EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 2, 2011, 162-173
ISSN 1307-5543 – www.ejpam.com

Fractional Calculus of a Class of Univalent Functions With Negative Coefficients Defined By Hadamard Product With Rafid -Operator

Waggas Galib Atshan*, Rafid Habib Buti

Department of Mathematics, College of Computer Science and Mathematics, University of Al-Qadisiya, Diwaniya, Iraq

Abstract. In our paper, we study a class $WR(\lambda, \beta, \alpha, \mu, \theta)$, which consists of analytic and univalent functions with negative coefficients in the open unit disk $U = \{z \in C : |z| < 1\}$ defined by Hadamard product (or convolution) with Rafid - Operator, we obtain coefficient bounds and extreme points for this class. Also distortion theorem using fractional calculus techniques and some results for this class are obtained.

2000 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Univalent Function, Fractional Calculus, Hadamard Product, Distortion Theorem, Rafid-Operator, Extreme Point.

1. Introduction

Let R denote the class of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{1}$$

which are analytic and univalent in the unit disk $U = \{z \in C : |z| < 1\}$. If $f \in R$ is given by (1) and $g \in R$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \; b_n \geq 0$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$f * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{2}$$

*Corresponding author.

Email addresses: `waggashnd@yahoo.com` (W. Atshan), `Rafidhb@yahoo.com` (R. Buti)

Lemma 1. *The Rafid -Operator of $f \in R$ for $0 \le \mu < 1$, $0 \le \theta \le 1$ is denoted by R_μ^θ and defined as following:*

$$\begin{aligned} R_\mu^\theta(f(z)) &= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)dt \\ &= z - \sum_{n=2}^{\infty} K(n,\mu,\theta) a_n z^n, \end{aligned} \tag{3}$$

where $K(n,\mu,\theta) = \frac{(1-\mu)^{n-1}\Gamma(\theta+n)}{\Gamma(\theta+1)}$.

Proof.

$$\begin{aligned} R_\mu^\theta(f(z)) &= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)dt \\ &= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} \left[zt - \sum_{n=2}^{\infty} a_n (zt)^n \right] dt \\ &= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \left[z \int_0^\infty t^{\theta} e^{-\left(\frac{t}{1-\mu}\right)} dt - \sum_{n=2}^{\infty} a_n z^n \int_0^\infty t^{\theta-1+n} e^{-\left(\frac{t}{1-\mu}\right)} dt \right] \end{aligned}$$

Let $x = \frac{t}{1-\mu}$, then if $t = 0$, we get $x = 0$, $t = \infty$, we get $x = \infty$ and $t = (1-\mu)x$, then $dt = (1-\mu)dx$. Thus

$$\begin{aligned} R_\mu^\theta(f(z)) &= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \left[z \int_0^\infty (1-\mu)^{1+\theta} e^{-x} x^\theta dx \right. \\ &\quad \left. - \sum_{n=2}^{\infty} a_n z^n \int_0^\infty (1-\mu)^{\theta+n} e^{-x} x^{\theta-1+n} dx \right] \\ &= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \left[z(1-\mu)^{1+\theta}\Gamma(\theta+1) - \sum_{n=2}^{\infty} a_n z^n (1-\mu)^{\theta+n}\Gamma(\theta+n) \right] \\ &= z - \sum_{n=2}^{\infty} \frac{(1-\mu)^{n-1}\Gamma(\theta+n)}{\Gamma(\theta+1)} a_n z^n \\ &= z - \sum_{n=2}^{\infty} K(n,\mu,\theta) a_n z^n. \end{aligned}$$

Definition 1. *A function $f(z) \in R$, $z \in U$ is said to be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ if and only if satisfies the inequality:*

$$Re\left\{ \frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''}{(1-\lambda)R_\mu^\theta((f*g)(z)) + \lambda z (R_\mu^\theta((f*g)(z)))'} \right\}$$

$$\geq \beta \left| \frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2(R_\mu^\theta((f*g)(z)))''}{(1-\lambda)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} - 1 \right| + \alpha, \tag{4}$$

where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $\beta \geq 0$, $z \in U$, $0 \leq \mu < 1$, $0 \leq \theta \leq 1$ *and* $g(z) \in R$ *given by*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0.$$

Lemma 2. *[1] Let* $w = u + iv$*. Then Re* $w \geq \sigma$ *if and only if* $|w - (1+\sigma)| \leq |w + (1-\sigma)|$*.*

Lemma 3. *[1] Let* $w = u + iv$ *and* σ, γ *are real numbers. Then Re* $w > \sigma|w-1| + \gamma$ *if and only if Re* $\{w(1+\sigma e^{i\phi}) - \sigma e^{i\phi}\} > \gamma$*.*

We aim to study the coefficient bounds, extreme points, application of fractional calculus and Hadamard product of the class $WR(\lambda, \beta, \alpha, \mu, \theta)$.

