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Statistical Approximation Properties of a Generalization of Positive Linear Operators

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Abstract. In the present paper, we introduce a generalization of positive linear operators and obtain its Korovkin type statistical approximation properties. The rates of statistical convergence of this generalization is also obtained by means of modulus of continuity and Lipschitz type maximal functions. Secondly, we construct a bivariate generalization of these operators and investigate the statistical approximation properties. We also get a partial differential equation such that the second moment of our bivariate operators is a particular solution of it. Finally, we obtain a Voronovskaja type formulae via statistical limit.
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1. Introduction

There are a lot of approximating operators that their Korovkin type error estimates, approximation properties and rates of convergence are investigated (see [1] for details).

In the present paper, Korovkin type statistical approximation properties of a generalization of positive linear operators including many well-known operators which was defined by Doğru in [4] are investigated.

These operators are introduced as

$$L_n(f;x) = \frac{1}{\varphi_n(x)} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{a_{n,\nu}}\right) \varphi_n^{(\nu)}(0) \frac{x^\nu}{\nu!} \tag{1}$$

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where $a_{n,\nu} = \frac{\varphi_n^{(\nu)}(0)}{\varphi_n^{(\nu-1)}(0)}$, $\lim\limits_{\nu\to\infty} \frac{a_{n,\nu}}{\nu} = \mu$, $0 \le x < \frac{1}{\mu}$ and $f \in C[0, \frac{1}{\mu})$.

Here $\varphi_n(x) \in C^\infty$ satisfies the following conditions:

(i) Every element of the sequence $\{\varphi_n\}$ is analytic on a domain D containing the disk $B = \left\{z \in \mathbb{C} : |z| < \frac{1}{\mu}\right\}$,

(ii) $\varphi_n^{(\nu)}(0) = \frac{d^\nu}{dx^\nu}\varphi_n(x)|_{x=0} > 0$ for $\nu = 1, 2, \ldots$,

(iii) $\varphi_n(x) > 0$ for each $x \in [0, \frac{1}{\mu})$,

(iv) There exists a sequence of $\{c_n\}$ such that $\left|\frac{\nu+1}{a_{n,\nu+1}} - \frac{\nu}{a_{n,\nu}}\right| \le c_n$

and $st \lim\limits_{n\to\infty} c_n = 0$.

Firstly, let us recall some notations and definitions on the concept of statistical convergence.

A sequence $x = (x_k)$ is said to be statistically convergent to a number of L if for every $\varepsilon > 0$,

$$\delta\left\{k \in \mathbb{N} : \left|x_k - L\right| \geqslant \varepsilon\right\} = 0$$

where $\delta(K) := \lim\limits_n \frac{1}{n}\{$ the number $k \leqslant n : k \in K\}$ whenever the limit exist [see e.g. 8]. For instance,

$$\delta(\mathbb{N}) = 1, \delta\{2k : k \in \mathbb{N}\} = \frac{1}{2} \text{ and } \delta\left\{k^2 : k \in \mathbb{N}\right\} = 0.$$

Notice that any convergent sequence is statistically convergent but not conversely. For example, the sequence

$$x_k = \begin{cases} L_1, & n = m^2, \\ & \\ L_2, & n \ne m^2 \end{cases} \quad (m = 1, 2, 3, \ldots)$$

is statistically convergent to L_2 but not convergent in ordinary sense when $L_1 \ne L_2$.

In this paper, we also define the bivariate operators for these operators and examine their statistical convergence and finally an application to partial differential equations is given.

2. Korovkin Type Statistical Approximation Properties

In [5], Gadjiev and Orhan proved the following Korovkin-type statistical approximation theorem for any sequence of positive linear operators.

Theorem 1 ([5]). *If the sequence of positive linear operators*

$$A_n \colon C_M[a, b] \to C[a, b]$$

satisfies the conditions

$$st{-}\lim_n \left\|A_n(e_\nu)-e_\nu\right\|_{C[a,b]}=0, with\ e_\nu(t)=t^\nu, for\ \nu=0,1,2,$$

then, for any function $f\in C_M[a,b]$, *we have*

$$st{-}\lim_n \left\|A_n(f)-f\right\|_{C[a,b]}=0.$$

The space of all functions f *which are continuous in* $[a,b]$ *and bounded all positive axis is denoted by* $C_M[a,b]$.

