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Applications of Fractional Differential Transform Method to Fractional Differential-Algebraic Equations

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Abstract. In this paper, we implement fractional differential transform method (FDTM), which is a semi analytical numerical technique, to fractional differential-algebraic equations (FDAEs). The fractional derivatives are described in the Caputo sense. The method provides the solution in the form of a rapidly convergent series. The method is illustrated by four examples of FDAEs and solutions are obtained. Comparisons are made between fractional differential transform method (FDTM), Homotopy Analysis Method (HAM) and the exact solutions. The results reveal that the proposed method is very effective and simple.

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1. Introduction

Fractional differential equations (FDEs) have been succesfully modelled for many physical and engineering phenomena such as seismic analysis, rheology, fluid flow, viscous damping, viscoelastic materials and polymer physics [9, 2, 31, 16, 32, 3]. Most nonlinear FDEs don't have exact analytic solutions, therefore approximation and numerical techniques must be used. Some of the recent analytic methods for solving nonlinear problems include the Adomian decomposition method (ADM) [12, 11, 30, 38, 35, 14], variational iteration method (VIM) [18, 19, 42, 36, 33], homotopy analysis method (HAM) [34, 13, 40, 37, 27] and fractional method [17]. Among these solution techniques, the VIM and the ADM are the most clear methods of solution of FDEs for providing instant and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to nonlinear differential equations without linearization or discretization.

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Many physical problems are governed by a system of differential-algebraic equations (DAEs), and the solution of these equations has been a subject of many investigators in recent years. Although many exact solutions for linear DAEs has been found, in general, there exists no method that yields an exact solution for nonlinear DAEs. Numerical approaches for approximating solutions of DAEs have been presented [39, 29, 28, 6, 5, 4, 8, 24, 25, 7, 10, 15].

Recently, many important mathematical models can be expressed in terms of differential-algebraic equations of fractional order. Homotopy analysis method was first introduced by Liao [34], who employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems. Zurigat, Momani and Alawneh [26] applied this method for fractional differential-algebraic equations (FDAEs).

The differential transform method (DTM) was first applied in the engineering domain in [20]. The DTM is numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. Arikoglu and Ozkol implement a new analytical technique for the field of fractional calculus, for solving fractional type differential equations that will be named as Fractional Differential Transform Method (FDTM)[1]. In this paper, fractional differential transform method (FDTM) is applied to solve fractional differential-algebraic equations (FDAEs) of form

$$D_*^{\alpha_i} x_i(t) = f(t, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n'),\ i = 1, 2, 3, \ldots, n-1,\ \ t \geq 0,\ \ 0 < \alpha_i \leq 1 \tag{1}$$

$$g(t, x_1, x_2, \ldots, x_n) = 0 \tag{2}$$

subject to the initial conditions

$$x_i(0) = a_i,\ \ i = 1, 2, \ldots, n \tag{3}$$

2. Basic Definitions

There are several definitions of a fractional derivative of order $\alpha > 0$ [17, 22].e.g.Riemann-Liouville, Grunwald-Letnikow, Caputo and Generalized Functions Approach. The most commonly used definitions are the Riemann-Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$. Clearly $C_\mu < C_\beta$ if $\beta < \mu$.*

Definition 2. *A function $f(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in N \cup \{0\}$ if $f^{(m)} \in C_\mu$.*

Definition 3. *The Riemann-Liouville fractional integral operator of order a $\alpha \geq 0$ of a function, $f \in C_\mu, \mu \geq -1$ is defined as [31]*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,\ \ \alpha > 0,\ x > 0 \tag{4}$$

$$J^0 f(x) = f(x) \tag{5}$$

Properties of the operator J^α can be found in [41, 23, 21], we mention only the following: For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \tag{6}$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \tag{7}$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \tag{8}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena using fractional differential equations. Therefore, we will introduce a modified fractional differential operator proposed by Caputo's work on the theory of viscoelasticity [22].

