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Morita Theory for Rings and Semigroups

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Abstract. The notion of Morita equivalence for rings defines a relationship between rings in terms of their module categories being equivalent in the sense of category theory. To characterise Morita equivalence for rings, Morita contexts and factors on various bimodules have emerged. As a generalisation of Morita equivalence, the concept of Morita-like equivalences was developed to investigate xst-rings. The study of Morita invariants is also an important branch in the Morita theory for rings.

Analogous to the Morita theory for rings, Morita equivalence and Morita invariants for semigroups have been developed. Four major approaches to the characterisations of Morita equivalence between semigroups have appeared. They are categories of acts over semigroups, Morita contexts, Cauchy completions and enlargements.

The aim of this article is to make a brief survey of Morita equivalence for rings not necessary with an identity and semigroups.

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1. Introduction

The classical Morita theory for rings has been recognised as one of the most important and fundamental tools in studying the structure of rings. In the paper [34] Morita firstly established the Morita equivalent theory for unital rings, that is, rings with identity. There exist two angles to generalise the Morita theory for rings. One is to investigate the Morita theory for different rings. In 1974, Fuller [10] made a first step in extending the theory of Morita equivalence to rings without identity. He considered the categorical equivalences between

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the complete additive subcategory of rings which do not necessarily possess an identity, and the category of unital modules over rings with identity. Such a result was further strengthened and enriched by Sato [41] and Azumaya [4]. Along this direction fruitful results about Morita theory for various rings have been obtained by many authors. In 1983, Abrams initiated the study of Morita theory for rings in which a set of commuting idempotents is given such that every element of the ring admits one of these idempotents as a two-sided unit, and the categories of all left modules over these rings which are unitary in a natural sense. Ánh and Márki [3] further generalised Abrams's result to rings with local units by weaken the condition of commutativity of idempotents. In 1991, Garcia and Simòn [14] made use of a completely new technique of non-commutative localisations to build the Morita theory for idempotent rings. In the last two decades, other generalisations of Morita theory have been widely studied by Komatsu [17], Kguno [18], Nobusawa [35], Pawathi and Ramakrishra Rao [36], Garcia [12], Xu, Shum and Turner-Smith [49] and so on. Recently, Quillen [38] studied non-unital rings and Morita invariance by homotopy categories.

The other way to generalise the Morita theory for rings is to describe the Morita equivalence in terms of different full subcategories of rings. For example, in [10] and [32] they chose a different bimodule P to describe the category equivalences of modules via the functors $P \otimes -$ and $\mathrm{Hom}(P, -)$. The bimodule P discussed in [10] is a quasi-progenerator and the bimodule P considered in [32] is a $*$-module. In [47], Trlifaj remarked that every $*$-module is finitely generated and Colpi [8] noticed that the tilting modules are closely related with the $*$-modules. This fact builds a connection between finite dimensional algebras via equivalent representable functor equivalences between categories of modules.

In [15], Jacobson remarked that classical Morita theory for rings can be expressed as a theory for equivalent matrix rings. In fact, in view of Morita equivalence, we can describe the common "two-sided" algebraic structures of various finite matrix rings with different ranks, for instance, the simple, left Artinean, left Notherian, primitive and semi-simple ring between rings, and also these ring properties are invariant and preserved if they are Morita equivalent. In [49], Xu, Shum and Turner-Smith generalised the classcal Morita theory for finite matrix rings to infinite matrix rings using the matrix approach and replacement techniques. They also defined a class of rings called xst-rings which contain the class of rings with local units. To investigate xst-rings, they defined a new equivalence, namely *Morita-like equivalence*, which is a generalisation of Morita equivalence.

The Morita theory for rings was first extended independently to monoids by Banaschewski [5] and Knauer [16]. In [5] Banaschewski showed that the generalisation of the Morita theory for rings to semigroups is in fact isomorphic in case **R-Act** and **S-Act** are equivalent, with no requirement that acts be unitary in any sense. So one is forced to define Morita equivalence in terms of subcategories if a notion differing from isomorphism is to be obtained. Based on the ideal of developing the Morita theory for rings with identity to the rings without identity, Tarlwar [43] initiated a new way to define the Morita theory for semigroups with local units, where a semigroup S is said to be with *local units* if for each $s \in S$ there exist idempotents e and f such that $es = s = sf$. To generalise the Morita theory for semigroups, there exist two directions. One is to a larger class of semigroups. Observe that if S is a semigroup with local units then it is a factorisable semigroup as it has the property $S^2 = S$. In [44, 45]

Tarlwar further generalised his theory to factorisable semigroups. Then Chen and Shum [6] studied the Mortia equivalence of factorisable semigroups using a different technique from Tarlwar's [45]. Subsequetly, Laan and Márki [20] investigated various classes of factorisable semigroups. The other way to generalise the Morita theory for semigroups with local units is to investigate some special kinds of semigroups with local units. Inverse semigroups can be considered as an important class of semigroups with local units. In [42] Steinberg introduced a strong Morita theory for inverse semigroups in terms of Morita contexts, which turns out to be equivalent with the usual Morita equivalence of inverse semigroups. Recently, Lawson [25] reformulate Tarlwar's theory for semigroups with local units and also gave equivalent characterisations of Morita equivalence in terms of categories of acts over them, Morita contexts, Cauchy completions and enlargements. To study Morita invariants for semigroups is also a focus which many authors pay attention to. In [25] Lawson showed that some important subclasses of regular semigroups are Morita invariant, under the assumption that these semigroups have local units. To get rid of the assumption that semigroups with local units Laan [19] has got some nice results. In [20] the lattice of congruences, the lattice of ideals and so on were discussed based on strongly Morita equivalent semigroups.