To obtain main results of this part, let us recall some lemmas given in [4]

Lemma 1 ([4])**.** *For all* $n\in N,\ x\in[0,a]$, $(0<a<\frac{1}{\mu})$, *we have*

$$L_n(e_0,x)=1. \tag{2}$$

Lemma 2 ([4])**.** *For all* $n\in N,\ x\in[0,a]$, $(0<a<\frac{1}{\mu})$, *we have*

$$L_n(e_1,x)=x. \tag{3}$$

Lemma 3 ([4])**.** *For all* $n\in N,\ x\in[0,a]$, $(0<a<\frac{1}{\mu})$, *we have*

$$\left|L_n(e_2,x)-x^2\right|\le c_n x. \tag{4}$$

Now, we can obtain the following main result for the operators given by (1).

Theorem 2. *For all* $f\in C_M[0,a]$, $(0<a<\frac{1}{\mu})$, *we have*

$$st-\lim_n \left\|L_n(f;.)-f\right\|_{C[0,a]}=0.$$

Proof. By Lemma 1 and Lemma 2 it is clear that,

$$st-\lim_n \left\|L_n(e_0;.)-e_0\right\|_{C[0,a]}=0 \tag{5}$$

and

$$st-\lim_n \left\|L_n(e_1;.)-e_1\right\|_{C[0,a]}=0. \tag{6}$$

From Lemma 3, we have

$$\left\|L_n(e_2;.)-e_2\right\|_{C[0,a]}\le c_n a. \tag{7}$$

Now, for a given $\varepsilon>0$, let us define the following sets:

$$T:=\left\{k:\left\|L_k(e_2;.)-e_2\right\|_{C[0,a]}\geqslant\varepsilon\right\}$$

and

$$T_1:=\left\{k:c_k a\geqslant\varepsilon\right\}.$$

We can see that $T \subseteq T_1$ by (7) so, we get

$$\delta\left\{k \leqslant n : \left\|L_k(e_2;.) - e_2\right\|_{C[0,a]} \geqslant \varepsilon\right\} \leqslant \delta\left\{k \leqslant n : c_k a \geqslant \varepsilon\right\}.$$

Using the $st - \lim_{n\to\infty} c_n = 0$ we have,

$$st - \lim_n \left\|L_n(e_2;.) - e_2\right\|_{C[0,a]} = 0. \tag{8}$$

Consequently, we can write

$$st - \lim_n \left\|L_n(e_\nu;.) - e_\nu\right\|_{C[0,a]} = 0, \text{ for } \nu = 0,1,2. \tag{9}$$

So the proof is completed from Theorem 1.

3. Rates of Statistical Convergence

Let $f \in C[0,a]$, the modulus of continuity of f, denoted by $\omega(f,\delta)$ is defined as

$$\omega(f,\delta) := \sup_{x,t\in[0,a],\ |t-x|\leq\delta} \left|f(t) - f(x)\right|. \tag{10}$$

At this point let us recall following theorems which were proved in [4].

Theorem 3 ([4]). *Let $f \in C[0,a]$. If L_n is defined by (1), then we have*

$$\left\|L_n(f;.) - f\right\| \leq (1+\sqrt{a})\omega(f,\sqrt{c_n}) \tag{11}$$

where $\omega(f,\sqrt{c_n})$ is modulus of continuity defined in (10) and $\lim_{n\to\infty} c_n = 0$.

The Lipschitz type maximal functions of order α introduced by Lenze [7] as follows

$$\widetilde{\omega}_\alpha(f,x) := \sup_{t\neq x;\ t\in[0,a]} \frac{\left|f(t) - f(x)\right|}{|t-x|^\alpha},\ x\in[0,a],\ \alpha\in(0,1].$$

Notice that, the boundedness of $\widetilde{\omega}_\alpha(f,x)$ is equivalent to $f \in Lip_M(\alpha)$.

Now let us compute the rate of convergence for the difference $\left|L_n(f;x) - f(x)\right|$ with the help of Lipschitz type maximal functions.

Theorem 4 ([4]). *If L_n is defined by (1), then we have*

$$\left|L_n(f;x) - f(x)\right| \leq \left(c_n x\right)^{\frac{\alpha}{2}} \widetilde{\omega}_\alpha(f,x). \tag{12}$$

Remark 1. *Achieving a fast order of statistical convergence is important in approximation by positive linear operators. If we replace $\lim_{n\to\infty} c_n = 0$ by $st - \lim_{n\to\infty} c_n = 0$ in Theorem 3 and Theorem 4, it is obvious that*

$$st - \lim_{n\to\infty} \omega(f,\sqrt{c_n}) = 0.$$

So, Theorems 3 and 4 give us the rates of statistical convergence of the operators $L_n(f;.)$ to f.

