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Coincidence & Fixed Points Of Nonexpansive Type Multi-Valued & Single Valued Maps

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Abstract. Fixed point theory of nonexpansive and nonexpansive type single and multivalued mappings provides techniques for solving a variety of applied problems in mathematical sciences and engineering. In this paper we consider the existence of coincidences and fixed points of nonexpansive type conditions satisfied by multivalued and single valued maps and prove some fixed point theorems for nonexpansive type single and multivalued mappings.

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1. Introduction & Preliminaries

Throughout this paper let (X,d) be a metric space and H denotes the Housdorff (resp. generalized Housdorff) metric on $CB(X)$ (resp. $CL(X)$) induced by the metric d, where $CB(X)$ (resp. $CL(X)$) is the collection of all nonempty closed and bounded (resp. closed), subsets of X. For these definitions one may refer [1, 3, 6, 7]. For $y \in X$ and $A \subset X$, $d\left(y,A\right)$ will denote the ordinary distance between y and A.
A map $T : X \to X$ is said to be nonexpansive if $d\left(Tx,Ty\right) \leq d\left(x,y\right)$ for all $x,y \in X$. Ćirić [4] investigated a class of self maps T of X which satisfy the following nonexpansive type condition:

$$d(Tx,Ty) \leq a\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\} + b\max\{d(x,Tx),d(y,Ty)\}+c[d(x,Ty)+d(y,Tx)] \tag{1}$$

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For all $x, y \in X$, where $a, b, c \geq 0$ such that $a + b + 2c = 1$.

M. Chandra et al [2] consider the following generalization of (1), let $T, f : X \to X$ satisfying:

$$\begin{aligned} d(Tx, Ty) \leq\ & a(x, y)d(fx, fy) + b(x, y)\max\{d(fx, Tx), d(fy, Ty)\} \\ & + c(x, y)[d(fx, Ty) + d(fy, Tx)] \end{aligned} \tag{2}$$

where, $a(x, y) \geq 0$, $\beta = \inf_{x,y\in X} b(x, y) > 0$, $\gamma = \inf_{x,y\in X} c(x, y) > 0$ and $\sup_{x,y\in X}[a(x, y) + b(x, y) + 2c(x, y)] = 1$ and prove some fixed point theorems for single valued and multi valued maps. They also prove that (1) contained in (2).

In this paper we use the following nonexpansive type condition: Let $T, f : X \to X$ be two self mappings satisfying the condition,

$$\begin{aligned} d(Tx, Ty) \leq\ & a(x, y)d(fx, fy) + b(x, y)\max\{d(fx, Tx), d(fy, Ty)\} \\ & + c(x, y)\max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\} \\ & + e(x, y)\max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty)\} \end{aligned} \tag{3}$$

Where $a(x, y), b(x, y), c(x, y), e(x, y) \geq 0$ and $\beta = \inf_{x,y\in X} e(x, y) > 0$,
$\gamma = \inf_{x,y\in X}(1 + b(x, y) + e(x, y)) > 0$ with $\sup_{x,y\in X}(a(x, y) + b(x, y) + c(x, y) + 2e(x, y)) = 1$.

Definition 1 ([5]). *Let f and g be two self maps of a metric space X. Then f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.*

2. Main Results

Theorem 1. *Let (X, d) be a metric space, T, f are self maps of X satisfying (3) with $T(X) \subseteq f(X)$ and either (a) X is complete and f is surjective; or (b) X is complete, f is continuous and T, f are compatible; or (c) $f(X)$ is complete ; or (d) $T(X)$ is complete. Then f and T have a coincidence point in X. Further, the coincidence value is unique, i.e. $fp = fq$ whenever $fp = Tp$ and $fq = Tq$ $(p, q \in X)$.*

Proof. Let $x_0 \in X$. Since $T(X) \subseteq f(X)$, choose x_1 so that $y_1 = fx_1 = Tx_0$. In general, choose x_{n+1} such that $y_{n+1} = fx_{n+1} = Tx_n$.