Definition 4. *The fractional derivative of $f(x)$ in the Caputo sense is defined as*

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{9}$$

for $m-1 < \alpha \leq m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 1. *If $m-1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m$, $m \geq -1$, then*

$$D_*^\alpha J^\alpha f(x) = f(x) \tag{10}$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \; x > 0 \tag{11}$$

3. Fractional Differential Transform Method (FDTM)

In this section, we introduce the fractional differential transform method used in this paper to obtain approximate analytical solutions for FDAEs in Eq.(1). This method has been developed in [1] as follows:
The fractional differentiation in Riemann-Liouville sense is defined by

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} D^m \left(\int_0^x (x-t)^{m-\alpha-1} f(t) dt \right) \tag{12}$$

for $m-1 \leq \alpha < m$, $m \in N$, $x > 0$. Let us expand the analytical and continuous function $f(x)$ in terms of a fractional power series as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k) x^{\frac{k}{\beta}} \tag{13}$$

where β is the order of fraction and $F(k)$ is the fractional differential transform of $f(x)$.
In order to avoid fractional initial and boundary conditions, we define the fractional derivative in the Caputo sense. The relation between the Riemann-Liouville operator and Caputo operator is given by

$$D_*^{\alpha} f(x) = D^{\alpha}\left(f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}\right) \tag{14}$$

Setting $f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}$ in Eq.(12)and using Eq.(14), we obtain fractional derivative in the Caputo sense as follows:

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} D^m \left[\int_0^x (x-t)^{m-\alpha-1}\left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!}\right) dt\right] \tag{15}$$

Since the initial conditions are implemented for the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$F(k) = \begin{cases} \frac{1}{(k/\beta)!}\left[\frac{d^{k/\beta} f(x)}{dx^{k/\beta}}\right]_{x=0} \text{ for } k = 0,1,2,\ldots,(\alpha\beta - 1) & ,k/\beta \in N^+ \\ 0 & ,k/\beta \notin N^+ \end{cases} \tag{16}$$

where, α is the order of fractional differential equation considered. The following theorems that can be deduced from Eqs.(12) and (13) are given below,for proofs and details see [1].

Theorem 1. *if $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$*

Theorem 2. *if $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$*

Theorem 3. *if $f(x) = g_1(x)g_2(x)\cdots g_{n-1}(x)g_n(x)$, then*

$$F(k) = \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}) \tag{17}$$

Theorem 4. *if $f(x) = x^p$, then $F(k) = \delta(k - \beta p)$*

$$\text{where, } \delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} \tag{18}$$

Theorem 5. *if $f(x) = D^{\alpha} g(x)$, then $F(k) = \frac{\Gamma(\alpha+1+k/\beta)}{\Gamma(1+k/\beta)} G(k + \alpha\beta)$*

Theorem 6. *if $f(x) = D^{\alpha_1} g_1(x) D^{\alpha_2} g_2(x)\cdots D^{\alpha_{n-1}} g_{n-1}(x) D^{\alpha_n} g_n(x)$, then*

$$F(k) = \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} \frac{\Gamma(\alpha_1+1+k_1/\beta)}{\Gamma(1+k_1/\beta)}\frac{\Gamma(\alpha_2+1+(k_2-k_1)/\beta)}{\Gamma(1+(k_2-k_1)/\beta)}\cdots$$

$$\times\frac{\Gamma(\alpha_n+1+(k-k_{n-1}/\beta)}{\Gamma(1+(k-k_{n-1}/\beta)} G_1(k_1+\alpha_1\beta)G_2(k_2-k_1+\alpha_2\beta)\cdots G_n(k-k_{n-1}+\alpha_n\beta)$$

where $\beta\alpha_i \in Z^+$ for $i = 1,2,\ldots,n$.

4. Numerical Examples

In order to demonstrate the effectiveness of the fractional differential transform method, we consider the following FDAEs. All the results are calculated by using the symbolic calculus software Maple.