4. Construction of the Bivariate Operators

Let $I^2 = [0,a] \times [0,a]$, $(0 < a < \frac{1}{\mu})$, and $f \in C([0,a]^2)$

$$L_n^x(f;x,y) = \frac{1}{\varphi_n(x)} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{a_{n,\nu}}, y\right) \varphi_n^{(\nu)}(0) \frac{x^\nu}{\nu!}$$

and

$$L_m^y(f;x,y) = \frac{1}{\varphi_m(y)} \sum_{\eta=0}^{\infty} f\left(x, \frac{\eta}{b_{m,\eta}}\right) \varphi_m^{(\eta)}(0) \frac{y^\eta}{\eta!}$$

where

$$a_{n,\nu} = \frac{\varphi_n^{(\nu)}(0)}{\varphi_n^{(\nu-1)}(0)}, \ \lim_{\nu\to\infty} \frac{a_{n,\nu}}{\nu} = \mu, \ 0 \le x < \frac{1}{\mu} \text{ and } f \in C([0,\frac{1}{\mu}) \times [0,\frac{1}{\mu}))$$

and

$$b_{m,\eta} = \frac{\varphi_m^{(\eta)}(0)}{\varphi_m^{(\eta-1)}(0)}, \ \lim_{\eta\to\infty} \frac{b_{m,\eta}}{\eta} = \mu, \ 0 \le y < \frac{1}{\mu} \text{ and } f \in C([0,\frac{1}{\mu}) \times [0,\frac{1}{\mu})).$$

Here $\varphi_n(x) \in C^\infty$ and $\varphi_m(y) \in C^\infty$ satisfy the following conditions:

(a) Every element of the sequence $\{\varphi_n\}$ and $\{\varphi_m\}$ are analytic on a domain D containing the disk $B = \left\{z \in \mathbb{C} : |z| < \frac{1}{\mu}\right\}$,

(b) $\varphi_n^{(\nu)}(0) = \frac{d^\nu}{dx^\nu}\varphi_n(x)|_{x=0} > 0$ for $\nu = 1,2,\ldots$, and $\varphi_m^{(\eta)}(0) = \frac{d^\eta}{dy^\eta}\varphi_m(y)\Big|_{y=0} > 0$ for $\eta = 1,2,\ldots$,

(c) $\varphi_n(x) > 0$ for each $x \in [0,\frac{1}{\mu})$, and $\varphi_m(y) > 0$ for each $y \in [0,\frac{1}{\mu})$,

(d) There exists a sequence of $\{c_n\}$ such that $\left|\frac{\nu+1}{a_{n,\nu+1}} - \frac{\nu}{a_{n,\nu}}\right| \le c_n$ and $st - \lim_{n\to\infty} c_n = 0$ and a sequence of $\{d_m\}$ such that $\left|\frac{\eta+1}{b_{m,\eta+1}} - \frac{\eta}{b_{m,\eta}}\right| \le d_m$ and $st - \lim_{m\to\infty} d_m = 0$.

Now we can define the following bivariate generalization of linear and positive operators

$$L_{n,m}(f;x,y) = \frac{1}{\varphi_n(x)} \frac{1}{\varphi_m(y)} \sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} f\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right) \varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0) \frac{x^\nu}{\nu!}\frac{y^\eta}{\eta!}. \tag{13}$$

Lemma 4. *For the operators* (13)*, we have*

$$L_{n,m}(f;x,y) = L_n^x\left(L_m^y(f;x,y)\right) = L_m^y\left(L_n^x(f;x,y)\right).$$

Proof. Following calculations reveal that

$$L_n^x\left(L_m^y(f;x,y)\right) = \frac{1}{\varphi_m(y)} \sum_{\eta=0}^{\infty} \frac{1}{\varphi_n(x)} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right) \varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}\varphi_m^{(\eta)}(0)\frac{y^\eta}{\eta!}$$

$$
\begin{aligned}
&= \frac{1}{\varphi_n(x)} \frac{1}{\varphi_m(y)} \sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty} f\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right) \varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0) \frac{x^\nu}{\nu!}\frac{y^\eta}{\eta!} \\
&= L_{n,m}(f;x,y).
\end{aligned}
$$

Similarly we can easily show that $L_m^y\left(L_n^x(f;x,y)\right) = L_{n,m}(f;x,y)$.