From (3), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) \leq\ & ad(fx_n, fx_{n+1}) + b\max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\ & + c\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\ & + e\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_{n+1})\} \\ \leq\ & ad(fx_n, Tx_n) + b\max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\ & + c\max\{d(fx_n, Tx_n), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\ & + e\max\{d(fx_n, Tx_n), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_n) + d(fx_{n+1}, Tx_{n+1})\} \end{aligned}$$

where a, b, c and e are evaluated at (x_n, x_{n+1}). Suppose that for some n, $d(fx_{n+1}, Tx_{n+1}) > d(fx_n, Tx_n)$. Then substituting in the above inequality we have

$$d(fx_{n+1}, Tx_{n+1}) < (a+b+c+2e)d(fx_{n+1}, Tx_{n+1})$$

a contradiction. Therefore, for all n we have

$$d(fx_{n+1}, Tx_{n+1}) \leq d(fx_n, Tx_n) \tag{4}$$

Again $d(y_{n-1}, Tx_n) = d(Tx_{n-2}, Tx_n)$
Using (3), (4) and triangle inequality we have

$$\begin{aligned} d(y_{n-1}, Tx_n) \leq\ & ad(fx_{n-2}, fx_n) + b\max\{d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)\} \\ & + c\max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)\} \\ & + e\max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n), d(fx_{n-2}, Tx_n)\} \\ \leq\ & 2ad(fx_{n-2}, Tx_{n-2}) + bd(fx_{n-2}, Tx_{n-2}) + 2cd(fx_{n-2}, Tx_{n-2}) \\ & + e\max\{2d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_n)\} \\ \leq\ & (2a+b+2c+3e)d(fx_{n-2}, Tx_{n-2}) \end{aligned}$$

implies that

$$d(y_{n-1}, Tx_n) \leq (1-b-e)d(fx_{n-2}, Tx_{n-2}) \tag{5}$$

Using (3), (4) and (5) we obtain,

$$\begin{aligned} d(y_n, Tx_n) &= d(Tx_{n-1}, Tx_n) \\ &\leq ad(fx_{n-1}, fx_n) + b\max\{d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\} \\ &\quad + c\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\} \\ &\quad + e\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), d(fx_{n-1}, Tx_n)\} \\ &\leq ad(fx_{n-2}, Tx_{n-2}) + bd(fx_{n-2}, Tx_{n-2}) + cd(fx_{n-2}, Tx_{n-2}) \\ &\quad + e(1-b-e)d(fx_{n-2}, Tx_{n-2}) \\ &\leq (1-e(1+b+e))d(fx_{n-2}, Tx_{n-2}) \\ &\leq (1-\beta\gamma)d(fx_{n-2}, Tx_{n-2}) \\ &\leq (1-\beta\gamma)^{n/2}d(y_0, y_1) \end{aligned}$$

where $\beta = \inf_{x,y\in X} e(x,y) > 0$, $\gamma = \inf_{x,y\in X}(1+b(x,y)+e(x,y)) > 0$ and $\{y_n\}$ is Cauchy, hence converges to a point p in X.

Case (a):- Suppose that f is surjective. Then there exists a point z in X such that $p = fz$.

From (3), we have

$$
\begin{aligned}
d(fz,Tz) &\le d(fz,y_{n+1})+d(y_{n+1},Tz)\\
&\le d(fz,y_{n+1})+ad(fx_n,fz)+b\max\{d(fx_n,Tx_n),d(fz,Tz)\}\\
&\quad +c\max\{(fx_n,fz),d(fx_n,Tx_n),d(fz,Tz)\}\\
&\quad +e\max\{d(fx_n,fz),d(fx_n,Tx_n),d(fz,Tz),d(fx_n,Tz)\}\\
&\le d(fz,fx_{n+1})+\sup_{x,y\in X}(b+c+e)\max\{\max\{d(fx_n,Tx_n),d(fz,Tz)\}\\
&\quad \max\{d(fx_n,fz),d(fx_n,Tx_n),d(fz,Tz)\},\\
&\quad \max\{d(fx_n,fz),d(fx_n,Tx_n),d(fz,Tz),d(fx_n,Tz)\}\}\\
&\quad +\sup_{x,y\in X} ad(fx_n,fz)
\end{aligned}
$$

Taking limit as $n\to\infty$, we get $d(fz,Tz)\le \sup_{x,y\in X}(b+c+e)d(fz,Tz)$ implies that $fz=Tz$.