2. Coefficient Bounds and Extreme Points

We obtain here a necessary and sufficient condition and extreme points for the functions $f(z)$ in the class $WR(\lambda, \beta, \alpha, \mu, \theta)$.

Theorem 1. *The function* $f(z)$ *defined by (1) is in the class* $WR(\lambda, \beta, \alpha, \mu, \theta)$ *if and only if*

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)a_n b_n \leq 1-\alpha, \tag{5}$$

where $0 \leq \alpha < 1$, $\beta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$ *and* $0 \leq \theta \leq 1$*.*

Proof. By Definition 1, we get

$$Re\left\{ \frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2(R_\mu^\theta((f*g)(z)))''}{(1-\lambda)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} \right\}$$
$$\geq \beta \left| \frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2(R_\mu^\theta((f*g)(z)))''}{(1-\lambda)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} - 1 \right| + \alpha.$$

Then by Lemma 3, we have

$$Re\left\{ \frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2(R_\mu^\theta((f*g)(z)))''}{(1-\lambda)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'}(1+\beta e^{i\phi}) - \beta e^{i\phi} \right\} \geq \alpha,$$

$-\pi < \phi \leq \pi$, or equivalently,

$$Re\left\{ \frac{(z(R_\mu^\theta((f*g)(z)))' + \lambda z^2(R_\mu^\theta((f*g)(z)))'')(1+\beta e^{i\phi})}{(1-\lambda)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} \right.$$

$$-\left.\frac{\beta e^{i\phi}((1-\lambda)(R_{\mu}^{\theta}((f*g)(z))+\lambda z^{2}(R_{\mu}^{\theta}((f*g)(z)))'))}{(1-\lambda)R_{\mu}^{\theta}((f*g)(z))+\lambda z(R_{\mu}^{\theta}((f*g)(z)))'}\right\}\geq\alpha. \tag{6}$$

Let

$$\begin{aligned} F(z) &= [z(R_{\mu}^{\theta}((f*g)(z)))'+\lambda z^{2}(R_{\mu}^{\theta}((f*g)(z)))''](1+\beta e^{i\phi}) \\ &\quad -\beta e^{i\phi}[(1-\lambda)(R_{\mu}^{\theta}((f*g)(z)))+\lambda z(R_{m}^{\theta}((f*g)(z)))'], \end{aligned}$$

and

$$E(z)=(1-\lambda)R_{\mu}^{\theta}((f*g)(z))+\lambda z(R_{\mu}^{\theta}((f*g)(z)))'.$$

By Lemma 2, (6) is equivalent to

$$|F(z)+(1-\alpha)E(z)|\geq|F(z)-(1+\alpha)E(z)| \text{ for } 0\leq\alpha<1.$$

But

$$\begin{aligned} &|F(z)+(1-\alpha)E(z)| \\ &=\Bigg|\left[z-\sum_{n=2}^{\infty}K(n,\mu,\theta)a_{n}b_{n}z^{n}-\lambda\sum_{n=2}^{\infty}n(n-1)K(n,\mu,\theta)a_{n}b_{n}z^{n}\right](1+\beta e^{i\phi}) \\ &-\beta e^{i\phi}\left[(1-\lambda)(z-\sum_{n=2}^{\infty}K(n,\mu,\theta)a_{n}b_{n}z^{n})+\lambda z-\lambda\sum_{n=2}^{\infty}nK(n,\mu,\theta)a_{n}b_{n}z^{n}\right] \\ &+(1-\alpha)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_{n}b_{n}z^{n}\right]\Bigg| \\ &=\Bigg|(2-\alpha)z-\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+(1-\alpha)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_{n}b_{n}z^{n} \\ &-\beta e^{i\phi}\sum_{n=2}^{\infty}[n+\lambda n(n-1)-(1-\lambda+n\lambda)]K(n,\mu,\theta)a_{n}b_{n}z^{n}\Bigg| \\ &\geq(2-\alpha)|z|-\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+(1-\alpha)(1-\lambda+\lambda n)]K(n,\mu,\theta)a_{n}b_{n}|z|^{n} \\ &-\beta\sum_{n=2}^{\infty}[n+\lambda n(n-2)-1+\lambda]K(n,\mu,\theta)a_{n}b_{n}|z|^{n}. \end{aligned}$$