In this paper we mainly make a survey of the Morita theory for rings and semigroups. The structure of this paper is as follows. In Section 2 we recall some basic concepts for rings and semigroups. To study Morita equivalence we also introduce several notions and terminology of acts over semigroups and modules. Section 3 discusses the Morita equivalence and Morita invariants for rings with local units and xst-rings. Morita equivalence for semigroups such as semigroups with local units, inverse semigroups, factorisable semigroups and so on is investigate in Section 4.

2. Preliminaries

In this section we mainly present a number of definitions and elementary observations concerning rings and semigroups. For further details of rings, we refer the reader to [22], for semigroup theory to [39] and for category theory to [33].

In this paper, let K be an algebra system. We denote by $E(K)$ the set of all its idempotents.

2.1. Rings and Modules

A ring is *unital* if it has an identity for multiplication. A ring is *commutative* if the multiplication is commutative. We say that R is a *ring with local units* if every finite subset of R is contained in a subring of the form eRe where $e \in E(R)$. A subset E of R is called a *set of local units* (slu) for R in case E is a set of commuting idempotents such that for each x in R there exists an e in E with $ex = xe = x$. Note that if R is a unital ring with identity 1, then $\{1\}$ is an slu for R. If R is a ring with slu then it is a ring with local units, but the converse is not true. In addition, a ring R with slu is a ring with local units whose local units commute.

Let R be a ring. If M is a left R-module we denote it by ${}_RM$. Let $m_1, m_2, \ldots, m_n$ be elements of a left R-module M. Then $m_1, m_2, \ldots, m_n$ are called *generators* of M if for each $m \in M$, there exists $r_1, r_2, \ldots, r_n \in R$ such that $m = r_1m_1 + r_2m_2 + \ldots + r_nm_n$, meanwhile M is called *finitely* *generated*. We call a left R-module *unitary* if $RM = M$, that is, for each $m \in M$, there exist $r_1, \ldots, r_n \in R$ and $m_1, \ldots, m_n \in M$ such that $r_1m_1 + \ldots + r_nm_n = m$. We call a bimodule *unitary* if it is unitary on both sides. A right R module M_R is called *s-unital* if for every $x \in M$ there exists $r \in R$ such that $xr = x$.

Let R be an arbitrary ring. For convenience, the category of left R-modules and the usual left R-homomorphisms is denoted by R-**Mod**. If M and N are left R-modules then $\mathrm{Hom}_R(M, N)$ denotes the set of all left R-homomorphisms from M to N. We use $_R\mathscr{C}$ to denote the full subcategory of unital left R-modules, which is complete additive, that is, it is closed under submodules, epimorphic images and direct sums. Dually, **Mod**-R and $\mathscr{C}_R$ denotes the category of right R-modules and the closed full subcategory unital ring R-modules. Notice that for a ring with local units the category of unital left R-modules is naturally closed and so we denote by R-**UMod** the category of unitary left R-modules and the usual left R-homomorphisms. Dually, **UMod**-R denotes the category of unitary right R-modules. Thus, for a ring R with local units we have $_R\mathscr{C} = R$-**UMod** and $\mathscr{C}_R =$ **UMod**-R.

Let $\mathscr{C}$ be an arbitrary subcategory of R-**Mod**. A left R-module $P \in \mathscr{C}$ is *projective* if for any $N, M \in \mathscr{C}$, every surjective left R-homomorphism $f : N \to M$ and every left R-homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $fh = g$.

Let R be a ring with slu. We say that a *progenerator* for R is a compatible set $\{X_i, \phi_{ij}, \psi_{ji} | i \in I\}$ in R-**UMod** such that

(1) for each $i \in I$, X_i is a finitely generated projective left R-module;

(2) $X = \varinjlim_I (X_i, \phi_{ij})$ is a generator for R-**UMod**.

Let R be a ring with local units. An R-module P is called *locally projective* in case there exists a compatible set $\{P_i, \phi_{ij}, \psi_{ji}, I\}$ such that each P_i is a finitely generated projective R-module, and $P = \varinjlim_I (P_i, \phi_{ij})$. For convenience, we denote a locally projective P by $\{P, \phi, \psi, I\}$.

Let $\{P, \phi, \psi, I\}$ and $\{Q, \tau, \sigma, K\}$ be locally projective R-modules, and let $f \in \mathrm{Hom}_R(P, Q)$. We call f a *localized morphism* from $\{P, \phi, \psi, I\}$ to $\{Q, \tau, \sigma, K\}$ if there exists $i \in I$ such that $f = \psi_i \phi_i f$.

Let R be a ring with slu. Abrams [1] showed that the collection of locally projective R-modules, together with localized morphisms forms a category with slu. Denote such a category by **LP**(R).