Case (b): Suppose f is continuous and f and T are compatible. Then since $\lim_{n\to\infty} y_n=p$, we have $\lim_{n\to\infty} fy_n=fp$.
Now,

$$
\begin{aligned}
d(fp,Tp) &\le d(fp,fy_{n+1})+d(fy_{n+1},Tp)\\
&\le d(fp,fy_{n+1})+d(fTx_n,Tfx_n)+d(Tfx_n,Tp)
\end{aligned}
$$

Note that since $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} Tx_n$ and f,T are compatible, $\lim_{n\to\infty} d\left(fTx_n,Tfx_n\right)=0$.
From (3),we have

$$
\begin{aligned}
d(Tfx_n,Tp) &\le ad(ffx_n,fp)+b\max\{d(ffx_n,Tfx_n),d(fp,Tp)\}\\
&\quad +c\max\{d(ffx_n,fp),d(ffx_n,Tfx_n),d(fp,Tp)\}\\
&\quad +e\max\{d(ffx_n,fp),d(ffx_n,Tfx_n),d(fp,Tp),d(ffx_n,Tp)\}\\
&\le \sup_{x,y\in X} a(x,y)d(ffx_n,fp)+\sup_{x,y\in X}(b(x,y)+c(x,y)+e(x,y))\\
&\quad \max\{\max\{d(ffx_n,Tfx_n),d(fp,Tp)\},\\
&\quad \max\{d(ffx_n,fp),d(ffx_n,Tfx_n),d(fp,Tp)\},\\
&\quad \max\{d(ffx_n,fp),d(ffx_n,Tfx_n),d(fp,Tp),d(ffx_n,Tp)\}\}
\end{aligned}
$$

Note that $d(ffx_n,Tfx_n)\le d(ffx_n,fTx_n)+d(fTx_n,Tfx_n)$. Using the continuity of f and compatibility of f and T, it follows that $\lim d(ffx_n,Tfx_n)=0$. Since $\lim ffx_n=fp$, it follows that $\lim Tfx_n=fp$.

Substituting into the above inequality and taking limit as $n\to\infty$, we get $d(fp,Tp)\le \sup_{x,y\in X}(b(x,y)+c(x,y)+e(x,y))d(fp,Tp)$ implies that $fp=Tp$.

Case (c): In this case $p\in f(X)$. Let $z\in f^{-1}p$. Then $p=fz$ and the proof is complete by case (a).

Case (d): In this case $p \in T(X) \subseteq f(X)$ and the proof is complete by case (c).

Uniqueness: Let q be another coincidence point of f and T, then from (3) with a, b, c and d evaluated at (p, q),

$$\begin{aligned} d(Tp, Tq) &\leq ad(fp, fq) + b\max\{d(fp, Tp), d(fq, Tq)\} \\ &\quad + c\max\{d(fp, fq), d(fp, Tp), d(fq, Tq)\} \\ &\quad + e\max\{d(fp, fq), d(fp, Tp), d(fq, Tq), d(fp, Tq)\} \\ &\leq (a + c + e)d(Tp, Tq) \end{aligned}$$

This implies that $Tp = Tq$ and hence $fp = fq$.

Corollary 1. *Let (X, d) be a complete metric space and T a self mapping of X satisfying (3) with $f = I$, the identity map on X and $\sup_{x,y \in X}(a(x, y) + 2b(x, y) + c(x, y) + e(x, y)) = 1$. Then T has a unique fixed point and at this fixed point T is continuous.*

Proof. The existence and uniqueness of the fixed point comes from Theorem 1 by setting $f = I$. To prove continuity , let $\{y_n\} \subset X$ with $\lim y_n = p$, p the unique fixed point of T.

Using (3), we have

$$\begin{aligned} d(Ty_n, Tp) &\leq ad(y_n, p) + b\max\{d(y_n, Ty_n), d(p, Tp)\} \\ &\quad + c\max\{d(y_n, p), d(y_n, Ty_n), d(p, Tp)\} \\ &\quad + e\max\{d(y_n, p), d(y_n, Ty_n), d(p, Tp), d(y_n, Tp)\} \\ &\leq ad(y_n, p) + b\max\{d(y_n, Ty_n), d(p, p\} \\ &\quad + c\max\{d(y_n, p), d(y_n, Ty_n), d(p, p)\} \\ &\quad + e\max\{d(y_n, p), d(y_n, Ty_n), d(p, p), d(y_n, p)\} \\ &\leq ad(y_n, p) + b[d(y_n, p) + d(p, Ty_n)] \\ &\quad + c\max\{d(y_n, p), d(y_n, p) + d(p, Ty_n)\} \\ &\quad + e\max\{d(y_n, p), d(y_n, p) + d(p, Ty_n)\} \end{aligned}$$

Hence

$$\begin{aligned} d(Ty_n, p) &\leq \sup_{x,y \in X}(a + b)d(y_n, p) + \sup_{x,y \in X}(b + c + e) \\ &\quad \max\{d(p, Ty_n), \max\{d(y_n, p), d(y_n, p) + d(p, Ty_n)\}\} \\ &\leq \sup_{x,y \in X}(\frac{b + c + e}{1 - a - b})d(y_n, p) \end{aligned}$$

Taking limit as $n \to \infty$ we get $\lim Ty_n = p = Tp$.