Also

$$\begin{aligned} &|F(z)-(1+\alpha)E(z)| \\ &=\Bigg|\left[z-\sum_{n=2}^{\infty}nK(n,\mu,\theta)a_{n}b_{n}z^{n}-\lambda\sum_{n=2}^{\infty}n(n-1)K(n,\mu,\theta)a_{n}b_{n}z^{n}\right](1+\beta e^{i\phi}) \end{aligned}$$

$$-\beta e^{i\phi}\left[z-(1-\lambda)\sum_{n=2}^{\infty}K(n,\mu,\theta)a_n b_n z^n-\lambda\sum_{n=2}^{\infty}nK(n,\mu,\theta)a_n b_n z^n\right]$$
$$\left.-(1+\alpha)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_n b_n z^n\right]\right|$$
$$=\left|-\alpha z-\sum_{n=2}^{\infty}[(n+\lambda n(n-1))-(1+\alpha)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n z^n\right.$$
$$\left.-\beta e^{i\phi}\sum_{n=2}^{\infty}[n+n\lambda(n-1)-(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n z^n\right|$$
$$\leq \alpha|z|+\sum_{n=2}^{\infty}[(n+n\lambda(n-1))-(1+\alpha)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n|z|^n$$
$$+\beta\sum_{n=2}^{\infty}[n+n\lambda(n-1)-(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n|z|^n$$

and so

$$|F(z)+(1-\alpha)E(z)|-|F(z)-(1+\alpha)E(z)|\geq 2(1-\alpha)|z|$$
$$-\sum_{n=2}^{\infty}[(2n+2n\lambda(n-1))-2\alpha(1-\lambda+n\lambda)-\beta(2n+2n\lambda(n-1)$$
$$-2(1-\lambda+n\lambda))]K(n,\mu,\theta)a_n b_n|z|^n\geq 0$$

or

$$\sum_{n=2}^{\infty}[n(1+\beta)+n\lambda(n-1)(1+\beta)-(1-\lambda+n\lambda)(\alpha+\beta)]K(n,\mu,\theta)a_n b_n\leq 1-\alpha.$$

This is equivalent to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)a_n b_n\leq 1-\alpha.$$

Conversely, suppose that (5) holds. Then we must show

$$Re\left\{\frac{(z(R_\mu^\theta((f*g)(z)))'+\lambda z^2(R_\mu^\theta((f*g)(z)))'')(1+\beta e^{i\phi})}{(1-\lambda)R_\mu^\theta((f*g)(z))+\lambda z(R_\mu^\theta((f*g)(z)))'}\right.$$
$$\left.-\frac{\beta e^{i\phi}((1-\lambda)(R_\mu^\theta((f*g)(z))+\lambda z(R_\mu^\theta((f*g)(z)))'))}{(1-\lambda)R_\mu^\theta((f*g)(z))+\lambda z(R_\mu^\theta((f*g)(z)))'}\right\}\geq\alpha.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$Re\left\{\frac{(1-\alpha)-\sum_{n=2}^{\infty}[n(1+\beta e^{i\phi})(1-\lambda+\lambda n)-(\alpha+\beta e^{i\phi})(1-\lambda+n\lambda)]K(n,\mu,\theta)a_nb_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_nb_nr^{n-1}}\right\} \geq 0.$$

Since $Re(-e^{i\phi}) \geq -|e^{i\phi}| = -1$, the above inequality reduces to

$$Re\left\{\frac{(1-\alpha)-\sum_{n=2}^{\infty}[n(1+\beta)(1-\lambda+\lambda n)-(\alpha+\beta)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_nb_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_nb_nr^{n-1}}\right\} \geq 0.$$

Letting $r \to 1^-$, we get desired conclusion.

Corollary 1. *Let $f(z) \in WR(\lambda,\beta,\alpha,\mu,\theta)$. Then*

$$a_n \leq \frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n},$$

where $0 \leq \alpha < 1$, $\beta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$, $0 \leq \theta \leq 1$.

Theorem 2. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n}z^n,$$

where $n \geq 2$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, $\beta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$ and $0 \leq \theta \leq 1$.

Then $f(z)$ is in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty}\sigma_n f_n(z),$$

where $\sigma_n \geq 0$ and $\sum_{n=1}^{\infty}\sigma_n = 1$ or $1 = \sigma_1 + \sum_{n=2}^{\infty}\sigma_n$.