Let R and S be rings. A six-tuple $\langle R, S, {}_RP_S, {}_SQ_R, \langle,\rangle, [,]\rangle$ is said to be a *Morita context* if the following conditions hold:

(1) $_RP_S$ is an R-S-bimodule and $_SQ_R$ is an S-R-bimodule;

(2) $\langle,\rangle$ is an R-R-homomorphism of $P \otimes_S Q$ into R, and $[,]$ is an S-S-homomorphism of $Q \otimes_R P$ into S;

(3) for all $p, p' \in P$ and $q, q' \in Q$ we have

$$\langle p, q\rangle p' = p[q, p'] \text{ and } q\langle p, q'\rangle = [q, p]q'.$$

For convenience the images on R and S of bimodule homomorphisms $\langle , \rangle$ and $[,]$ are called the *traces* of the context.

For rings with local units R and S, we call R and S are *Morita equivalent* if R-**UMod** is equivalent to S-**UMod**. Notice that if R and S are unital then R and S are Morita equivalent if and only if R-**Mod** is equivalent to S-**Mod**.

We say that arbitrary rings R and S are *Morita-like equivalent* if $\mathscr{C}_R$ and $\mathscr{C}_S$ are equivalent. Clearly, for rings R with local units we have $\mathscr{C}_R = \mathbf{UMod}\text{-}R$. Thus, the concept of Morita-like equivalent is a generalisation of Morita equivalence for rings with local units.

The Morita theory is not only expressed in terms of categories and Mortia contexts as above, but also described by the matrix formulation.

Let R be a ring and Γ be an arbitrary indexing set. We define

(1) $M_\Gamma(R)$ to be the matrix ring of all $\Gamma \times \Gamma$ row-finite matrices over R (i.e., if $m \in M_\Gamma(R)$ then each row of m has at most a finite number of non-zero entries);

(2) $M_\Gamma^0(R)$ to be the subring of those matrices of $M_\Gamma(R)$ with at most a finite number of non-zero columns (we call such matrices *almost zero-column matrices*).

Notice that if Γ is finite then $M_\Gamma(R) = M_\Gamma^0(R)$ is the matrix ring of all $\Gamma \times \Gamma$ matrices over R. In [15] Morita's definition of equivalence may now be stated as follows:

Rings R and S are Morita equivalent if there exists a natural number n and an idempotent matrix $l \in M_n(R)$ such that

(1) $S \cong lM_n(R)l$;

(2) $M_n(R)lM_n(R) = M_n(R)$.

2.2. Semigroups and Acts

We begin with recalling some definitions needed in the sequel.

An element s of a semigroup S is called *regular* if there exists $s' \in S$ such that $s = ss's$ and $s' = s'ss'$. Here s' is called an *inverse* of s. A semigroup S is *regular* if every element of S is regular. A regular semigroup S is said to be *inverse* if each element of S has a unique inverse. A semigroup with identity is called a *monoid*. In [21] a semigroup S is said to have *local units* if for every $s \in S$ there exist $e, f \in E(S)$ such that $s = es = sf$. Certainly, a monoid is a semigroup with local units. In addition, we have:

Definition 1 ([43]). *Let S be a semigroup. Then*

(1) S is said to be a semigroup with weak local units if for every $s \in S$ there exist $u, v \in S$ such that $s = us = sv$(these semigroups are called semigroups satisfying Condition (P) in [24] and like unity semigroups in [6]);

(2) S is said to be a semigroup with common two-sided local units (called simply "local units" in the ring case [3]) if for every finite subset $S' \subseteq S$ there exists an idempotent $e \in S$ such that $S' \subseteq eSe$; that is, $s = es = se$ for every $s \in S'$;

(3) *S is said to be a semigroup with common two-sided weak local units if for all $s, s' \in S$ there exists $u \in S$ such that $s = us = su$ and $s' = us' = s'u$;*

(4) *S is said to be a semigroup with common joint weak local units if for all $s, s' \in S$ there exist $u, v \in S$ such that $s = usv$ and $s' = us'v$.*

We pause here to make a short observation that if a semigroup has common two-sided local units then it has local units and common two-sided weak local units; the latter property implies having weak local units as well as common joint weak local units; having local units implies having weak local units.

A semigroup T is said to be an *enlargement* of its subsemigroup S if $S = STS$ and $T = TST$. Let S, T and R be semigroups. We say that R is a *joint enlargement* of S and T if it is an enlargement of subsemigroups S' and T' which are isomorphic to S and T respectively. If R is a regular semigroups we say that it is a regular joint enlargement.

Recall that the *Cauchy completion* of a semigroup S is a category with object set $E(S)$, homomorphism sets

$$C(S) = \{(e, s, f) \in E(S) \times S \times E(S) : esf = s\}$$

and composition

$$(e, s, f)(f, t, h) = (e, st, h).$$

Let S be an inverse semigroup. We can construct a left cancellative category with object set S, homomorphism sets

$$L(S) = \{(e, s) \in E(S) \times S : se = s\}$$

and composition $(e, s)(f, t) = (e, st)$ whenever $s * s = f$. In addition, the regular elements of $C(S)$ form an inverse category, $I(S)$, given by

$$I(S) = \{(e, a, f) \in C(S) : a \in \mathrm{Reg}(S)\},$$

where $\mathrm{Reg}(S)$ is the set of regular elements of S.