Example 1. *We consider the following fractional differential-algebraic equations.*

$$D_*^{\alpha}x(t) - ty'(t) + x(t) - (1+t)y(t) = 0,\ \ 0 < \alpha \leq 1 \tag{19}$$

$$y(t) - sin(t) = 0 \tag{20}$$

with initial conditions as

$$x(0) = 1, y(0) = 0 \tag{21}$$

For the special case when $\alpha = 1$ the exact solution is $x(t) = e^{-t} + tsin(t),\ y(t) = sin(t)$. Eqs.(19)-(20) are transformed by using Theorems 1, 2, 4, 5 and Eq.(13) as follows:

$$X(k+\alpha\beta) = \frac{\Gamma(1+k/\beta)}{\Gamma(\alpha+1+k/\beta)}\left[\sum_{l=0}^{k}\left[\frac{\Gamma(2+l/\beta)}{\Gamma(1+l/\beta)}Y(l+\beta)+Y(l)\right]\delta(k-l-\beta) - X(k) + Y(k)\right] \tag{22}$$

$$Y(k) = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)!}\delta(k-\beta(2i+1)) \tag{23}$$

where β is the unknown value of the fractions. Initial conditions in Eq.(21) are transformed by using Eq.(16) as follows:

$$X(0) = 1,\ Y(0) = 0,\ X(k) = Y(k) = 0 \text{ for } k = 1,2,\ldots,\alpha\beta - 1 \tag{24}$$

From Eqs.(22)-(24), $X(k)$ and $Y(k)$ are obtained for different values of α and using the inverse transformation in Eq.(13), $x(t)$ and $y(t)$ are evaluated.Numerical results with comparison to Ref. [26] is given in Table 1.

Example 2. *Consider the following fractional differential-algebraic equations.*

$$D_*^{\alpha_1}x(t) - x(t) - z(t)x(t) = 1 \tag{25}$$

$$D_*^{\alpha_2}z(t) - y(t) + x^2(t) + z(t) = 0,\ \ 0 < \alpha_1, \alpha_2 \leq 1 \tag{26}$$

$$y(t) - x^2(t) = 0 \tag{27}$$

with initial conditions as

$$x(0) = y(0) = z(0) = 1 \tag{28}$$

For $\alpha_1 = \alpha_2 = 1$ the exact solution is $x(t) = e^t$, $y(t) = e^{2t}$, $z(t) = e^{-t}$. By using Theorems 1, 2, 4, 5 and Eq.(13), Eqs.(25)-(27) are transformed to,

$$X(k+\alpha_1\beta_1) = \frac{\Gamma(1+k/\beta_1)}{\Gamma(\alpha_1+1+k/\beta_1)}\left[X(k) - \sum_{l=0}^{k}Z(l)X(k-l) + \delta(k)\right] \tag{29}$$

Table 1: Numerical results with comparison to Ref. [26] in Example 1

	$\alpha=0.5$		$\alpha=0.75$		$\alpha=1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.7642925	0.7642925	0.8492995	0.8492996	0.9148208	0.9148208	0.9148208
0.2	0.7545097	0.7545096	0.8016696	0.8016697	0.8584646	0.8584646	0.8584646
0.3	0.7903162	0.7903162	0.7979000	0.7978999	0.8294743	0.8294743	0.8294743
0.4	0.8524950	0.8524950	0.8250873	0.8250871	0.8260874	0.8260874	0.8260874
0.5	0.9323247	0.9323247	0.8760146	0.8760144	0.8462434	0.8462434	0.8462434
0.6	1.0242052	1.0242052	0.9454582	0.9454582	0.8875971	0.8875971	0.8875971
0.7	1.1237906	1.1237906	1.0290755	1.0290757	0.9475377	0.9475377	0.9475377
0.8	1.2273291	1.2273291	1.1229592	1.1229595	1.0232138	1.0232138	1.0232138
0.9	1.3313916	1.3313915	1.2234363	1.2234368	1.1115639	1.1115639	1.1115639
1.0	1.4327552	1.4327552	1.3269757	1.3269767	1.2093505	1.2093504	1.2093504