Proof. Let $f(z) = \sum_{n=1}^{\infty}\sigma_n f_n(z)$, where $\sigma_n \geq 0$ and $\sum_{n=1}^{\infty}\sigma_n = 1$. Then

$$f(z) = z - \sum_{n=2}^{\infty}\frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n}\sigma_n z^n,$$

and we get

$$\sum_{n=2}^{\infty}\left(\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n}{1-\alpha}\right)\times$$

$$\left(\sigma_n \frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n}\right)$$
$$=\sum_{n=2}^{\infty}\sigma_n = 1-\sigma_1 \le 1 \text{ (by Theorem 1).}$$

By virtue of Theorem 1, we can show that $f(z) \in WR(\lambda, \beta, \alpha, \mu, \theta)$. Conversely, assume that $f(z)$ of the form (1) belongs to $WR(\lambda, \beta, \alpha, \mu, \theta)$. Then

$$a_n \le \frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n},\ n \in \mathbb{N},\ n \ge 2.$$

Setting

$$\sigma_n = \frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)a_n b_n}{1-\alpha}$$

and $\sigma_1 = 1 - \sum\limits_{n=2}^{\infty}\sigma_n$, we obtain

$$f(z) = \sum_{n=1}^{\infty}\sigma_n f_n(z) = \sigma_1 f_1() + \sum_{n=2}^{\infty}\sigma_n f_n(z).$$

This completes the proof.

3. Application of the Fractional Calculus

Various operators of fractional calculus (that is, fractional derivative and fractional integral) have been rather extensively studied by many researcher (c.f. [3-5]). However, we try to restrict ourselves to the following definitions given by Owa [2] for convenience.

Definition 2 (Fractional integral operator)**.** *The fractional integral of order δ is defined, for a function $f(z)$, by*

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)}\int_0^z \frac{f(t)}{(z-t)^{1-\delta}}dt \ (\delta > 0), \tag{7}$$

where $f(z)$ is an analytic function in a simply - connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ *to be real, when* $(z-t) > 0$.

Definition 3 (Fractional derivative operator)**.** *The fractional derivative of order δ is defined, for a function $f(z)$ by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\delta}}dt` \ (0 \le \delta < 1), \tag{8}$$

where $f(z)$ is as in Definition 2.

Definition 4 (Under the condition of Definition 3). *The fractional derivative of order $k+\delta$ $(k = 0, 1, 2, \cdots)$ is defined by*

$$D_z^{k+\delta} f(z) = \frac{d^k}{dz^k} D_z^{\delta} f(z), \ (0 \le \delta < 1). \tag{9}$$

From Definition 2 and 3 by applying a simple calculation we get

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta}, \tag{10}$$

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta}. \tag{11}$$

Now making use of above (10), (11), we state and prove the theorems :

Theorem 3. *Let $f(z) \in WR(\lambda, \beta, \alpha, \mu, \theta)$. Then*

$$|D_z^{-\delta} f(z)| \le \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \left[1 + \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2} |z| \right], \tag{12}$$

and

$$|D_z^{-\delta} f(z)| \ge \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \left[1 - \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2} |z| \right], \tag{13}$$

The inequalities in (12) and (13) are attained for the function given by

$$f(z) = z - \frac{1-\alpha}{(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2} z^2 \tag{14}$$

Proof. By using Theorem 1, we have

$$\sum_{n=2}^{\infty} a_n \le \frac{1-\alpha}{(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}. \tag{15}$$

By (10), we have

$$\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) = z - \sum_{n=2}^{\infty} \ell(n,\delta) a_n z^n, \tag{16}$$

such that

$$\ell(n,\delta) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}, \ n \ge 2.$$

we know that $\ell(n,\delta)$ is a decreasing function of n and

$$0 < \ell(n,\delta) \le \ell(2,\delta) = \frac{2}{2+\delta}.$$

Using (15) and (16), we have

$$\begin{aligned}|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| &\leq |z|+\ell(2,\delta)|z|^2\sum_{n=2}^{\infty}a_n\\ &\leq |z|+\frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}|z|^2,\end{aligned}$$

which gives (12); we also have

$$\begin{aligned}|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| &\geq |z|-\ell(2,\delta)|z|^2\sum_{n=2}^{\infty}a_n\\ &\geq |z|-\frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}|z|^2,\end{aligned}$$

which gives (13).

