Inequalities Involving Certain Integral Operator

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Abstract. The object of this paper is to give several strict inequalities associated with the operator $I_{\lambda p, n}(a, b; c)$, defined by X.-L. Fu and M.-S. Liu, Some subclasses of analytic functions involving the generalized Noor integral operator [see 3].

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1. Introduction

Let $\mathcal{A}_n(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p}z^{k+p} \quad (p, n \in \mathbb{N} = \{1, 2, \ldots\}),$$

which are analytic and $p$-valent in the open unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$. For functions $f$ given by (1) and $g \in \mathcal{A}_n(p)$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p}z^{k+p} \quad (z \in U)$$

the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f \ast g)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p}b_{k+p}z^{k+p} = (g \ast f)(z).$$

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For real or complex numbers $a, b, c$ other than $0, -1, -2, \ldots$, the Gaussian hypergeometric series is defined by
\[
_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(1)_k} z^k, \tag{3}
\]
where
\[
(d)_k = \begin{cases} 1 & (k = 0; d \in \mathbb{C} \setminus \{0\}), \\ d(d+1) \ldots (d+k-1) & (k \in \mathbb{N}; d \in \mathbb{C}), \end{cases}
\]
we note that the series (3) converges absolutely for all $z \in U$ so that it represents an analytic function in $U$ (see [8]).

With the aid of the Gaussian hypergeometric function $_2F_1(a, b; c; z)$, let us consider a family of linear operators $I_{p,n}^{\lambda} : \mathcal{A}_n(p) \to \mathcal{A}_n(p)$ as follows:
\[
I_{p,n}^{\lambda}(a, b; c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(c)_k(\lambda+p)_k}{(a)_k(b)_k} a_{k+p} z^{k+p} = z^p _2F_1(c, 1; \lambda; z) * z^p _2F_1(\lambda+p, 1; b; z) \quad (a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -p; z \in U). \tag{4}
\]

The operator $I_{p,n}^{\lambda}$ was introduced and studied by Fu and Liu [3].

We note that:

(i) $I_{1,1}^{n}(a, n+1; a)f(z) = I_n f(z) \quad (n > -1)$, where $I_n$ is the Noor integral operator of $n \text{-th}$ order (see [6]);

(ii) $I_{1,1}^{\lambda}(\mu + 2, 1; 1)f(z) = I_{\mu,\lambda} f(z) \quad (\mu > -2, \lambda > -1)$, where $I_{\mu,\lambda}$ is the Choi–Saigo–Srivastava operator [see 2];

(iii) $I_{p,1}^{\lambda}(\lambda + p + 1, b; b)f(z) = F_{\lambda,p}(f)(z) \quad (\lambda > -p)$, where $F_{\lambda,p}(f)(z)$ is the generalized Bernardi–Libera–Livingston operator [see 2];

(iv) $I_{p,1}^{\lambda}(a, 1; c)f(z) = I_{p}^{\lambda}(a, c)f(z) \quad (a, c \in \mathbb{R} \setminus \mathbb{Z}^-_0, \lambda > -p)$, where $I_{p}^{\lambda}(a, c)$ is the Cho–Kwon–Srivastava operator [see 1];

(v) $I_{p,1}^{\lambda}(n + p, c; c)f(z) = I_{n,p} f(z) \quad (n > -p)$, where $I_{n,p}$ is the Noor integral operator of $(n + p - 1) \text{-th}$ order (see Liu and Noor [4] and Patel and Cho [7]).

Also it is easy to show that [see 3]:
\[
I_{p,n}^{\lambda}(a, \lambda + p; a)f(z) = I_{p,n}^{1}(p + 1, b; b)f(z) = f(z) \quad \text{and} \quad I_{p,n}^{1}(a, p; a)f(z) = \frac{zf''(z)}{p},
\]
\[
z \left( I_{p,n}^{\lambda}(a, b; c)f(z) \right)' = (\lambda + p)I_{p,n}^{\lambda+1}(a, b; c)f(z) - \lambda I_{p,n}^{\lambda}(a, b; c)f(z) \tag{5}
\]
and
\[
z \left( I_{p,n}^{\lambda}(a + 1, b; c)f(z) \right)' = aI_{p,n}^{\lambda}(a, b; c)f(z) - (a - p)I_{p,n}^{\lambda}(a + 1, b; c)f(z). \tag{6}
\]

By using the operator $I_{p,n}^{\lambda}(a, b; c)$, we define the following classes of functions:
Definition 1. Let $\Phi$ be the set of complex-valued functions $\varphi(r, s, t)$,

$$
\varphi(r, s, t) : \mathbb{C}^3 \rightarrow \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane})
$$
such that