$$Z(k+\alpha_2\beta_2)=\frac{\Gamma(1+k/\beta_2)}{\Gamma(\alpha_2+1+k/\beta_2)}\left[Y(k)-\sum_{l=0}^{k}X(l)X(k-l)-Z(k)\right] \tag{30}$$

$$Y(k)=\sum_{l=0}^{k}X(l)X(k-l) \tag{31}$$

where β_1 and β_2 are the unknown values of the fractions and $\beta = LCM(\beta_1,\beta_2)$. From Eq.(16), initial conditions in Eq.(28) can be transformed as follows:

$$X(0)=Z(0)=1, X(k)=0,\ k=1,2,\ldots,\alpha_1\beta_1-1,\ Z(k)=0,\ k=1,2,\ldots,\alpha_2\beta_2-1 \tag{32}$$

From Eqs.(29)-(32), $X(k)$, $Y(k)$ and $Z(k)$ are calculated and using the inverse transformation rule in Eq.(13), $x(t)$,$y(t)$ and $z(t)$ are calculated for different values of α_1 and α_2. Numerical comparisons are given in Table 2-3-4.

Table 2: Numerical results of $x(t)$ with comparison to HAM in Example 2

	$\alpha=0.5$		$\alpha=0.75$		$\alpha=1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.4678849	1.4678849	1.2187069	1.2187069	1.1051709	1.1051709	1.1051709
0.2	1.7411322	1.7411322	1.4000280	1.4000280	1.2214028	1.2214028	1.2214028
0.3	1.9927891	1.9927891	1.5841270	1.5841270	1.3498588	1.3498588	1.3498588
0.4	2.2392557	2.2392557	1.7769089	1.7769089	1.4918247	1.4918247	1.4918247
0.5	2.4871415	2.4871415	1.9813870	1.9813870	1.6487213	1.6487213	1.6487213
0.6	2.7401183	2.7401183	2.1997453	2.1997453	1.8221188	1.8221188	1.8221188
0.7	3.0006469	3.0006469	2.4338838	2.4338838	2.0137527	2.0137527	2.0137527
0.8	3.2706054	3.2706054	2.6856249	2.6856249	2.2255409	2.2255409	2.2255409
0.9	3.5515666	3.5515666	2.9568131	2.9568125	2.4596031	2.4596031	2.4596031
1.0	3.8450351	3.8450346	3.2493750	3.2493684	2.7182818	2.7182818	2.7182818

Table 3: Numerical results of $y(t)$ with comparison to HAM in Example 2

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	2.1546862	2.1546862	1.4852465	1.4852465	1.2214028	1.2214028	1.2214028
0.2	3.0315412	3.0315412	1.9600784	1.9600784	1.4918247	1.4918247	1.4918247
0.3	3.9712084	3.9712084	2.5094586	2.5094586	1.8221188	1.8221188	1.8221188
0.4	5.0142660	5.0142660	3.1574052	3.1574052	2.2255409	2.2255409	2.2255409
0.5	6.1858731	6.1858732	3.9258942	3.9258942	2.7182815	2.7182818	2.7182815
0.6	7.5082482	7.5082482	4.8388794	4.8388794	3.3201150	3.3201169	3.3201169
0.7	9.0038821	9.0038821	5.9237902	5.9237903	4.0551908	4.0552000	4.0552000
0.8	10.6968505	10.696866	7.2125789	7.2125809	4.9529970	4.9530324	4.9530324
0.9	12.6037326	12.603625	8.7427133	8.7427398	6.0495302	6.0496475	6.0496475
1.0	14.7830711	14.784077	10.558121	10.558395	7.3887125	7.3890561	7.3890561

