Some properties for certain general integral operator

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Abstract

In this paper we consider some subclasses of the class of analytic functions defined in the open unit disk of the complex plane and we study some properties for an integral operator on these classes. Particular results are presented.

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1 Introduction

Let $\mathcal{A}$ denote the class of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We also denote by S the subclass of $\mathcal{A}$ consisting of functions which are univalent in U.

A function $f \in \mathcal{A}$ is said to be *convex of order* α, $0 \leq \alpha < 1$ if it satisfies the condition

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad (z \in U)$$

and we denote this class by $K(\alpha)$.

A function $f \in \mathcal{A}$ is said to be *starlike of order* α, $0 \leq \alpha \leq 1$ if it satisfies the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U)$$

and denote this class by $S^*(\alpha)$.

Let $\mathcal{N}(\rho)$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \rho, \;\; \rho > 1, \quad (z \in U).$$

This class was studied by S. Owa and H.M. Srivastava in [5].

A. Mohammed et al. considered in [4] $\mathcal{MT}(\mu,\beta)$ the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\beta\left|\mu\frac{zf'(z)}{f(z)}+1\right|,\ 0<\beta\leq 1,\ 0\leq\mu<1,\ (z\in U).$$

Also, Frasin and Jahangiri introduced in [2] the family $\mathcal{B}(\mu,\alpha)$, $\mu\geq 0$, $0\leq\alpha\leq 1$, consisting of the functions f which satisfy the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|<1-\alpha,\quad (z\in U).$$

This family is a comprehensive class of analytic functions that includes various classes of analytic functions. We have $\mathcal{B}(1,\alpha)\equiv S^*(\alpha)$ and $\mathcal{B}(0,\alpha)\equiv R(\alpha)$.

Let $\beta-S_p(\alpha)$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}-\alpha\right)\geq\beta\left|\frac{zf'(z)}{f(z)}-1\right|,\ -1\leq\alpha\leq 1,\ \beta>0,\quad (z\in U).$$

This class was studied by M. Darus in [1].

A function f is said to be in the class $KD(\mu,\alpha)$ if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)\geq\mu\left|\frac{zf''(z)}{f'(z)}\right|+\alpha,\ \mu\geq 0,\ 0\leq\alpha<1,\quad (z\in U).$$

This class was studied by S. Shams et al. in [6].

In the present paper we study some properties for the integral operator G_n defined by

$$G_n(z)=\int_0^z\prod_{i=1}^{n}\left(f_i(t)\right)^{\gamma_i-1}\left(g_i'(t)\right)^{\eta_i}\mathrm{d}t \tag{1}$$

on the classes presented above.

In order to prove our main results we need the following lemma:

Lemma 1.1 (General Schwarz Lemma). *[3]. Let the function f be regular in the disk $U_R=\{z\in\mathbb{C}:|z|<R\}$, with $|f(z)|<M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z=0$, then*

$$|f(z)|\leq\frac{M}{R^m}\cdot|z|^m\quad (z\in U_R).$$

The equality can hold only if

$$f(z)=e^{i\theta}\cdot\frac{M}{R^m}\cdot z^m,$$

where θ is constant.

2 Main results

Theorem 2.1. *Let* $\gamma_i \in \mathbb{R}$, $\gamma_i > 1$, $\eta_i \in \mathbb{R}$, $\eta_i > 0$ *for all* $i = 1, 2, ..., n$, *the functions* $f_i \in \mathcal{MT}(\mu_i, \beta_i)$, $0 < \beta_i \leq 1$, $0 \leq \mu_i < 1$ *and* $g_i \in \mathcal{A}$ *for all* $i = 1, 2, ..., n$ *satisfying the conditions*

$$\left|\frac{f_i'(z)}{f_i(z)}\right| < M_i, \ (M_i \geq 1) \text{ for all } i = 1, 2, ..., n \tag{2}$$

and

$$\left|\frac{g_i''(z)}{g_i'(z)}\right| < N_i, \ (N_i \geq 1) \text{ for all } i = 1, 2, ..., n. \tag{3}$$