1. $\varphi(r, s, t)$ is continuous in a domain $D \subset \mathbb{C}^3$;
2. $(0, 0, 0) \in D$ and $|\varphi(0, 0, 0)| < 1$;
3. $\left| \varphi \left( e^{i\theta}, f(\lambda, \zeta, \theta, p), g(\lambda, \zeta, \theta, p, M) \right) \right| > 1$

whenever

$$
\left( e^{i\theta}, f(\lambda, \zeta, \theta, p), g(\lambda, \zeta, \theta, p, M) \right) \in D,
$$
with $\Re \{e^{-i\theta} M\} \geq \zeta(\zeta - 1)$, for all $\theta \in \mathbb{R}$, and for all $\zeta \geq p \geq 1$, where

$$
f(\lambda, \zeta, \theta, p) = \left( \frac{\zeta + \lambda}{\lambda + p} \right) e^{i\theta}
$$

and

$$
g(\lambda, \zeta, \theta, p, M) = \frac{(\lambda + 1)(\lambda + 2\zeta)e^{i\theta} + M}{(\lambda + p)(\lambda + p + 1)}.
$$

Definition 2. Let $\mathcal{H}$ be the set of complex-valued functions $h(r, s, t)$;

$$
h(r, s, t) : \mathbb{C}^3 \rightarrow \mathbb{C}
$$
such that

1. $h(r, s, t)$ is continuous in a domain $D \subset \mathbb{C}^3$;
2. $(1, 1, 1) \in D$ and $|g(1, 1, 1)| < J (J > 1)$;
3. $\left| h \left( J e^{i\theta}, f(\lambda, \zeta, \theta, p, J), g(\lambda, \zeta, \theta, p, J) \right) \right| \geq J$

whenever

$$
\left( J e^{i\theta}, f(\lambda, \zeta, \theta, p, J), g(\lambda, \zeta, \theta, p, J, L) \right) \in D,
$$
with $\Re \{L\} \geq \zeta(\zeta - 1)$ for all $\theta \in \mathbb{R}$ and for all $\zeta \geq \frac{J - 1}{J + 1}$, where

$$
f(\lambda, \zeta, \theta, p, J) = \frac{1 + \zeta + (\lambda + p - 1)J e^{i\theta}}{(\lambda + p)}
$$

and

$$
g(\lambda, \zeta, \theta, p, J, L) = \frac{1}{(\lambda + p + 1)} \left\{ 2 + \zeta + (\lambda + p - 1)J e^{i\theta} + \frac{\zeta - \zeta^2 + (\lambda + p + 1)J e^{i\theta} + L}{\zeta + (\lambda + p - 1)J e^{i\theta}} \right\}.
$$
2. Main Results

We recall the following lemma due to Miller and Mocanu [5].

**Lemma 1.** Let \( w(z) = a + bw_0z + \ldots \) be regular in \( U \) with \( v \in \mathbb{N} \). If \( z_0 = r_0e^{i\theta} \) \( (0 < r_0 < 1) \) and \( |w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| \), then

\[
z_0w'(z_0) = \zeta w(z_0)
\]

(7)

and

\[
\Re \left( 1 + \frac{z_0w''(z_0)}{w'(z_0)} \right) \geq \zeta,
\]

(8)

where \( \zeta \) is a real number and

\[
\zeta \geq \nu \left| \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \right| \geq \nu \left| \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|} \right|.
\]

(9)

Note that if \( \nu = 0 \) then the condition (9) becomes \( \zeta \geq \nu \geq 1 \).

**Theorem 1.** Let \( \varphi(r, s, t) \in \Phi \) and let \( f \) in the class \( \mathcal{A}_n(p) \) satisfy

\[
\left( I_{p,n}^\lambda(a, b; c)f(z), I_{p,n}^{\lambda+1}(a, b; c)f(z), I_{p,n}^{\lambda+2}(a, b; c)f(z) \right) \in D \subseteq \mathbb{C}^3
\]

(10)

and

\[
\left| \varphi \left( I_{p,n}^\lambda(a, b; c)f(z), I_{p,n}^{\lambda+1}(a, b; c)f(z), I_{p,n}^{\lambda+2}(a, b; c)f(z) \right) \right| < 1
\]

(11)

for \( a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p, p \in \mathbb{N} \) and \( z \in U \). Then we have

\[
\left| I_{p,n}^\lambda(a, b; c)f(z) \right| < 1 \quad (z \in U).
\]

(12)

**Proof.** We define the function \( w \) by

\[
w(z) = I_{p,n}^\lambda(a, b; c)f(z) \quad \left( a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p; p \in \mathbb{N} \right)
\]