We shall build semigroups from (small) categories using the following technique. A category C is said to be *strongly connected* if for each pair of identities e and f there is an arrow from e to f. Let C be a strongly connected category. A *consolidation* for C is a function $p : C_0 \times C_0 \to C$, $p(e, f) = p_{e,f}$, where $p_{e,f}$ is an arrow from e to f and $p_{e,e} = e$. Given a category C equipped with a consolidation p we can define a binary operation $\circ$ on C by $x \circ y = xp_{e,f}y$ where x has codomain e and y has domain f. It is easily checked that this converts C into a semigroup. We denote this semigroup by C^p. If we omit $\circ$ then the product is in the category.

We pause here to recall a concept. Let C be a category. We say that $C = [A, B]$ is *bipartite* (with left part A and right part B) if it satisfies the following conditions:

(B1) C has full disjoint subcategories A and B such that $C_0 = A_0 \cup B_0$;

(B2) for each identity $e \in A_0$ there exists an isomorphism x with domain e and codomain in B_0; for each identity $f \in B_0$ there exists an isomorphism y with domain f and codomain in A_0.

In [37], the categories A and B are equivalent if and only if there is a bipartite category with left part A and right part B.

Lemma 1. *[25, Theorem 3.7] Let $C = [A, B]$ be a bipartite category and let r be a consolidation defined on C. Then C_r is an enlargement of both A_r and B_r.*

An inverse semigroup S can also be regarded as an inductive groupoid $G(S)$. Inductive groupoids are ordered groupoid in which the set of identities forms a semilattice. Let S and T be inverse semigroups with associated inductive groupoids $G(S)$ and $G(T)$. A bipartite ordered groupoid enlargement of $G(S)$ and $G(T)$ is an ordered groupoid $[G(S), G(T)]$ such that the set of identities of $[G(S), G(T)]$ is the disjoint union of the set of identities of $G(S)$ and $G(T)$ and for each $e \in G(S)_0$ there exists an arrow x such that the domain of x is e and the codomain of x is contained in $G(T)_0$ and dually.

Let S and T be semigroups with local units. A homomorphism $\theta : S \to T$ is said to be a *local isomorphism* if the following conditions are satisfied:

(LI1) the function θ restricted to eSf induces an isomorphism with $\theta(e)T\theta(f)$ for all idempotents e and f in S;

(LI2) idempotents lift along θ meaning that if e is an idempotent in the image of θ then there is an idempotent e in S such that $\theta(e) = e$;

(LI3) for each idempotent $e \in T$ there exists an idempotent $f \in T$ in the image of θ such that $e \mathscr{D} f$.

This is a generalisation of the classical definition of a local isomorphism between regular semigroups [27, 28] and has its origins in [26] and [24] as well as topos theory. When S is regular, surjective local isomorphisms are precisely the surjective homomorphisms that are injective when restricted to each local submonoid [24].

Let S be a semigroup. If the action of S on the left of the set X we say that X is a *left S-act*. If M and N are left S-acts then $\mathrm{Hom}_S(M, N)$ denotes the set of all left S-homomorphisms from M to N. If M is a right S-act then $\mathrm{Hom}_S(M, N)$ becomes a left S-act when we define $s \cdot f$ by $(m)(s \cdot f) = (ms)f$. In particular, $\mathrm{Hom}_S(S, M)$ is a left S-act. We denote by S-**Act** the category of left S-acts and left S-homomorphisms. A left S-act X is said to be *left unitary* if and only if $SX = X$. If S has local units and X is a unitary left S-act, then it is easy to check that for each $x \in X$ there exists $e \in E(S)$ such that $ex = x$. We denote by S-**UAct** the category consisting of unitary left S-acts and S-homomorphisms.

Let X be a left S-act. The action of S on X induces a map $\mu_X : S \otimes X \to X$, where $S \otimes X$ is the tensor product of S and X. In [25] Lawson showed that μ_X is surjective if and only if X is left unitary. The left S-act X is said to be *closed* if μ_X is surjective and injective. All the closed left S-acts form a full-subcategory of S-**Act**, denoted by S-**FAct** (it is denoted by S-**FixAct** in [43]). Dually we have right S-acts and also we define (S, T)-biacts in the usual way. A biact is *unitary* if it is left-right unitary. A biact is *closed* if it is closed as a left and as a right act. Notice that S-**FAct** $\subseteq$ S-**UAct**, where the inclusion is as full subcategories.

The full subcategory which consists of all objects that are in S-**UAct** and are fixed by the functor $S\mathrm{Hom}_S(S, -)$ is denoted by S-**UFAct**.

We say that a closed left S-act M is *indecomposable* if M is not isomorphic to any coproduct $N \sqcup N$ where N and N are non-empty closed left S-acts.