5. Statistical Approximation Properties of the Bivariate Operators

If we have

$$st - \lim_{n,m} \left\| f_{n,m} - f \right\|_{C([a,b]\times[c,d])} = 0$$

then we say that the sequence of functions $\left\{f_{n,m}\right\}$ statistically convergent to f uniformly. Where

$$\left\| f \right\|_{C([a,b]\times[c,d])} = \max_{(x,y)\in[a,b]\times[c,d]} \left| f(x,y) \right|.$$

Volkov [9] gave the first Korovkin type theorem for bivariate functions. Subsequently, H.H. Gonska, C. Badea and I. Badea established a simpler form of Volkov's theorem as follows:

Theorem 5 ([6]). *Let a, b, c, d be real numbers satisfying the inequalities $a < b$, $c < d$ and let*

$$L_{n,m} : C([a,b]\times[c,d]) \to C([a,b]\times[c,d])$$

be a positive linear operators having the properties for any $(x,y) \in [a,b]\times[c,d]$

(1) $L_{n,m}(e_{00};x,y) = 1 + u_{n,m}(x,y)$,

(2) $L_{n,m}(e_{10};x,y) = x + v_{n,m}(x,y)$,

(3) $L_{n,m}(e_{01};x,y) = y + w_{n,m}(x,y)$,

(4) $L_{n,m}(e_{20}+e_{02};x,y) = x^2 + y^2 + h_{n,m}(x,y)$.

If the sequences $\left\{u_{n,m}(x,y)\right\}$, $\left\{v_{n,m}(x,y)\right\}$, $\left\{w_{n,m}(x,y)\right\}$, $\left\{h_{n,m}(x,y)\right\}$ converge to zero uniformly on $[a,b]\times[c,d]$, then $(L_{n,m}f)$ converges to f uniformly on $[a,b]\times[c,d]$ for any $f \in C([a,b]\times[c,d])$ where $e_{i,j} = x^i y^j$ are two dimensional test functions.

Lemma 5. *The bivariate operators in* (13) *satisfy the following items:*

(i) $L_{n,m}(e_{00};x,y) = 1$,

(ii) $L_{n,m}(e_{10};x,y) = x$,

(iii) $L_{n,m}(e_{01};x,y) = y$,

(iv) $\left|L_{n,m}(e_{20}+e_{02};x,y) - x^2 - y^2\right| \leqslant c_n x + d_m y$ *where c_n and d_n satisfy the properties in* (d).

Proof.

(i) It is obvious that

$$\begin{aligned} L_{n,m}\left(e_{00};x,y\right) &= L_{n,m}\left(1;x,y\right) \\ &= \frac{1}{\varphi_n\left(x\right)}\frac{1}{\varphi_m\left(y\right)}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^{\nu}}{\nu!}\frac{y^{\eta}}{\eta!}. \end{aligned}$$

By using Lemma 1, we have $L_{n,m}\left(e_{00};x,y\right)=1$.

(ii)

$$L_{n,m}\left(e_{10};x,y\right)=\frac{1}{\varphi_n\left(x\right)}\frac{1}{\varphi_m\left(y\right)}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{\nu}{a_{n,\nu}}\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^{\nu}}{\nu!}\frac{y^{\eta}}{\eta!}$$

by using Lemma 2, we can easily see that $L_{n,m}\left(e_{10};x,y\right)=x$.

(iii) It is proven by similarly way like (ii).

(iv) Since

$$\begin{aligned} L_{n,m}\left(e_{20}+e_{02};x,y\right) &= \frac{1}{\varphi_n\left(x\right)}\frac{1}{\varphi_m\left(y\right)}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\left[\left(\frac{\nu}{a_{n,\nu}}\right)^2+\left(\frac{\eta}{b_{m,\eta}}\right)^2\right] \\ &\quad\times\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^{\nu}}{\nu!}\frac{y^{\eta}}{\eta!}, \end{aligned}$$

by using Lemma 3 the proof is completed.

Theorem 6. *The sequence* $\left(L_{n,m}f\right)$ *defined by* (13) *converges statistically to* $f\in C\left([0,a]\times[0,a]\right)$ *uniformly in* $[0,a]\times[0,a]$.