Next we establish some results when T is a multi-valued map from a metric space X to the collection of nonempty subset of X, and f is a self map of X. Let $C(X)$ denote the collection of all nonempty compact subset of X.

Definition 2 ([3]). *An orbit of the multi-valued map T at a point x_0 in X is a sequence* $\{x_n : x_n \in Tx_{n-1}\}$. *A space X is $T-$orbitally complete if every Cauchy sequence of the form* $\{x_{n_i} : x_{n_i} \in Tx_{n_i-1}\}$ *converges in X.*

Definition 3 ([7]). *If for a point x_0 in X, there exists a sequence $\{x_n\} \subset X$ such that $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \cdots$, then $O_f(x_0) = \{fx_n : n = 1, 2, \cdots\}$ is an orbit of (T, f) at x_0. A space X is called $(T, f)-$orbitally complete if every Cauchy sequence of the form* $\{fx_{n_i} : fx_{n_i} \in Tx_{n_i-1}\}$ *converges in X.*

Theorem 2. *Let X be a metric space, T a multi-valued map from X to $C(X)$. Let f be a self map of X such that $T(X) \subseteq f(X)$ and either of the following conditions is satisfied:*

1. *X is $(T, f)-$orbitally complete and f is surjective;*
2. *$f(X)$ is $(T, f)-$orbitally complete;*
3. *$T(X)$ is $(T, f)-$orbitally complete.*

Suppose that T and f satisfy the condition:

$$\begin{aligned} H(Tx, Ty) \le a(x, y)d(fx, fy) &+ b(x, y)\max\{d(fx, Tx), d(fy, Ty)\} \\ &+ c(x, y)\max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\} \\ &+ e(x, y)\max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty)\} \end{aligned} \tag{6}$$

Where a, b, c and e are nonnegative functions from $X \times X \to [0, 1)$ such that $\inf_{x,y\in X} e(x, y) > 0$, $\inf_{x,y\in X}(1 + b(x, y) + e(x, y)) > 0$ *and* $\sup_{x,y\in X}(a + b + c + 2e)(x, y) = 1$. *Then f and T have a coincidence, i.e. there exists a point z in X such that $fz \in Tz$.*

Proof. Choose $x_0 \in X$. We construct sequences $\{x_n\}$ and $\{y_n\}$ as follows: Since $T(X) \subseteq f(X)$, we can choose $y_1 = fx_1 \in Tx_0$. If $Tx_0 = Tx_1$, Choose $y_2 = fx_2 \in Tx_1$ such that $y_1 = y_2$. If $Tx_0 \neq Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that $d(y_1, y_2) \le H(Tx_0, Tx_1)$. Such a choice is possible since Tx is compact for each x in X. In general, choose $y_{n+2} = fx_{n+2} \in Tx_{n+1}$ such that $y_{n+1} = y_{n+2}$ if $Tx_n = Tx_{n+1}$ and $d(y_{n+1}, y_{n+2}) \le H(Tx_n, Tx_{n+1})$ otherwise. From (6) with a, b, c and e evaluated at (x_n, x_{n+1}),

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\le H(Tx_n, Tx_{n+1}) \\ &\le ad(fx_n, fx_{n+1}) + b\max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\ &\quad + c\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\ &\quad + e\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_{n+1})\} \\ &\le ad(y_n, y_{n+1}) + b\max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\} \\ &\quad + c\max\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\} \\ &\quad + e\max\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\} \end{aligned}$$

If for some n, $d(y_{n+1}, y_{n+2}) > d(y_n, y_{n+1})$, the above inequality gives

$$d(y_{n+1}, y_{n+2}) < (a + b + c + 2e)d(y_{n+1}, y_{n+2})$$

a contradiction. Therefore, for each n, we have

$$d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}) \tag{7}$$

Again from (6) with a, b, c and e evaluated at (x_{n-2}, x_n)

$$\begin{aligned} d(y_{n-1}, Tx_n) \leq\ & H(Tx_{n-2}, Tx_n) \\ & \leq ad(fx_{n-2}, fx_n) + b\max\{d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)\} \\ & \quad + c\max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n)\} \\ & \quad + e\max\{d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n), d(fx_{n-2}, Tx_n)\} \end{aligned}$$