Proposition 1. *[25, Proposon 3.3] In the category S-**FAct**, P is indecomposable and projective if and only if P is isomorphic to Se for some idempotent e.*

In 1972, Banaschewski showed that the generalisation of the Morita theory for rings to semigroups is in fact isomorphic in case **R-Act** and **S-Act** are equivalent, with no requirement that acts be unitary in any sense. So one is forced to define Morita equivalence in terms of subcategories if a notion differing from isomorphism is to be obtained. In 1990, Knauer and Normak showed that monoids M and N are Morita equivalent if and only if M-**FAct** and N-**FAct** are equivalent categories. As a generalisation of Morita equivalence for rings and monoids Tarlwar first defined Morita theory for semigroups with local units. Semigroups R and S with local units are *Morita equivalent* if S-**FAct** is equivalent to R-**FAct**. If R and S both have a zero, then we shall require that S^0-**FAct** be equivalent to R^0-**FAct**.

Let R and S be semigroups. A six-tuple $\langle R, S,_R P_S,_S Q_R, \langle,\rangle, [,]\rangle$ is said to be a *Morita context* if the following conditions hold:

(1) ${}_R P_S$ is an R-S-biact and ${}_S Q_R$ is an S-R-biact;

(2) $\langle,\rangle$ is an R-R-morphism of $P \otimes_S Q$ into R, and $[,]$ is an S-S-morphism of $Q \otimes_R P$ into S;

(3) for all $p, p' \in P$ and $q, q' \in Q$ we have

$$\langle p, q\rangle p' = p[q, p'] \text{ and } q\langle p, q'\rangle = [q, p]q'.$$

We say that a Morita context $\langle R, S,_R P_S,_S Q_R, \langle,\rangle, [,]\rangle$ is *unitary* if ${}_R P \in S$-**FAct**, ${}_S Q \in S$-**FAct** and the biacts ${}_R P_S$ and ${}_S Q_R$ are unitary.

Two semigroups R and S are *strongly$_1$ Morita equivalent* if there exists a unitary Morita context such that $\langle,\rangle$ and $[,]$ are surjective.

The subscript is used to distinguish this meaning of the word "strongly" from both general semigroups and inverse semigroups which will occur below.

A 5-tuple $(S, T, X, \langle,\rangle, [,])$, where S and T are inverse semigroups and X is a S-T-biact, is said to be an *inverse Morita context* if the following conditions holds:

(1) the left action of S on X and the right action of T on X commute;

(2) $\langle,\rangle : X \times X \to S$ and $[,] : X \times X \to T$ are surjective functions;

(3) for any $x, y, z \in X$, $s \in S$ and $t \in T$ we have:

  (i) $\langle sx, y\rangle = s\langle x, y\rangle$;
  (ii) $\langle y, x\rangle = \langle x, y\rangle^*$;

(iii) $\langle x,x\rangle x = x$;

(iv) $[x, yt] = [x, y]t$;

(v) $[y, x] = [x, y]^*$;

(vi) $x[x, x] = x$;

(vii) $\langle x, y\rangle z = x[y,z]$,

where a^* is the inverse of a for all $a \in S$.

Here we call X (together with the two "inner products") an *equivalence bimodule*. Two inverse semigroups S and T are said to *strongly$_2$ Morita equivalent* if there exists an equivalence bimodule for them, that is, there exists an inverse Morita context $(S, T, X, \langle,\rangle, [,])$.

3. Morita Theory for Rings

The main purpose of this section is to make a survey of Morita theory for rings. The Morita theory for rings has been widely studied and so fruitful results have been obtained. Considering the length of the paper, we focus on rings with local units and xst-rings.

3.1. Rings with Local Units

We begin with a well-known theorem for unital rings. Let R be a unital ring and let $\mathbf{p}(R)$ denote the full subcategory of R-**Mod** consisting of the finitely generated projective left R-modules.

Theorem 1. *[34, Theorem 3.4] Let R and S be unital rings. Then the following statements are equivalent:*

(1) R and S are Morita equivalent;

*(2) there exists a finitely generated projective generator ${}_RP$ for R-**Mod** such that $S \cong End_R(P)$;*

*(3) there exists P in $\mathbf{p}(R)$, with P a generator for R-**Mod**, such that $S \cong End_{\mathbf{p}(R)}(P)$.*

In 1983, Abrams extent Theorem 1 from unital rings to rings with slu. We now give a brief description.

Let R and S be Mortia equivalent rings with slu. B.Abrams [1] further showed that S can be considered as $\mathrm{Lim}_I\mathrm{End}(Q_i)$, where $\{Q_i, \phi_{ij}, \psi_{ji} | i \in I\}$ is a progenerator for R.

Conversely, if $S \cong \mathrm{Lim}_I\mathrm{End}(Q_i)$, where $\{Q_i, \phi_{ij}, \psi_{ji} | i \in I\}$ is a progenerator. We define

$$S_i = \mathrm{End}_R(X_i).$$

Then

$$S \cong \varinjlim_I((S_i, 1_j), \Omega_{ij})$$

where $\Omega_{ij} : S_i \to S_j$ defined by $r \mapsto \psi_{ji} r \phi_{ij}$. Let f_i denote $[1_{X_i}]$ in $\mathrm{Lim}_I\mathrm{End}(X_i)$. Then

$$\mathrm{End}_R(X_i) \cong f_i S f_i$$

given by $r \mapsto [r]$.