Proof.

$$st-\lim_{n,m}\left\|L_{n,m}\left(e_{00};.,.\right)-e_{00}\right\|=0, \tag{14}$$

$$st-\lim_{n,m}\left\|L_{n,m}\left(e_{10};.,.\right)-e_{10}\right\|=0, \tag{15}$$

$$st-\lim_{n,m}\left\|L_{n,m}\left(e_{01};.,.\right)-e_{01}\right\|=0 \tag{16}$$

and from the property (d), we can easily obtain

$$st-\lim_{n,m}\left\|L_{n,m}\left(e_{20}+e_{02};.,.\right)-e_{20}-e_{02}\right\|=0. \tag{17}$$

Using (14), (15), (16), (17), in the light of Theorem 3, we have

$$st-\lim_{n,m}\left\|L_{n,m}\left(f;.,.\right)-f\right\|=0. \tag{18}$$

6. Estimation of the Rate of Statistical Convergence of the Bivariate Operators

Definition 1 ([2]). *$I^2=[0,a]\times[0,a]$, $f\in C\left(I^2\right)$ for any $\delta_1>0$, $\delta_2>0$*

$$\omega\left(f;\delta_1,\delta_2\right)=\sup_{\substack{(t,s)\in I^2,(x,y)\in I^2\\ |t-x|\le\delta_1,|s-y|\le\delta_2}}\left|f\left(t,s\right)-f\left(x,y\right)\right|. \tag{19}$$

Theorem 7. *If $\left(L_{n,m}f\right)$ is defined by* (13) *then we have*

$$\left\|L_{n,m}\left(f;.,.\right)-f\right\|_{C\left(I^2\right)}\leqslant\omega\left(f;\sqrt{c_n},\sqrt{d_m}\right)\left(\sqrt{a}+1\right)^2. \tag{20}$$

Proof. Using the properties for modulus (19), we have

$$\left|f\left(t,s\right)-f\left(x,y\right)\right|\leqslant\omega\left(f;\delta_1,\delta_2\right)\left(\frac{|t-x|}{\delta_1}+1\right)\left(\frac{|s-y|}{\delta_2}+1\right). \tag{21}$$

On the other hand, for any $n\in\mathbb{N}$, $\left(x,y\right)\in I^2$ we have

$$\left|L_{n,m}\left(f;x,y\right)-f\left(x,y\right)\right|\leqslant L_{n,m}\left(\left|f\left(t,s\right)-f\left(x,y\right)\right|;x,y\right). \tag{22}$$

If we use (21) in (22), then we get

$$\begin{aligned}
\left|L_{n,m}\left(f;x,y\right)-f\left(x,y\right)\right| &\leqslant \omega\left(f;\delta_1,\delta_2\right)\\
&\quad\times L_{n,m}\left(\left(\frac{|t-x|}{\delta_1}+1\right)\left(\frac{|s-y|}{\delta_2}+1\right);x,y\right)\\
&= \omega\left(f;\delta_1,\delta_2\right)\frac{1}{\delta_1\delta_2}\frac{1}{\varphi_n\left(x\right)}\frac{1}{\varphi_m\left(y\right)}\\
&\quad\times\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\left|\frac{\nu}{a_{n,\nu}}-x\right|\left|\frac{\eta}{b_{m,\eta}}-y\right|\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\\
&\quad\times\frac{x^\nu}{\nu!}\frac{y^\eta}{\eta!}\\
&\quad+\omega\left(f;\delta_1,\delta_2\right)\frac{1}{\delta_1}\frac{1}{\varphi_n\left(x\right)}\frac{1}{\varphi_m\left(y\right)}\\
&\quad\times\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\left|\frac{\nu}{a_{n,\nu}}-x\right|\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^\nu}{\nu!}\frac{y^\eta}{\eta!}\\
&\quad+\omega\left(f;\delta_1,\delta_2\right)\frac{1}{\delta_2}\frac{1}{\varphi_n\left(x\right)}\frac{1}{\varphi_m\left(y\right)}\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}\left|\frac{\eta}{b_{m,\eta}}-y\right|\\
&\quad\times\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^\nu}{\nu!}\frac{y^\eta}{\eta!}+\omega\left(f;\delta_1,\delta_2\right).
\end{aligned}$$