Using (7) and triangle inequality we get

$$\begin{aligned} d(y_{n-1}, Tx_n) \leq\ & 2ad(y_{n-2}, y_{n-1}) + bd(y_{n-2}, y_{n-1}) + 2cd(y_{n-2}, y_{n-1}) \\ & + e\max\{2d(y_{n-2}, y_{n-1}), d(y_{n-2}, y_{n-1}) + d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n)\} \\ & + (2a + b + 2c)d(y_{n-2}, y_{n-1}) + e\max\{2d(y_{n-2}, y_{n-1}), 3d(y_{n-2}, y_{n-1})\} \\ & + (2a + b + 2c + 2e)d(y_{n-2}, y_{n-1}) \end{aligned}$$

Implies that

$$d(y_{n-1}, Tx_n) \leq (1 - b - e)d(y_{n-1}, y_{n-2}) \tag{8}$$

Again from (6), (7) and using (8)

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ & \leq H(Tx_{n-1}, Tx_n) \\ & \leq ad(fx_{n-1}, fx_n) + b\max\{d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\} \\ & \quad + c\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\} \\ & \quad + e\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), d(fx_{n-1}, Tx_n)\} \\ & \leq ad(fx_{n-2}, Tx_{n-2}) + bd(fx_{n-2}, Tx_{n-2}) + cd(fx_{n-2}, Tx_{n-2}) \\ & \quad + e(1 - b - e)d(fx_{n-2}, Tx_{n-2}) \\ & \leq [1 - e(1 + b + e)]d(fx_{n-2}, Tx_{n-2}) \\ & \leq (1 - \beta\gamma)d(y_{n-2}, y_{n-1}) \\ & \leq (1 - \beta\gamma)^{n/2} d(y_0, y_1) \end{aligned}$$

and $\{y_n\}$ is Cauchy, hence convergent to a point p in X in Cases (i)-(iii).

If f is surjective, there exists a point z such that $p = fz$. This is obviously true in cases (ii) and (iii) as well,

$$\begin{aligned}
d(fz, Tz) &\leq d(fz, fx_{n+1}) + d(fx_{n+1}, Tz)\\
&\leq d(fz, fx_{n+1}) + H(Tx_n, Tz)\\
&\leq d(fz, fx_{n+1}) + ad(fx_n, fz)\\
&\quad + b\max\{d(fx_n, Tx_n), d(fz, Tz)\}\\
&\quad + c\max\{d(fx_n, fz, d(fx_n, Tx_n), d(fz, Tz)\}\\
&\quad + e\max\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz), d(fx_n, Tz)\}\\
&\leq d(fz, fx_{n+1}) + \sup_{x,y\in X} ad(fx_n, fz)\\
&\quad + \sup_{x,y\in X}(b+c+e)\max\{\max\{d(fx_n, Tx_n), d(fz, Tz)\},\\
&\quad \max\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz)\},\\
&\quad \max\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz), d(fx_n, Tz)\}\}
\end{aligned}$$

Taking limit as $n \to \infty$

$$\begin{aligned}
d(fz, Tz) &\leq d(fz, fz) + \sup_{x,y\in X} ad(fz, fz)\\
&\quad + \sup_{x,y\in X}(b+c+e)\max\{\max\{d(fz, Tz), d(fz, Tz)\}\\
&\quad ,\max\{d(fz, fz), d(fz, Tz), d(fz, Tz)\}, \max\{d(fz, fz), d(fz, Tz), d(fz, Tz), d(fz, Tz)\}\}
\end{aligned}$$

Implies that $d(fz, Tz) \leq \sup_{x,y\in X}(b+c+e)d(fz, Tz)$and hence $fz = Tz$.

Corollary 2. *Let X be a metric space, T a multi-valued map from X to $C(X)$. If X is T-orbitally complete and for each $x, y \in X$*

$$\begin{aligned}
H(Tx, Ty) \leq a(x,y)d(x,y) &+ b(x,y)\max\{d(x,Tx), d(y,Ty)\}\\
&+ c(x,y)\max\{d(x,y), d(x,Tx), d(y,Ty)\}\\
&+ e(x,y)\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty)\}
\end{aligned} \tag{9}$$

Where $a, b, c, e : X \times X \to [0,1)$ satisfying $a(x,y) \geq 0$, $\inf_{x,y\in X} e(x,y) > 0$

$$\inf_{x,y\in X}(1 + b(x,y) + e(x,y)) > 0 \text{ and } \sup_{x,y\in X}(a(x,y) + b(x,y) + c(x,y) + 2e(x,y)) = 1$$

Then T has a fixed point in X.