(13)

for \( f \) belonging to the class \( \mathcal{A}_n(p) \). Then, it follows that \( w \in \mathcal{A}_n(p) \) and \( w(z) \neq 0 \) for \( z \in U \setminus \{0\} \). With the aid of (5), we have

\[
I_{p,n}^{\lambda+1}(a, b; c)f(z) = \frac{1}{(\lambda + p)} [zw'(z) + \lambda w(z)]
\]

(14)

and

\[
I_{p,n}^{\lambda+2}(a, b; c)f(z) = \frac{z^2w'(z) + 2(\lambda + 1)zw'(z) + \lambda(\lambda + 1)w(z)}{(\lambda + p)(\lambda + p + 1)}
\]

(15)

Suppose that \( z_0 = r_0e^{i\theta} \) \( (0 < r_0 < 1; \theta \in \mathbb{R}) \) and

\[
w(z_0) = \max_{|z| \leq |z_0|} |w(z)| = 1.
\]

(16)
Then, letting \( w(z_0) = e^{i\theta} \) and using (7) of Lemma 1, we obtain
\[
I^\lambda_{p,n}(a, b; c)f(z_0) = e^{i\theta}, \quad (17)
\]
\[
I^{\lambda+1}_{p,n}(a, b; c)f(z_0) = \frac{1}{\lambda + p} \left[ z_0w'(z_0) + \lambda w(z_0) \right] = \frac{(\zeta + \lambda)}{\lambda + p} e^{i\theta} \quad (18)
\]
and
\[
I^{\lambda+2}_{p,n}(a, b; c)f(z_0) = \frac{1}{\lambda + p}(\lambda + p + 1) \left[ (\lambda + 1)(\lambda + 2\zeta)e^{i\theta} + z_0^2w'(z_0) \right] \\
= \frac{(\lambda + 1)(\lambda + 2\zeta)e^{i\theta} + M}{(\lambda + p)(\lambda + p + 1)}, \quad (19)
\]
where \( M = z_0^2w''(z_0) \) and \( \zeta \geq p \geq 1 \). Further, an application of (8) in Lemma 1, gives
\[
\Re \left\{ \frac{z_0w''(z_0)}{w'(z_0)} \right\} = \Re \left\{ \frac{z_0^2w''(z_0)}{\zeta e^{i\theta}} \right\} \geq \zeta - 1, \quad (20)
\]
\[
\Re \left\{ e^{-i\theta}M \right\} \geq \zeta(\zeta - 1) \quad (\theta \in \mathbb{R}; \zeta \geq 1). \quad (21)
\]
Since \( \varphi(r, s, t) \in \Phi \), we also have
\[
\left| \varphi \left( I^\lambda_{p,n}(a, b; c)f(z), I^{\lambda+1}_{p,n}(a, b; c)f(z), I^{\lambda+2}_{p,n}(a, b; c)f(z) \right) \right| = \left| \varphi \left( e^{i\theta}, \frac{\zeta + \lambda}{\lambda + p} e^{i\theta}, \frac{1}{\lambda + p}(\lambda + p + 1) \left[ (\lambda + 1)(\lambda + 2\zeta)e^{i\theta} + M \right] \right) \right| > 1 \quad (22)
\]
which contradicts the condition (11) of Theorem 1. Therefore, we conclude that
\[
|w(z)| = \left| I^\lambda_{p,n}(a, b; c)f(z) \right| < 1 \quad (z \in U). \quad (23)
\]
This completes the proof of Theorem 1.

**Corollary 1.** Let \( \varphi_1(r, s, t) = s \) and let \( f \in \mathcal{A}(p) \) satisfy the conditions in Theorem 1 for \( a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p, p \in \mathbb{N} \) and \( z \in U \). Then
\[
|I^{\lambda+i}_{p,n}(a, b; c)f(z)| < 1 \quad (i = 0, 1, 2, \ldots; a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -p; p \in \mathbb{N}; z \in U).
\]