By using Cauchy-Schwarz inequality and Lemmas 1, 2 and 3, then we obtain

$$\begin{aligned}
\left|L_{n,m}(f;x,y)-f(x,y)\right| &\leqslant \frac{\omega(f;\delta_1,\delta_2)}{\delta_1\delta_2} \\
&\quad\times\left[\frac{1}{\varphi_n(x)}\sum_{\nu=0}^{\infty}\left(\frac{\nu}{a_{n,\nu}}-x\right)^2\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}\right]^{\frac{1}{2}} \\
&\quad\times\left[\frac{1}{\varphi_m(y)}\sum_{\eta=0}^{\infty}\left(\frac{\eta}{b_{m,\eta}}-y\right)^2\varphi_m^{(\eta)}(0)\frac{y^\eta}{\eta!}\right]^{\frac{1}{2}} \\
&\quad+\frac{\omega(f;\delta_1,\delta_2)}{\delta_1} \\
&\quad\times\left[\frac{1}{\varphi_n(x)}\sum_{\nu=0}^{\infty}\left(\frac{\nu}{a_{n,\nu}}-x\right)^2\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}\right]^{\frac{1}{2}} \\
&\quad+\frac{\omega(f;\delta_1,\delta_2)}{\delta_2} \\
&\quad\times\left[\frac{1}{\varphi_m(y)}\sum_{\eta=0}^{\infty}\left(\frac{\eta}{b_{m,\eta}}-y\right)^2\varphi_m^{(\eta)}(0)\frac{y^\eta}{\eta!}\right]^{\frac{1}{2}} \\
&\quad+\omega(f;\delta_1,\delta_2) \\
&= \frac{\omega(f;\delta_1,\delta_2)}{\delta_1\delta_2}(c_nx)^{\frac{1}{2}}(d_my)^{\frac{1}{2}}+\frac{\omega(f;\delta_1,\delta_2)}{\delta_1}(c_nx)^{\frac{1}{2}} \\
&\quad+\frac{\omega(f;\delta_1,\delta_2)}{\delta_2}(d_my)^{\frac{1}{2}}+\omega(f;\delta_1,\delta_2).
\end{aligned}$$

If we choose $\delta_1=\sqrt{c_n}$, $\delta_2=\sqrt{d_m}$ in the last inequality then we have

$$\begin{aligned}
\left\|L_{n,m}(f;.,.)-f\right\|_{C(I^2)} &\leqslant \omega\left(f;\sqrt{c_n},\sqrt{d_m}\right)\sqrt{a}\sqrt{a}+\omega\left(f;\sqrt{c_n},\sqrt{d_m}\right)\sqrt{a} \\
&\quad+\omega\left(f;\sqrt{c_n},\sqrt{d_m}\right)\sqrt{a}+\omega\left(f;\sqrt{c_n},\sqrt{d_m}\right) \\
&= \omega\left(f;\sqrt{c_n},\sqrt{d_m}\right)\left(a+2\sqrt{a}+1\right) \\
&= \omega\left(f;\sqrt{c_n},\sqrt{d_m}\right)\left(\sqrt{a}+1\right)^2.
\end{aligned}$$

Remark 2. *Since c_n and d_m satisfy $st-\lim_{n\to\infty}c_n=0$ and $st-\lim_{m\to\infty}d_m=0$, we can easily say that $\left(\left(L_{n,m}f\right)\right)$ is statistically convergent to f on I^2.*

7. Application to Partial Differential Equations

Let $\left(\left(L_{n,m}f\right)\right)$ be as in (13) then we can give the following theorem.

Theorem 8. *Let*

$$\frac{d}{dx}\varphi_n(x) = h_n(x)\varphi_n(x), \tag{23}$$

$$\frac{d}{dy}\varphi_m(y) = h_m(y)\varphi_m(y) \tag{24}$$

and

$$g\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right) = \frac{\nu}{s_n} + \frac{\eta}{t_m}. \tag{25}$$

Then we have

$$\begin{aligned}\frac{x}{s_n}\frac{\partial}{\partial x}L_{n,m}(f;x,y) + \frac{y}{t_m}\frac{\partial}{\partial y}L_{n,m}(f;x,y) = &\left[-\frac{x}{s_n}h_n(x) - \frac{y}{t_m}h_m(y)\right] \\ &\times L_{n,m}(f;x,y) + L_{n,m}(fg;x,y).\end{aligned} \tag{26}$$