Theorem 3. *Let X, T and f satisfy the hypotheses of Theorem 2 with $C(X)$ replaced by $CL(X)$ and a, b, c, e satisfy $\delta = \sup_{x,y\in X}(a(x,y) + b(x,y) + c(x,y) + 2e(x,y) < 1$. Then T and f have a coincidence.*

Proof. Let $x_0 \in X$, and construct sequences $\{x_n\}$ and $\{y_n\}$ as follows: since $T(X) \subseteq f(X)$, choose $y_1 = fx_1 \in Tx_0$. If $Tx_0 = Tx_1$ choose $y_2 = fx_2 \in Tx_1$ such that $y_1 = y_2$. If

$Tx_0 \neq Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that $d(y_1, y_2) \leq \lambda H(Tx_0, Tx_1)$, where $\lambda > 1$ and $\lambda\delta < 1$. In general, choose $y_{n+2} \in Tx_{n+1}$ such that $d(y_{n+1}, y_{n+2}) \leq \lambda H(Tx_n, Tx_{n+1})$. From (6), we have

$$\begin{aligned}
d(y_{n+1}, y_{n+2}) &= d(fx_{n+1}, fx_{n+2}) \\
&\leq H(Tx_n, Tx_{n+1}) \\
&\leq \lambda a(x,y)d(fx_n, fx_{n+1}) + \lambda b(x,y)\max\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\
&\quad + \lambda c(x,y)\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\} \\
&\quad + \lambda e(x,y)\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_{n+1})\} \\
&\leq \lambda a d(y_n, y_{n+1}) + \lambda b \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\} \\
&\quad + \lambda c \max\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\} \\
&\quad + \lambda e \max\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\}
\end{aligned}$$

If there exists an n such that $d(y_{n+1}, y_{n+2}) > d(y_n, y_{n+1})$, we have

$$d(y_{n+1}, y_{n+2}) < \lambda(a + b + c + 2e)d(y_{n+1}, y_{n+2})$$

a contradiction. Therefore for all n we get $d\left(y_{n+1}, y_{n+2}\right) \leq d\left(y_n, y_{n+1}\right)$ and we obtain

$$\begin{aligned}
d(y_{n+1}, y_{n+2}) &\leq \lambda a d(y_n, y_{n+1}) + \lambda b d(y_n, y_{n+1}) + \lambda c d(y_n, y_{n+1}) + 2\lambda e d(y_n, y_{n+1}) \\
&\leq \lambda(a + b + c + 2e)d(y_n, y_{n+1}) \\
&\leq k d(y_n, y_{n+1}) \leq k^n d(y_0, y_1) \text{ where } k = \sup_{x,y\in X} \lambda(a + b + c + 2e)
\end{aligned}$$

Therefore $\{y_n\}$ is Cauchy, hence convergent to some point p in X. Since f is surjective, there exists a z such that $fz = p$.
Now

$$\begin{aligned}
d(fz, Tz) &\leq d(fz, fx_{n+1}) + d(fx_{n+1}, Tz) \\
&\leq d(fz, fx_{n+1}) + H(Tx_n, Tz) \\
&\leq d(fz, fx_{n+1}) + a d(fx_n, fz) \\
&\quad + b\max\{d(fx_n, Tx_n), d(fz, Tz)\} \\
&\quad + c\max\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz)\} \\
&\quad + e\max\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz), d(fx_n, Tz)\} \\
&\leq d(fz, fx_{n+1}) + \sup_{x,y\in X} a d(fx_n, fz) \\
&\quad + \sup_{x,y\in X}(b + c + e)\max\{\max\{d(fx_n, Tx_n), d(fz, Tz)\}, \\
&\quad \max\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz)\}, \\
&\quad \max\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz), d(fx_n, Tz)\}\}
\end{aligned}$$

Taking limit as $n \to \infty$ we get $d\left(fz, Tz\right) \leq \sup_{x,y\in X}\left(b + c + e\right)d\left(fz, Tz\right)$ implies that $fz \in Tz$.

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