Note that \( \varphi_1(r, s, t) = s \) is in \( \Phi \), with the aid of Theorem 1, we have
\[
|I^\lambda_{p,n}(a, b; c)f(z)| < 1 \Rightarrow |I^{\lambda+1}_{p,n}(a, b; c)f(z)| < 1 \\
\Rightarrow |I^{\lambda+i}_{p,n}(a, b; c)f(z)| < 1 \quad (i = 0, 1, 2, \ldots; a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -p; p \in \mathbb{N}; z \in U).
Theorem 2. Let $h(r, s, t) \in \mathcal{H}$, and let $f \in \mathcal{A}_n(p)$ satisfy
\[
\left( \frac{I_{p,n}^\lambda(a, b; c)f(z)}{I_{p,n}^{\lambda-1}(a, b; c)f(z)} \right) \left( \frac{I_{p,n}^{\lambda+1}(a, b; c)f(z)}{I_{p,n}^\lambda(a, b; c)f(z)} \right) \left( \frac{I_{p,n}^{\lambda+2}(a, b; c)f(z)}{I_{p,n}^{\lambda+1}(a, b; c)f(z)} \right) \in D \subset \mathbb{C}^3
\] (24)
and
\[
\left| h \left( \frac{I_{p,n}^\lambda(a, b; c)f(z)}{I_{p,n}^{\lambda-1}(a, b; c)f(z)} \right) \left( \frac{I_{p,n}^{\lambda+1}(a, b; c)f(z)}{I_{p,n}^\lambda(a, b; c)f(z)} \right) \left( \frac{I_{p,n}^{\lambda+2}(a, b; c)f(z)}{I_{p,n}^{\lambda+1}(a, b; c)f(z)} \right) \right| < J
\] (25)
for some $a, b, c, \lambda, p, n, J \left(a, b, c \in \mathbb{R} \backslash \mathbb{Z}; \lambda > 1; p, n \in \mathbb{N}; J > 1\right)$ and for all $z \in U$. Then we have
\[
\left| \frac{I_{p,n}^\lambda(a, b; c)f(z)}{I_{p,n}^{\lambda-1}(a, b; c)f(z)} \right| < J \quad (z \in U).
\] (26)

Proof: We define the function $w$ by
\[
w(z) = \frac{I_{p,n}^\lambda(a, b; c)f(z)}{I_{p,n}^{\lambda-1}(a, b; c)f(z)}
\] (27)
for $f$ belonging to the class $\mathcal{A}_n(p)$. Then, it follows that $w$ is either analytic or meromorphic in $U$, $w(0) = 1$, and $w(z) \neq 1$. With the aid of the identity (5), we have
\[
\frac{I_{p,n}^{\lambda+1}(a, b; c)f(z)}{I_{p,n}^\lambda(a, b; c)f(z)} = \frac{1}{\lambda + p} \left[ 1 + (\lambda + p - 1)w(z) + \frac{zw'(z)}{w(z)} \right]
\] (28)
and
\[
\frac{I_{p,n}^{\lambda+2}(a, b; c)f(z)}{I_{p,n}^{\lambda+1}(a, b; c)f(z)} = \frac{1}{\lambda + p + 1} \left\{ 2 + (\lambda + p - 1)w(z) + \frac{zw'(z)}{w(z)} + \frac{(\lambda + p - 1)zw'(z) + zw'(z)}{w(z)} \right. \\
\left. + \frac{(\lambda + p - 1)zw'(z) + z^2w'(z)}{w(z)} \right\}
\] (29)

Suppose that $z_0 = r_0e^{i\theta} (0 < r_0 < 1; \theta \in \mathbb{R})$ and $|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| = J$. Letting $w(z_0) = Je^{i\theta}$ and using Lemma 1 with $a = \nu = 1$, we see that
\[
\frac{I_{p,n}^{\lambda+1}(a, b; c)f(z_0)}{I_{p,n}^\lambda(a, b; c)f(z_0)} = \frac{1}{\lambda + p} \left[ 1 + \zeta + (\lambda + p - 1)Je^{i\theta} \right]
\] (30)
and
\[
\frac{I_{p,n}^{\lambda+2}(a, b; c)f(z)}{I_{p,n}^{\lambda+1}(a, b; c)f(z)} = \frac{1}{\lambda + p + 1} \left\{ 2 + \zeta + (\lambda + p - 1)Je^{i\theta} + \frac{\zeta - \zeta^2 + (\lambda + p - 1)Je^{i\theta} + 1}{\zeta + (\lambda + p - 1)Je^{i\theta}} \right\},
\] (31)
where \( L = \frac{z_0^2 w''(z_0)}{w'(z_0)} \) and \( \zeta \geq \frac{J - 1}{J + 1} \). Further, an application of (16) in Lemma 1 gives

\[ \Re \{ L \} \geq \zeta (\zeta - 1). \]

Since \( h(r, s, t) \in \mathcal{H} \), we have

\[
\begin{align*}
&\left| h \left( J e^{i\theta}, \frac{1 + \zeta + (\lambda + p - 1) e^{i\theta}}{(\lambda + p)} + \frac{1}{(\lambda + p + 1)} \left\{ 2 + \zeta + (\lambda + p - 1) e^{i\theta} \right\} \right) \right| \geq J,
&\text{which contradicts condition (25) of Theorem 2. Therefore, we conclude that}
&\left| w(z) \right| = \left| \frac{I^\lambda_{p, n}(a, b; c) f(z)}{I^{-\lambda-1}_{p, n}(a, b; c) f(z)} \right| < J
\end{align*}
\]

for some \( a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^{-} \), \( \lambda > 1 \), \( p, n \in \mathbb{N} \), \( J > 1 \) and for all \( z \in U \). This completes the proof of Theorem 2.

References