Suppose that M is a left R-modules. Let $i \leq j$ in I. Then there exists a unique left S-module homomorphism

$$g_{ij}^M : Sf_i \otimes_i \ \mathrm{Hom}_R(X_i, M) \to Sf_j \otimes_j \ \mathrm{Hom}_R(X_j, M)$$

such that $g_{ij}^M : sf_i \otimes_i v \mapsto sf_i \otimes_j \psi_{ji} v$. In fact, for $i \leq j \leq k$ in I, $g_{ij}^M \circ g_{jk}^M = g_{ik}^M$. We define

$$G(M) = \underrightarrow{\mathrm{Lim}}_I (Sf_i \otimes_i \ \mathrm{Hom}_R(X_i, M), g_{ij}^M).$$

Then $G(M)$ is a left S-module. For each $i \in I$ let

$$g_i^M : Sf_i \otimes_i \ \mathrm{Hom}_R(X_i, M) \to MG$$

denote the limit map; so for each $i \leq j$, we have $g_{ij}^M \circ g_j^M = g_i^M$.

Suppose that M and M' are left R-modules, and $\alpha \in \ \mathrm{Hom}_R(M, M')$. For $i \in I$, we define

$$\alpha_*^i : \mathrm{Hom}_R(X_i, M) \to \ \mathrm{Hom}_R(X_i, M')$$

via $v\alpha_*^i = v\alpha$, where $v \in \ \mathrm{Hom}_R(X_i, M)$. This induces the map

$$1 \otimes \alpha_*^i : Sf_i \otimes_i \ \mathrm{Hom}_R(X_i, M) \to Sf_i \otimes_i \ \mathrm{Hom}_R(X_i, M').$$

Thus, we get a unique S-homomorphism $G(\alpha)$ with certain properties as required.

Dually , we can define a functor $H : S$-**Mod** → R-**Mod** which is an inverse for G. So R-**Mod** and S-**Mod** are equivalent. Thus, we obtain an analogue of Theorem 1.

Theorem 2. *[1, Theorem 4.2] Let R and S be rings with slu. Then the following statements are equivalent:*

(1) R and S are Morita equivalent;

(2) there exists a progenerator $\{x_i : \phi_{ij}, \psi_{ji} | i \in I\}$ *for R such that*

$$S \cong \ Lim_I \ End(x_i).$$

(3) there exists a locally projective R-module (P, ϕ, ψ, I)*, with P a generator for R-***Mod***, such that* $S \cong End_{\mathbf{LP}(R)}((P, \phi, \psi, I))$.

In the following we describe Morita equivalence in terms of Mortia context.

Let R and S be Morita equivalent rings with local units via inverse equivalences $G : R$-**Mod** → S-**Mod** and $H : S$-**Mod** → R-**Mod**. Set

$$P = H({}_S S) \text{ and } Q = G({}_R R).$$

Then P and Q are naturally unitary bimodules ${}_R P_S$ and ${}_S Q_R$. We define two bilinear products

$$(-,-) : P \times Q \to R : (p, q) = pq \in R,$$

$$\langle -,- \rangle : Q \times P \to S : \langle q, p \rangle = (-, q)p \in S.$$

Put

$$(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \otimes \langle q_1, p_2 \rangle q_2$$

and

$$(q_1 \otimes p_1)(q_2 \otimes p_2) = q_1 \otimes (p_1, q_2)p_2.$$

Then, $P \otimes_S Q$ and $Q \otimes_R P$ become rings, and we have $R \cong P \otimes_S Q$ and $S \cong Q \otimes_R P$. So $(R, S,_R P_S,_S Q_R, \langle , \rangle, [,])$ forms a Mortia context with surjective mappings.

Conversely, let R, S, ${}_RP_S$, ${}_SQ_R$, $(,) : P \times Q \to R$, $\langle , \rangle : Q \times P \to S$ be a Morita context where R, S are rings with local units and P, Q are unitary bimodules. Then $P \otimes_S - : S\text{Mod} \to R\text{Mod}$ and $Q \otimes_R - : R\text{Mod} \to S\text{Mod}$ are equivalence inverse to each other if and only if both $(,)$ and $\langle , \rangle$ are surjective.

Thus, we have:

Theorem 3. *[3] Let R and S be rings with local units. Then the following statements are equivalent:*

(1) R and S are Morita equivalent;

(2) there exists a Morita context $(R, S, P, Q, \langle , \rangle, [,])$ with surjective mappings.

We pause here to remark that Morita contexts with surjective mappings are shown to yield Morita equivalence, and vice versa, for central separable algebras over a commutative ring with identity. However, central separable algebras need not have local units and the converse implication does not hold either. More details are referred to [46].

Corollary 1. *[3, Corollary 2.3] For any rings R, S with local units, R-***UMod** *and S-* **UMod** *are equivalent if and only if* **UMod***- R and* **UMod***- S are equivalent.*

We recall that a unitary bimodule ${}_RM_S$ is *balanced* if the canonical homomorphisms $S \to \text{End}_R M$ and $R \to \text{End}_S M$ are injective and, identifying R and S with the corresponding subrings of endomorphisms of M, it holds $S\text{End}_S M = S$ and $(\text{End}_S M)R = R$.

