to (16) it holds that

$${}_1F_1(a;b;\Delta A){}_1F_1(c;d;\Omega B) = {}_1F_1(a+c;b+d;(\Delta+\Omega)(A+B))$$

and hence

$$\begin{aligned} &\int_{A+B=D} \exp\{-\frac{1}{2}tr(A+B)\}|A|^{\frac{1}{2}(n-p-1)}|B|_1^{\frac{1}{2}(q-b-1)}F_1(\frac{1}{2}(n-1);\frac{1}{2}n;\Lambda A) \\ &{}_1F_1(\frac{1}{2}(q-1);\frac{1}{2}q;\theta B)oF_1(\frac{1}{2}n;\frac{1}{4}\Delta A)oF_1(\frac{1}{2}q;\Omega B)dAdB \\ = \ &K\exp\{\frac{1}{2}tr(D)\}|D|^{\frac{1}{2}(n-q-b-1)}{}_1F_1(\frac{1}{2}(n+q-2);\frac{1}{2}(n+q);(\Delta+\theta)D) \\ &oF_1(\frac{1}{2}(n+q);\frac{1}{4}(\Delta+\Omega)D). \end{aligned}$$

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