Table 4: Numerical results of $z(t)$ with comparison to HAM in Example 2

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.7235784	0.7235784	0.8282505	0.8282505	0.9048374	0.9048374	0.9048374
0.2	0.6437883	0.6437883	0.7325847	0.7325847	0.8187308	0.8187308	0.8187308
0.3	0.5920184	0.5920184	0.6603375	0.6603375	0.7408182	0.7408182	0.7408182
0.4	0.5536063	0.5536063	0.6021211	0.6021211	0.6703201	0.6703201	0.6703201
0.5	0.5231566	0.5231566	0.5536026	0.5536026	0.6065307	0.6065307	0.6065307
0.6	0.4980246	0.4980246	0.5122851	0.5122851	0.5488116	0.5488116	0.5488116
0.7	0.4767027	0.4767027	0.4765549	0.4765549	0.4965853	0.4965853	0.4965853
0.8	0.4582460	0.4582460	0.4452924	0.4452924	0.4493290	0.4493290	0.4493290
0.9	0.4420214	0.4420214	0.4176820	0.4176821	0.4065697	0.4065697	0.4065697
1.0	0.4275836	0.4275836	0.3931083	0.3931083	0.3678795	0.3678794	0.3678794

Example 3. *Consider the following fractional differential- algebraic equations.*

$$x(t) + y(t) = e^{-t} + sin(t) \tag{33}$$

$$D_*^{\alpha} x(t) + x(t) - y(t) + sin(t) = 0,\ 0 < \alpha \le 1 \tag{34}$$

with initial conditions as

$$x(0) = 1, y(0) = 0 \tag{35}$$

For the special case when $\alpha = 1$ the exact solution is $x(t) = e^{-t}$, $y(t) = sin(t)$. Eqs.(33)-(34) are transformed by using Theorems 1, 4 and 5 as follows:

$$Y(k) = -X(k) + E(k) + S(k) \tag{36}$$

$$X(k + \alpha\beta) = \frac{\Gamma(1 + k/\beta)}{\Gamma(\alpha + 1 + k/\beta)} [-X(k) + Y(k) - S(k)] \tag{37}$$

where β is the unknown value of the fractions, $E(k)$ and $S(k)$ are the fractional differential transform of $x(t)=e^{-t}$ and $y(t)=sin(t)$ that can be evaluated using Eq.(13) as

$$E(k)=\begin{cases}\frac{(-1)^{k/\beta}}{(k/\beta)!} & \text{if } k/\beta\in Z^{+}\\ 0 & \text{if } k/\beta\notin Z^{+}\end{cases} \tag{38}$$

$$S(k)=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{(2i+1)!}\delta(k-\beta(2i+1)) \tag{39}$$

From Eq.(16), initial conditions in Eq.(35) can be transformed as follows:

$$X(0)=1, Y(0)=0,\ X(k)=Y(k)=0,\ \text{for } k=1,2,\dots,\alpha\beta-1 \tag{40}$$

From Eqs.(38)-(40), $X(k)$ and $Y(k)$ are calculated and using the inverse transformation rule in Eq.(13), $x(t)$ and $y(t)$ are calculated for different values of. Numerical comparisons are given in Table 5-6.

Table 5: Numerical results of $x(t)$ with comparison to HAM in Example 3

	$\alpha=0.5$		$\alpha=0.75$		$\alpha=1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.7608910	0.7608910	0.8373931	0.8373931	0.9048374	0.9048374	0.9048374
0.2	0.6909262	0.6909262	0.7494391	0.7494391	0.8187308	0.8187308	0.8187308
0.3	0.6396502	0.6396502	0.6816129	0.6816129	0.7408182	0.7408182	0.7408182
0.4	0.5970878	0.5970877	0.6250322	0.6250322	0.6703201	0.6703201	0.6703201
0.5	0.5599926	0.5599926	0.5760122	0.5760122	0.6065307	0.6065307	0.6065307
0.6	0.5268894	0.5268894	0.5326238	0.5326238	0.5488116	0.5488116	0.5488116
0.7	0.4969640	0.4969640	0.4937128	0.4937128	0.4965853	0.4965853	0.4965853
0.8	0.4697022	0.4697024	0.4585197	0.4585198	0.4493290	0.4493290	0.4493290
0.9	0.4447448	0.4447444	0.4265076	0.4265076	0.4065697	0.4065697	0.4065697
1.0	0.4218207	0.4218206	0.3972736	0.3972738	0.3678794	0.3678795	0.3678794