Then the integral operator G_n *defined in (1) is in* $\mathcal{N}(\rho)$, *where*

$$\rho = 1 + \sum_{i=1}^{n} [(\gamma_i - 1)(\beta_i \mu_i M_i + \beta_i + 1) + \eta_i N_i]$$

Proof. From (1), we have

$$G_n'(z) = \prod_{i=1}^{n} (f_i(z))^{\gamma_i - 1} (g_i'(z))^{\eta_i}$$

and

$$\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} (\gamma_i - 1) \frac{zf_i'(z)}{f_i(z)} + \sum_{i=1}^{n} \eta_i \frac{zg_i''(z)}{g_i'(z)}.$$

Thus, we have

$$\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1\right) = \sum_{i=1}^{n} (\gamma_i - 1) \operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \eta_i \operatorname{Re}\left(\frac{zg_i''(z)}{g_i'(z)}\right) + 1.$$

Since Re $w \leq |w|$, then

$$\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1\right) \leq \sum_{i=1}^{n} (\gamma_i - 1) \left|\frac{zf_i'(z)}{f_i(z)}\right| + \sum_{i=1}^{n} \eta_i \left|\frac{zg_i''(z)}{g_i'(z)}\right| + 1. \tag{4}$$

Using that $f_i \in \mathcal{MT}(\mu_i, \beta_i)$ for all $i = 1, 2, ..., n$ in relation (4), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1\right) &\leq \sum_{i=1}^{n} (\gamma_i - 1) \left(\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + 1\right) + \sum_{i=1}^{n} \eta_i \left|\frac{zg_i''(z)}{g_i'(z)}\right| + 1 \\
&< \sum_{i=1}^{n} (\gamma_i - 1) \beta_i \left|\mu_i \frac{zf_i'(z)}{f_i(z)} + 1\right| + \sum_{i=1}^{n} (\gamma_i - 1) + \\
&+ \sum_{i=1}^{n} \eta_i \left|\frac{zg_i''(z)}{g_i'(z)}\right| + 1
\end{aligned}$$

and using the hypothesis (2) and (3) in this last relation we have

$$\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right) \quad < \quad \sum_{i=1}^{n}[(\gamma_i-1)(\beta_i\mu_iM_i+\beta_i+1)+\eta_iN_i]+1=\rho$$

This completes the proof of our theorem. $\square$

Letting $n=1$, $\gamma_1=\gamma$, $\eta_1=\eta$, $M_1=M$, $N_1=N$, $\mu_1=\mu$, $\beta_1=\beta$, $f_1=f$ and $g_1=g$ in Theorem 2.1, we have

Corollary 2.1. *Let $\gamma\in\mathbb{R}$, $\gamma>1$, $\eta\in\mathbb{R}$, $\eta>0$, the functions $f\in\mathcal{MT}(\mu,\beta)$, $0<\beta\le 1$, $0\le\mu<1$ and $g\in\mathcal{A}$ satisfying the conditions*

$$\left|\frac{f'(z)}{f(z)}\right|<M,\ (M\ge 1)\quad \text{and}\quad \left|\frac{g''(z)}{g'(z)}\right|<N,\ (N\ge 1).$$

Then the integral operator

$$G_1(z)=\int_0^z (f(t))^{\gamma-1}(g'(t))^{\eta}\mathrm{d}t$$

is in $\mathcal{N}(\rho)$, where $\rho=(\gamma-1)(\beta\mu M+\beta+1)+\eta N+1$.

Letting $\gamma=2$, $\eta=1$ in Corollary 2.1, we have

Corollary 2.2. *Let $f\in\mathcal{MT}(\mu,\beta)$, $0<\beta\le 1$, $0\le\mu<1$ and $g\in\mathcal{A}$ satisfying the conditions*

$$\left|\frac{f'(z)}{f(z)}\right|<M,\ (M\ge 1)\quad \text{and}\quad \left|\frac{g''(z)}{g'(z)}\right|<N,\ (N\ge 1).$$

Then the integral operator

$$G(z)=\int_0^z f(t)g'(t)\mathrm{d}t$$

is in $\mathcal{N}(\rho)$, where $\rho=\beta(\mu M+1)+N+2$.