Theorem 4. *Let $f(z)\in WR(\lambda,\beta,\alpha,\mu,\theta)$. Then*

$$|D_z^{\delta}f(z)|\leq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1+\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}|z|\right], \tag{17}$$

and

$$|D_z^{\delta}f(z)|\geq\frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1-\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}|z|\right]. \tag{18}$$

The inequalities (17) and (18) are attained for the function $f(z)$ given by (14).

Proof. By (11), we have

$$\begin{aligned}\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z) &= z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_nz^n\\ &= z-\sum_{n=2}^{\infty}\Phi(n,\delta)a_nz^n,\end{aligned}$$

where $\Phi(n,\delta)=\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$. For $n\geq 2,\Phi(n,\delta)$ is a decreasing function of n, then

$$\Phi(n,\delta)\leq\Phi(2,\delta)=\frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)}=\frac{2\Gamma(2)\Gamma(2-\delta)}{(2-\delta)\Gamma(2-\delta)}=\frac{2}{2-\delta}.$$

Also by using (15), we have

$$\begin{aligned}|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)| &\leq |z|+\Phi(2,\delta)|z|^2\sum_{n=2}^{\infty}a_n\\ &\leq |z|+\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|^2.\end{aligned}$$

Then

$$|D_z^{\delta} f(z)| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1+\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right],$$

and by the same way, we obtain

$$|D_z^{\delta} f(z)| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1-\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right].$$

Corollary 2. *For every $f \in WR(\lambda,\beta,\alpha,\mu,\theta)$, we have*

$$\frac{|z|^2}{2}\left[1-\frac{2(1-\alpha)}{3(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right] \tag{19}$$

$$\leq \left|\int_0^z f(t)dt\right| \leq \frac{|z|^2}{2}\left[1+\frac{2(1-\alpha)}{3(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right], \tag{20}$$

and

$$|z|\left[1-\frac{(1-\alpha)}{(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right] \leq |f(z)| \tag{21}$$

$$\leq |z|\left[1+\frac{(1-\alpha)}{(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right], \tag{22}$$

Proof.

(i) By Definition 2 and Theorem 3 for $\delta = 1$ we have $D_z^{-1} f(z) = \int_0^z f(t)dt$, the result is true.

(ii) By Definition 3 and Theorem 4 for $\delta = 0$, we have

$$D_z^0 f(z) = \frac{d}{dz}\int_0^z f(t)dt = f(z),$$

the result is true.

Corollary 3. *$D_z^{-\delta} f(z)$ and $D_z^{\delta} f(z)$ are included in the disk with center at the origin and radii*

$$\frac{1}{\Gamma(2+\delta)}\left[1+\frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}\right],$$

and

$$\frac{1}{\Gamma(2-\delta)}\left[1+\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}\right].$$

4. Hadamard Product

Theorem 5. *Let*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; g(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

belong to $WR(\lambda, \beta, \alpha, \mu, \theta)$*. Then the Hadamard product of* f *and* g *given by* $(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$ *belongs to* $WR(\lambda, \beta, \alpha, \mu, \theta)$*.*

Proof. Since f and $g \in WR(\lambda, \beta, \alpha, \mu, \theta)$, we have

$$\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)b_n}{1-\alpha} \right] a_n \leq 1$$

and

$$\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)a_n}{1-\alpha} \right] b_n \leq 1$$

and by applying the Cauchy- Schwarz inequality, we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n} \\ &\leq \left(\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)b_n}{1-\alpha} \right] a_n \right)^{1/2} \\ &\times \left(\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)a_n}{1-\alpha} \right] b_n \right)^{1/2}. \end{aligned}$$

However, we obtain

$$\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n} \leq 1.$$

Now, we want to prove

$$\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha} \right] a_n b_n \leq 1.$$

Since

$$\begin{aligned} &\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha} \right] a_n b_n \\ &= \sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n}. \end{aligned}$$

Hence, we get the required result.

References

[1] E. S. Aqlan, *Some Problems Connected with Geometric Function Theory*, Ph.D. Thesis, Pune University, Pune (unpublished), (2004).

[2] S. Owa, On the distortion theorems, *Kyungpook Math. J.*, 18: 53-59, 1978.

[3] H. M. Srivastava and R. G. Buschman, *Convolution integral equation with special function kernels*, John Wiley and Sons, New York, London, Sydney and Toronto, 1977.

[4] H. M. Srivastava and S. Owa, An application of the fractional derivative, *Math. Japon*, 29:384-389, 1984.

[5] H. M. Srivastava and S. Owa, (Editors), *Univalent Functions, Fractional Calculus and Their Applications*, Halsted press (Ellis Harwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.