Example 4. *We consider the following fractional differential-algebraic equations.*

$$D_*^{\alpha_1}x(t)-t^2x(t)+y(t)-2t=0 \tag{41}$$

$$D_*^{\alpha_2}y(t)-2z(t)+2(t+1)=0,\ 0<\alpha_1,\alpha_2\le 1 \tag{42}$$

$$z(t)-y(t)-2tx(t)+t^4-t-1=0 \tag{43}$$

with initial conditions as

$$x(0)=0, y(0)=0, z(0)=1 \tag{44}$$

For $\alpha_1=\alpha_2=1$ the exact solution is $x(t)=t^2$, $y(t)=t^4$, $z(t)=2t^3+t+1$. By using Theorems 1, 2, 4, 5 and Eq.(13), Eqs.(41)-(43) are transformed to,

$$X(k+\alpha_1\beta_1)=\frac{\Gamma(1+k/\beta_1)}{\Gamma(\alpha_1+1+k/\beta_1)}\left[\sum_{l=0}^{k}\delta(l-2\beta_1)X(k-l)-Y(k)+2\delta(k-\beta_1)\right] \tag{45}$$

Table 6: Numerical results of $y(t)$ with comparison to HAM in Example 3

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.2437798	0.2437798	0.1672777	0.1672777	0.0998334	0.0998334	0.0998334
0.2	0.3264739	0.3264739	0.2679610	0.2679610	0.1986693	0.1986693	0.1986693
0.3	0.3966883	0.3966883	0.3547256	0.3547256	0.2955202	0.2955202	0.2955202
0.4	0.4626507	0.4626507	0.4347062	0.4347062	0.3894183	0.3894183	0.3894183
0.5	0.5259636	0.5259636	0.5099441	0.5099441	0.4794255	0.4794255	0.4794255
0.6	0.5865647	0.5865647	0.5808303	0.5808303	0.5646425	0.5646425	0.5646425
0.7	0.6438391	0.6438391	0.6470902	0.6470902	0.6442177	0.6442177	0.6442177
0.8	0.6969830	0.6969827	0.7081653	0.7081653	0.7173561	0.7173561	0.7173561
0.9	0.7451519	0.7451522	0.7633890	0.7633890	0.7833269	0.7833269	0.7833269
1.0	0.7875300	0.7875299	0.8120769	0.8120766	0.8414710	0.8414710	0.8414710

$$Y(k+\alpha_2\beta_2) = \frac{\Gamma(1+k/\beta_2)}{\Gamma(\alpha_2+1+k/\beta_2)}\left[2Z(k) - \delta(k-\beta_2) - 2\delta(k)\right] \tag{46}$$

$$Z(k) = Y(k) + 2\sum_{l=0}^{k}\delta(l-\beta)X(k-l) - 4\delta(k-4\beta) + \delta(k-\beta) + \delta(k) \tag{47}$$

where β_1 and β_2 are the unknown value of the fractions and $\beta = LCM(\beta_1, \beta_2)$.From Eq.(16), initial conditions in Eq.(44) can be transformed as follows:

$$X(0) = 0,\ k = 0,1,2,\ldots,\alpha_1\beta_1 - 1,\ \ Y(0) = 0,\ k = 0,1,2,\ldots,\alpha_2\beta_2 - 1,\ \ Z(0) = 1 \tag{48}$$

From Eqs.(45)-(48), $X(k)$, $Y(k)$ and $Z(k)$ are obtained up to and using the inverse transformation rule in Eq.(13), $x(t)$,$y(t)$ and $z(t)$ are calculated for different values of α_1 and α_2. Numerical comparisons are given in Table 7-8-9.