Theorem 2.2. *Let $\gamma_i\in\mathbb{R}$, $\gamma_i>1$, $\eta_i\in\mathbb{R}$, $\eta_i>0$ for all $i=1,2,...,n$, the functions $f_i\in\mathcal{B}(\mu_i,\alpha_i)$, $\mu_i\ge 0$, $0\le\alpha_i<1$ satisfying the conditions $|f_i(z)|\le M_i$, $(M_i\ge 1)$ and $g_i\in\mathcal{N}(\rho_i)$, $\rho_i>1$ for all $i=1,2,...,n$. If*

$$\sum_{i=1}^{n}\left[(\gamma_i-1)(2-\alpha_i)M_i^{\mu_i-1}+\eta_i(\rho_i-1)\right]<1,$$

then the integral operator G_n defined in (1) is in $K(\delta)$, where

$$\delta=1-\sum_{i=1}^{n}\left[(\gamma_i-1)(2-\alpha_i)M_i^{\mu_i-1}+\eta_i(\rho_i-1)\right].$$

Proof. From (1), we have

$$G_n'(z) = \prod_{i=1}^{n} (f_i(z))^{\gamma_i - 1} (g_i'(z))^{\eta_i}$$

and

$$\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} (\gamma_i - 1) \frac{zf_i'(z)}{f_i(z)} + \sum_{i=1}^{n} \eta_i \frac{zg_i''(z)}{g_i'(z)}.$$

Hence

$$\begin{aligned} \left| \frac{zG_n''(z)}{G_n'(z)} \right| &\le \sum_{i=1}^{n} (\gamma_i - 1) \left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) \left| \frac{f_i(z)}{z} \right|^{\mu_i - 1} \\ &+ \sum_{i=1}^{n} \eta_i \left(\left| \frac{zg_i''(z)}{g_i'(z)} + 1 \right| - 1 \right) \end{aligned} \tag{5}$$

Since $|f_i(z)| \le M_i$ for all $i = 1, 2, ..., n$, applying the General Schwarz Lemma, it results

$$\left| \frac{f_i(z)}{z} \right| \le M_i \text{ for all } i = 1, 2, ..., n. \tag{6}$$

From (5) and (6), using that $f_i \in \mathcal{B}(\mu_i, \alpha_i)$ and $g_i \in \mathcal{N}(\rho_i)$ for all $i = 1, 2, ..., n$, we obtain

$$\left| \frac{zG_n''(z)}{G_n'(z)} \right| < \sum_{i=1}^{n} \left[(\gamma_i - 1)(2 - \alpha_i) M_i^{\mu_i - 1} + \eta_i(\rho_i - 1) \right] = 1 - \delta$$

This completes the proof of our theorem. □

Letting $n = 1$, $\gamma_1 = \gamma$, $\eta_1 = \eta$, $M_1 = M$, $\mu_1 = \mu$, $\alpha_1 = \alpha$, $\rho_1 = \rho$, $f_1 = f$ and $g_1 = g$ in Theorem 2.2, we have

Corollary 2.3. *Let $\gamma \in \mathbb{R}$, $\gamma > 1$, $\eta \in \mathbb{R}$, $\eta > 0$, the functions $f \in \mathcal{B}(\mu, \alpha)$, $\mu \ge 0$, $0 \le \alpha < 1$, satisfying the condition $|f(z)| \le M$, $(M \ge 1)$ and $g \in \mathcal{N}(\rho)$, $\rho > 1$. If*

$$(\gamma - 1)(2 - \alpha) M^{\mu - 1} + \eta(\rho - 1) < 1$$

then the integral operator

$$G_1(z) = \int_0^z (f(t))^{\gamma - 1} (g'(t))^{\eta} \mathrm{d}t$$

is in $K(\delta)$, where $\delta = 1 + (\gamma - 1)(\alpha - 2) M^{\mu - 1} + \eta(1 - \rho)$.