Proof. Using the equalities

$$\begin{aligned}\frac{\partial}{\partial x}L_{n,m}(f;x,y) &= \frac{-\varphi_n^{'}(x)}{\varphi_n^2(x)}\frac{1}{\varphi_m(y)} \\ &\times\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}f\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right)\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^\nu}{\nu!}\frac{y^\eta}{\eta!} \\ &+\frac{1}{\varphi_n(x)\varphi_m(y)} \\ &\times\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}f\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right)\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{\nu x^{\nu-1}}{\nu!}\frac{y^\eta}{\eta!}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}L_{n,m}(f;x,y) &= \frac{-\varphi_m^{'}(y)}{\varphi_m^2(y)}\frac{1}{\varphi_n(x)} \\ &\times\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}f\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right)\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^\nu}{\nu!}\frac{y^\eta}{\eta!} \\ &+\frac{1}{\varphi_n(x)\varphi_m(y)} \\ &\times\sum_{\nu=0}^{\infty}\sum_{\eta=0}^{\infty}f\left(\frac{\nu}{a_{n,\nu}}, \frac{\eta}{b_{m,\eta}}\right)\varphi_n^{(\nu)}(0)\varphi_m^{(\eta)}(0)\frac{x^\nu}{\nu!}\frac{\eta y^{\eta-1}}{\eta!},\end{aligned}$$

we get the proof immediately.

8. Voronovskaja Type Approximation Properties

It can be given the following theorem for Voronovskaja type operators via statistical limit.

Lemma 6. *It can be easily showed that*

$$L_n\left(t^3;x\right) \leqslant x^3+3x^2c_n+xc_n^2 \tag{27}$$

and

$$L_n\left(t^4;x\right) \leqslant x^4+6c_nx^3+4c_n^2x^2+c_n^3x \tag{28}$$

Theorem 9. *Let* $\left(L_n\left(f;x\right)\right)$ *as in* (1)

$$st-\lim_{n\longrightarrow\infty}\frac{1}{c_n}\left[L_n\left(t;x\right)-f\left(x\right)\right]=\frac{f^{''}\left(x\right)}{2}x \tag{29}$$

Proof. Proof. Necessity. It will be used the same technique in [3] for this proof. It is known from the Taylor expansion

$$f\left(t\right)=f\left(x\right)+f^{'}\left(x\right)\left(t-x\right)+\frac{f^{''}\left(x\right)}{2}\left(t-x\right)^2\eta\left(t-x\right) \tag{30}$$

where $\eta\left(t-x\right)=\frac{f^{'''}(x)}{3!}\left(t-x\right)+\ldots$ and it is a continuous function and tends to zero for $t\rightarrow x$.

Let's choose $t=\frac{\nu}{a_{n,\nu}}$ in (30) then

$$f\left(\frac{\nu}{a_{n,\nu}}\right)=f\left(x\right)+f'\left(x\right)\left(\frac{\nu}{a_{n,\nu}}-x\right)+\frac{f^{''}\left(x\right)}{2}\left(\frac{\nu}{a_{n,\nu}}-x\right)^2+\left(\frac{\nu}{a_{n,\nu}}-x\right)^2\eta\left(\frac{\nu}{a_{n,\nu}}-x\right). \tag{31}$$

Since η is a continuous function, it is bounded and there exists a positive constant H, so for all h, we can write $\left|\eta\left(h\right)\right|\leqslant H$. If (31) is multiplied with $\frac{1}{\varphi_n(x)}\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}$ and taken sum from $\nu=0$ to infinity from both side of it, we have

$$\begin{aligned}\frac{1}{\varphi_n(x)}\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{a_{n,\nu}}\right)\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!} &= f\left(x\right)L_n\left(1;x\right)+f'\left(x\right)L_n\left(t-x;x\right)\\ &+\frac{f^{''}\left(x\right)}{2}L_n\left((t-x)^2;x\right)\\ &+\frac{1}{\varphi_n(x)}\sum_{\nu=0}^{\infty}\left(\frac{\nu}{a_{n,\nu}}-x\right)^2\eta\left(\frac{\nu}{a_{n,\nu}}-x\right)\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}\end{aligned}$$

so

$$L_n\left(f;x\right)=f\left(x\right)+f'\left(x\right)\left[L_n\left(t;x\right)-x\right]+\frac{f^{''}\left(x\right)}{2}\left[L_n\left(t^2;x\right)-2xL_n\left(t;x\right)+x^2\right]+I. \tag{32}$$

where

$$\begin{aligned}I =& \frac{1}{\varphi_n(x)}\sum_{\nu=0}^{\infty}(\frac{\nu}{a_{n,\nu}}-x)^2\eta\left(\frac{\nu}{a_{n,\nu}}-x\right)\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}\\ =& \frac{1}{\varphi_n(x)}\sum_{\substack{\nu=0\\ \left|\frac{\nu}{a_{n,\nu}}-x\right|\leqslant\delta}}^{\infty}(\frac{\nu}{a_{n,\nu}}-x)^2\eta\left(\frac{\nu}{a_{n,\nu}}-x\right)\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}\\ &+\frac{1}{\varphi_n(x)}\sum_{\substack{\nu=0\\ \left|\frac{\nu}{a_{n,\nu}}-x\right|>\delta}}^{\infty}(\frac{\nu}{a_{n,\nu}}-x)^2\eta\left(\frac{\nu}{a_{n,\nu}}-x\right)\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}\end{aligned}\tag{33}$$