Table 7: Numerical results of $x(t)$ with comparison to HAM in Example 4

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.0468839	0.0468839	0.0220868	0.0220868	0.0100000	0.0100000	0.0100000
0.2	0.1256709	0.1256709	0.0738823	0.0738823	0.0400000	0.0400000	0.0400000
0.3	0.2069285	0.2069285	0.1483883	0.1483883	0.0900000	0.0900000	0.0900000
0.4	0.2635406	0.2635406	0.2403223	0.2403223	0.1600000	0.1600000	0.1600000
0.5	0.2692381	0.2692381	0.3435494	0.3435494	0.2500000	0.2500000	0.2500000
0.6	0.2064374	0.2064374	0.4503451	0.4503451	0.3600000	0.3600000	0.3600000
0.7	0.0771395	0.0771393	0.5510811	0.5510811	0.4900000	0.4900000	0.4900000
0.8	−0.0874721	−0.0874740	0.6342822	0.6342821	0.6400000	0.6400000	0.6400000
0.9	−0.2250269	−0.2249716	0.6872107	0.6872098	0.8100000	0.8100000	0.8100000
1.0	−0.2550524	−0.2536490	0.6972563	0.6972492	1.0000000	1.0000000	1.0000000

Table 8: Numerical results of $y(t)$ with comparison to HAM in Example 4

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.0047962	0.0047962	0.0006640	0.0006640	0.0001000	0.0001000	0.0001000
0.2	0.0446486	0.0446486	0.0079990	0.0079990	0.0016000	0.0016000	0.0016000
0.3	0.1654479	0.1654479	0.0347313	0.0347313	0.0081000	0.0081000	0.0081000
0.4	0.4099255	0.4099255	0.0988115	0.0988115	0.0256000	0.0256000	0.0256000
0.5	0.7956628	0.7956628	0.2220253	0.2220253	0.0625000	0.0625000	0.0625000
0.6	1.2918230	1.2918231	0.4277910	0.4277910	0.1296000	0.1296000	0.1296000
0.7	1.8063925	1.8063925	0.7376692	0.7376692	0.2401000	0.2401000	0.2401000
0.8	2.1992173	2.1992062	1.1662343	1.1662344	0.4096000	0.4096000	0.4096000
0.9	2.3319582	2.3316970	1.7141421	1.7141424	0.6561000	0.6561000	0.6561000
1.0	2.1509580	2.1479845	2.3596212	2.3596233	1.0000000	1.0000000	1.0000000

Table 9: Numerical results of $z(t)$ with comparison to HAM in Example 4

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
t	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{HAM}	x_{FDTM}	x_{Exact}
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.1140730	1.1140730	1.1049813	1.1049813	1.1020000	1.1020000	1.1020000
0.2	1.2933169	1.2933169	1.2359520	1.2359520	1.2160000	1.2160000	1.2160000
0.3	1.5815050	1.5815050	1.4156643	1.4156643	1.3540000	1.3540000	1.3540000
0.4	1.9951580	1.9951580	1.6654692	1.6654692	1.5280000	1.5280000	1.5280000
0.5	2.5024009	2.5024009	2.0030745	2.0030746	1.7500000	1.7500000	1.7500000
0.6	3.0099479	3.0099480	2.4386051	2.4386051	2.0320000	2.0320000	2.0320000
0.7	3.3742876	3.3742875	2.9690827	2.9690826	2.3860000	2.3860000	2.3860000
0.8	3.4496824	3.4496478	3.5714858	3.5714850	2.8240000	2.8240000	2.8240000
0.9	3.1713371	3.1706481	4.1950213	4.1950119	3.3580000	3.3580000	3.3580000
1.0	2.6477139	2.6406864	4.7541339	4.7540532	4.0000000	4.0000000	4.0000000

5. Conclusion

In this paper, fractional differential transform method (FDTM) is extended to solve fractional differential-algebraic equations (FDAEs). The results of this method are in good agreement with those obtained by using the homotopy analysis method (HAM). The study emphasized our belief that the method is a reliable technique to handle fractional differential-algebraic equations and the FDTM offer significant advantages in terms of its straightforward applicability, its computational effectiveness and its accuracy. In general, FDTM can be used as a powerful solver for the solution of fractional differential-algebraic equations.

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