Letting $\mu_i = 0$ and $M_i = M$ for all $i = 1, 2, ..., n$ in Theorem 2.2, we have

Corollary 2.4. *Let $\gamma_i \in \mathbb{R}$, $\gamma_i > 1$, $\eta_i \in \mathbb{R}$, $\eta_i > 0$ for all $i = 1, 2, ..., n$, the functions $f_i \in R(\alpha_i)$, $0 \leq \alpha_i < 1$, satisfying the conditions $|f_i(z)| \leq M$, $(M \geq 1)$ and $g_i \in \mathcal{N}(\rho_i)$, $\rho_i > 1$ for all $i = 1, 2, ..., n$. If*

$$\sum_{i=1}^{n} \left[(\gamma_i - 1)(2 - \alpha_i) \frac{1}{M} + \eta_i(\rho_i - 1) \right] < 1$$

then the integral operator G_n defined in (1) is in $K(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n} \left[(\gamma_i - 1)(2 - \alpha_i) \frac{1}{M} + \eta_i(\rho_i - 1) \right].$$

Letting $\mu_i = 1$ and $M_i = M$ for all $i = 1, 2, ..., n$ in Theorem 2.2, we have

Corollary 2.5. *Let $\gamma_i \in \mathbb{R}$, $\gamma_i > 1$, $\eta_i \in \mathbb{R}$, $\eta_i > 0$ for all $i = 1, 2, ..., n$, the functions $f_i \in S^*(\alpha_i)$, $0 \leq \alpha_i < 1$, satisfying the conditions $|f_i(z)| \leq M$, $(M \geq 1)$ and $g_i \in \mathcal{N}(\rho_i)$, $\rho_i > 1$ for all $i = 1, 2, ..., n$. If*

$$\sum_{i=1}^{n} [(\gamma_i - 1)(2 - \alpha_i) + \eta_i(\rho_i - 1)] < 1$$

then the integral operator G_n defined in (1) is in $K(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n} [(\gamma_i - 1)(2 - \alpha_i) + \eta_i(\rho_i - 1)].$$

Theorem 2.3. *Let $\gamma_i \in \mathbb{R}$, $\gamma_i > 1$, $\eta_i \in \mathbb{R}$, $\eta_i > 0$ for all $i = 1, 2, ..., n$, the functions $f_i \in \rho_i - S_p(\varepsilon_i)$, $-1 \leq \varepsilon_i \leq 1$, $\rho_i > 0$ and $g_i \in KD(\mu_i, \alpha_i)$, $0 \leq \alpha_i < 1$, $\mu_i \geq 0$ for all $i = 1, 2, ..., n$. If*

$$0 < \sum_{i=1}^{n} [(1 - \gamma_i)\varepsilon_i + \eta_i(1 - \alpha_i)] \leq 1$$

then the integral operator G_n defined in (1) is in $K(\delta)$, where

$$\delta = 1 + \sum_{i=1}^{n} [(\gamma_i - 1)\varepsilon_i + \eta_i(\alpha_i - 1)].$$

Proof. Following the same steps as in Theorem 2.1, we obtain that

$$\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} (\gamma_i - 1) \frac{zf_i'(z)}{f_i(z)} + \sum_{i=1}^{n} \eta_i \frac{zg_i''(z)}{g_i'(z)}$$

and hence

$$\begin{aligned}\frac{zG_n''(z)}{G_n'(z)}+1 &= \sum_{i=1}^{n}\left[(\gamma_i-1)\left(\frac{zf_i'(z)}{f_i(z)}-\varepsilon_i\right)+(\gamma_i-1)\varepsilon_i\right]+\\ &+ \sum_{i=1}^{n}\left[\eta_i\left(\frac{zg_i''(z)}{g_i'(z)}+1\right)-\eta_i\right]+1.\end{aligned}$$