It is known [9] that untial rings R and S are Morita equivalent if and only if there exists a balanced bimodule ${}_RP_S$ such that

(1) P_S and ${}_RP$ are progenerators;

(2) the functor pair $(\otimes_R P, \otimes_S Q)$ defines an equivalence of the categories R-**Mod** and S-**Mod**, where $Q = \text{Hom}_S(P, S)$.

Such a result was extended by Fuller [9]. Fuller investigated the categorical equivalence between R-**Mod** and R-**UMod**. He showed that if R-**Mod** is equivalent to S-**Mod** then there exists a bimodule ${}_RU_S$ such that

(1) U_S is finitely generated and quasi-projective and generates each of its submodules;

(2) ${}_RU$ is faithful and flat;

(3) the functor pair $(\otimes_R U, \mathrm{Hom}_S(U,-))$ defines an equivalence between R-**Mod** and S-**UMod**.

Analogous to such a result we proceed to study Morita equivalence in terms of locally projective generators instead of progenerators.

Theorem 4. *[3, Theorem 2.4] Let R, S be rings with local units, and let $G : R - \textbf{UMod} \to S - \textbf{UMod}$, $H : S - \textbf{UMod} \to R - \textbf{UMod}$ be additive functors. Then G and H are equivalences inverse to each other if and only if there exists a unitary bimodule ${}_RP_S$ such that*

(1) both ${}_RP$, P_S are locally projective generators;

(2) ${}_RP_S$ is balanced;

(3) $G \cong SHom_R(P,-)$ and $H \cong P \otimes_S -$.

Moreover, if P satisfies these conditions then, putting $Q = SHom_R(P,R)$, ${}_SQ_R$ is a balanced bimodule, both ${}_SQ$ and Q_R are locally projective generators, $H \cong RHom_S(Q,-)$ and $G \cong Q \otimes_R -$.

Building on this idea Garcia [12] reintroduced matrices by showing that if $\mathbb{N}$ is the set of natural numbers and T is the matrix ring $M^0_{\mathbb{N}}(R)$, then, R-**Mod** is equivalent to T-**UMod**.

Xu, Shum and Turner-Smith further generalised Garcia's result by replacing the index set $\mathbb{N}$ with an arbitrary set Γ as follows:

Theorem 5. *[49, Theorem 3.2] Let R be a unital ring. If $l \in M_\Gamma(R)$ is idempotent and such that $M^0_\Gamma(R)/M^0_\Gamma(R) = M^0_\Gamma(R)$, then $lM^0_\Gamma(R)l$ and $M^0_\Gamma(R)$ are Morita-like equivalent. Moreover, the functor pair $(\otimes_S lM^0_\Gamma(R), \otimes_T M^0_\Gamma(R)l)$ defines an equivalence between S-**Mod** and T-**Mod**, where $S = lM^0_\Gamma(R)l$ and $T = M^0_\Gamma(R)$.*

The following corollary not only includes the result of Xu [48] but also that of Garcia [12], who considers $\Gamma = \mathbb{N}$.

Corollary 2. *[49, Corollary 3.3] Let R be a unital ring. Then R and $M^0_\Gamma(R)$ are Morita-like equivalent.*

If $|\Gamma| = n$, then we have $M_n(R) = M^0_n(R)$. Thus, we have:

Corollary 3. *[49, Corollary 3.3] Let R be a unital ring, and let l be an idempotent in $M_n(R)$ such that $M_n(R)lM_n(R) = M_n(R)$. Then R and $lM_n(R)l$ are Morita equivalent.*

We now list from [3] two properties of Morita equivalent rings with local units.

Proposition 2. *Let R and S be Morita equivalent rings with local units. Then*

(1) if R is regular then S is also regular;

(2) R is primitive or a ring with zero Jacobson radical if and only if S is such.

It is a good place to cite from [3] very interesting results characterising rings which are Morita equivalent to rings of certain kinds. We say that a unital ring R is *primary* if the factor $R/J(R)$ is a simple artinian ring such that the idempotents can be lifted, where $J(R)$ is its Jacobson radical. In addition, if $R/J(R)$ is a division ring then R is said to be a *local ring*. A ring R is said to be a *strongly locally matrix* ring over a (unital) ring S if every finite subset $U \subseteq R$ there is an idempotent $e \in R$ such that $U \subseteq eRe$ and eRe is isomorphic to the matrix ring S_n for some n.

Proposition 3. *Let R be a ring with local units. Then*

(1) R is Morita equivalent to a untial ring if and only if there exists an idempotent e in R with $R = ReR$. Moreover, R is Morita equivalent to eRe;

(2) R is Mortia equivalent to a division ring if and only if it is a simple ring with minimal one-sided ideals;

(3) R is Morita equivalent to a primary ring if and only if R is isomorphic to a strongly locally matrix ring over a local ring.

3.2. XST-rings

The aim of this subsection is to give a number of important and useful results concerning xst-rings.

We recall that a ring R is called a *right xst-ring* if every submodule of a unitary right R-module is again unitary. A ring R is an *xst-ring* if it is both right and left xst. In [13], a ring R is right xst if and only if **UMod**-R is complete additive, if and only if every unitary ring R-module is s-unital. In addition, if R is a right xst ring then **Mod**-R coincides with $\mathscr{C}_R$. Thus rings R and S are right Morita-like equivalent if and only if R and S are right xst and **Mod**-R and **Mod**-S are equivalent. Observe that for right xst-rings R and S, they are right Morita-like equivalent if and only if **Mod**-R and **Mod**-S are equivalent, that is, R and S are Morita equivalent. So for xst-rings, Morita-like equivalence and Morita equivalence coincide.