Because η is a continuous function for every $\varepsilon>0$ there exists a $\delta(\varepsilon)$, $\left|\eta\left(\frac{\nu}{a_{n,\nu}}-x\right)\right|\leqslant\varepsilon$ and η is bounded for $\left|\frac{\nu}{a_{n,\nu}}-x\right|>\delta$ we have $\left|\eta\left(\frac{\nu}{a_{n,\nu}}-x\right)\right|<H$. If these expressions are used in (33), we have

$$I\leqslant\varepsilon\frac{1}{\varphi_n(x)}\sum_{\nu=0}^{\infty}(\frac{\nu}{a_{n,\nu}}-x)^2\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}+HJ \tag{34}$$

where

$$J=\frac{1}{\varphi_n(x)}\sum_{\substack{\nu=0\\ \left|\frac{\nu}{a_{n,\nu}}-x\right|>\delta}}^{\infty}(\frac{\nu}{a_{n,\nu}}-x)^2\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!} \tag{35}$$

Due to $\left|\frac{\nu}{a_{n,\nu}}-x\right|>\delta$, $\frac{(\frac{\nu}{a_{n,\nu}}-x)^2}{\delta^2}>1$. So

$$J\leqslant\frac{1}{\delta^2}\frac{1}{\varphi_n(x)}\sum_{\nu=0}^{\infty}(\frac{\nu}{a_{n,\nu}}-x)^4\varphi_n^{(\nu)}(0)\frac{x^\nu}{\nu!}. \tag{36}$$

By using (34) and (36)

$$I\leqslant\varepsilon L_n\left((t-x)^2;x\right)+H\frac{1}{\delta^2}L_n\left((t-x)^4;x\right) \tag{37}$$

and using (28) in (37) we obtain

$$L_n(t;x)-f(x)\leqslant c_n\left[\frac{f''(x)}{2}x+\varepsilon x+\frac{H}{\delta^2}xc_n^2\right],$$

so

$$L_n(t;x)-f(x)=o(c_n)\left[\frac{f''(x)}{2}x+\varepsilon x+\frac{H}{\delta^2}xc_n^2\right]$$

Because $st-\lim_{n\to\infty}c_n=0$ and ε is an arbitrary positive constant, we have the proof.

Remark 3. *It is obvious that since c_n tends to zero statistically, we have a better order of approximation in Theorem 9 than Theorem 3.*

References

[1] F. Altomore and M. Campiti. *Korovkin Type Approximation Theory and its Applications*. Walter de Gruyter, Berlin, 1994.

[2] G. Anastassiou and S. Gal. *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*. Birkhäuser, Boston, 2000.

[3] O. Doğru. Necessary conditions to obtain Voronovskaja type asymptotic formulae via statistical limit. *Proc. of the 12th WSEAS Int. Conf. on Applied Mathematics.,*pages 128-131, Cairo, Egypt, 2007.

[4] O. Doğru. Approximation properties of a generalization of positive linear operators. *J. Math. Anal. Appl.*, 342:161-170, 2008.

[5] A. Gadjiev and C. Orhan. Some Approximation Theorems via statistical convergence. *Rocky Mountain J. Math.*, 32:129-138, 2002.

[6] H. Gonska, C. Badea and I. Badea. A Test Function Theorem and Approximation by Pseudopolynomials. *Bull. Austral. Math. Soc.*, 34:53-64, 1986.

[7] B. Lenze. Bernstein-Baskakov-Kantorovich Operators and Lipschitz-Type maximal functions. *in: Approx. Th. Kecskemét, Hungary, Colloq. Math. Soc. János Bolyai,*. 58:469-496, 1990.

[8] I. Niven, H. Zuckerman and H. Montgomery. *An Introduction to the Theory of Numbers*. 5th Edition, Wiley, New York, 1991.

[9] V. Volkov. On the Convergence Sequences of Linear Positive Operators in the Space of Continuous Functions of Two Variables. *(Russian) Dokl. Akad. Nauk. SSSR (N. S.)*. 115:17-19, 1957.