We calculate the real part from both terms of the above expression and obtain

$$\begin{aligned}\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right) &= \sum_{i=1}^{n}(\gamma_i-1)\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}-\varepsilon_i\right)+\sum_{i=1}^{n}(\gamma_i-1)\varepsilon_i\\ &+ \sum_{i=1}^{n}\eta_i\operatorname{Re}\left(\frac{zg_i''(z)}{g_i'(z)}+1\right)-\sum_{i=1}^{n}\eta_i+1.\end{aligned} \tag{7}$$

From (7), using that $f_i\in\rho_i-S_p(\varepsilon_i)$ and $g_i\in KD(\mu_i,\alpha_i)$ for all $i=1,2,...,n$, we have

$$\begin{aligned}\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right) &\geq \sum_{i=1}^{n}(\gamma_i-1)\rho_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\sum_{i=1}^{n}(\gamma_i-1)\varepsilon_i+\\ &+ \sum_{i=1}^{n}\eta_i\left(\mu_i\left|\frac{zg_i''(z)}{g_i'(z)}\right|+\alpha_i\right)-\sum_{i=1}^{n}\eta_i+1.\end{aligned}$$

Then

$$\begin{aligned}\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right) &\geq \sum_{i=1}^{n}(\gamma_i-1)\rho_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\sum_{i=1}^{n}(\gamma_i-1)\varepsilon_i+\\ &+ \sum_{i=1}^{n}\eta_i\mu_i\left|\frac{zg_i''(z)}{g_i'(z)}\right|+\sum_{i=1}^{n}\eta_i\left(\alpha_i-1\right)+1.\end{aligned} \tag{8}$$

Since $(\gamma_i-1)\rho_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|>0$ and $\eta_i\mu_i\left|\frac{zg_i''(z)}{g_i'(z)}\right|\geq 0$ for all $i=1,2,...,n$, we obtain from (8) that

$$\operatorname{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right) > \sum_{i=1}^{n}\left[(\gamma_i-1)\varepsilon_i+\eta_i(\alpha_i-1)\right]+1=\delta.$$

This completes the proof of our theorem. □

Letting $n=1$, $\gamma_1=\gamma$, $\eta_1=\eta$, $\rho_1=\rho$, $\varepsilon_1=\varepsilon$ $\mu_1=\mu$, $\alpha_1=\alpha$, $f_1=f$ and $g_1=g$ in Theorem 2.3, we have

Corollary 2.6. *Let $\gamma\in\mathbb{R}$, $\gamma>1$, $\eta\in\mathbb{R}$, $\eta>0$, the functions $f\in\rho-S_p(\varepsilon)$, $-1\leq\varepsilon\leq 1$, $\rho>0$ and $g\in KD(\mu,\alpha)$, $0\leq\alpha<1$, $\mu\geq 0$. If*

$$0<(1-\gamma)\varepsilon+\eta(1-\alpha)\leq 1$$

then the integral operator

$$G_1(z)=\int_0^z\left(f(t)\right)^{\gamma-1}\left(g'(t)\right)^{\eta}\mathrm{d}t$$

is in $K(\delta)$, where $\delta=1+(\gamma-1)\varepsilon+\eta(\alpha-1)$.

Letting $\gamma = 2$ and $\eta = 1$ in Corollary 2.6, we have

Corollary 2.7. *Let the functions* $f \in \rho - S_p(\varepsilon)$, $-1 \le \varepsilon \le 1$, $\rho > 0$ *and* $g \in KD(\mu, \alpha)$, $0 \le \alpha < 1$, $\mu \ge 0$. *If*

$$0 < 1 - \alpha - \varepsilon \le 1$$

then the integral operator

$$G(z) = \int_0^z f(t)g'(t)\mathrm{d}t$$

is in $K(\delta)$, *where* $\delta = \varepsilon + \alpha$.

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