Theorem 5 can be restated in the following form cited from [13].

Theorem 6. *Let R be a unital ring, Λ any non-empty set, ${}_RF$ a free left R-module of rank Λ, $E = End({}_RF)$ its endomorphism ring, and $E_f = fEnd({}_RF)$ the ring of the finite endomorphisms of F. Moreover, let ${}_RP$ be a projective generator which is isomorphic to a direct summand of ${}_RF$. Then*

(1) E_f and $fEnd({}_RP)$ are right xst-rings;

(2) **Mod**-E_f *and* **Mod**-$fEnd({}_RP)$ *are equivalent categories.*

That is, E_f and $fEnd({}_RP)$ are right Morita-like equivalent rings.

The following theorem identifies all rings which are right Morita-like equivalent to unital rings.

Theorem 7. *[13, Corollary 3] Let R be an idempotent ring. Then R is a right xst-ring such that R is right Morita-like equivalent to a unital ring A if and only if there exists a generator $_AM$ of A-**Mod** such that $\frac{R}{T(_RR)}$ is isomorphic to a dense right ideal of End$(_AM)$ which is contained in fEnd$(_AM)$, where $\frac{R}{T(_RR)}$ is the quotient module of R factored by $T(_RR)$, and fEnd$(_AM)$ is the ring of the finite endomorphisms of M.*

If R is an xst-ring, we shall use $U(R)$ to denote the ideal of R. Then

Proposition 4. *[13, Proposition 8] Let R and S be xst-rings. Then the following statements are equivalent:*

(1) **Mod**-R *and* **Mod**-S *are equivalent;*

(2) **Mod**-$U(R)$ *and* **Mod**-$U(S)$ *are equivalent;*

(3) R-**Mod** *and* S-**Mod** *are equivalent;*

(4) $U(R)$-**Mod** *and* $U(S)$-**Mod** *are equivalent.*

Proposition 5. *[13, Proposition 9] Let R and S be xst-rings. Then* **Mod**-R *and* **Mod**-S *are equivalent if and only if there exists a Morita context between R and S, given by (on both sides) bimodules $_RP_S$ and $_SQ_R$ such that the traces of the context are, respectively, $U(R)$ and $U(S)$.*

Let R be a ring, $_RP$ a unitary left R-module and $S = \mathrm{End}(_RP)$. We denote by $f\mathrm{End}(_RP)$ to the following (non-unital, in general) subring of $_RP$:

$$\{\alpha \in S \mid \exists x_i \in P, \exists f_i \in \mathrm{Hom}_R(P,R), \forall u \in P, u\alpha = \sum_{i=1}^{n}(uf_i)x_i\}.$$

Theorem 8. *[13, Theorem 6] Let R and S be xst-rings. R and S are Morita-like equivalent if and only if there exists a generator $_RP$ of the category R-**Mod** such that $U(S)$ is isomorphic to a dense right ideal T of End$_RP$, such that $T \subseteq f$End$(_RP)$.*

4. Morita Theory for Semigroups

In this section we describe the development of Morita theory for semigroups. There mainly exist four ways of investigating Morita equivalence for semigroups: using categories of acts over them, Morita contexts, Cauchy completions and enlargements.

4.1. Semigroups with Local Units

In [25], Lawson gave a list of equivalent characterisations of Morita equivalence for semigroups with local units. We reformulate them in Theorem 9 appearing in the follow.

Theorem 9. *[25, Theorem 1.1] Let S and T be semigroups with local units. Then the following statements are equivalent:*

(1) S and T are Morita equivalent;

(2) the categories $C(S)$ and $C(T)$ are equivalent;

(3) S and T have a joint enlargement which can be chosen to be regular if S and T are both regular;

(4) there exists a unitary Morita context $(S,T,P,Q,\langle,\rangle,[,])$ with surjective mappings.

It is now worth to give a short remark about the proof of Theorem 9.

From (1) to (2). Let S and T be Morita equivalent via inverse functors $G : S-\mathbf{FAct} \to T-\mathbf{FAct}$ and $H : T-\mathbf{FAct} \to S-\mathbf{FAct}$. Notice that if M is an indecomposable projective in S-**FAct** then $G(M)$ is an indecomposable projective in T-**FAct**. Thus G maps the full subcategory of indecomposable projectives in S-**FAct** to the full subcategory of indecomposable projectives in T-**FAct**, and H does the same in the opposite direction. So the full subcategory of indecomposable projectives in S-**FAct** is equivalent to the full subcategory of indecomposable projectives in T-**FAct**. By Proposition 1, every indecomposable projective in S-**FAct** is isomorphic to one of the form Se for some idempotent e. Let $\mathbf{IP}_S$ be the full subcategory of S-**FAct** whose objects are all the left closed S-acts of the form Se where e ranges over all idempotents of S. Then the full subcategory of indecomposable projectives in S-**FAct** is equivalent to $\mathbf{IP}_S$. Similarly, the full subcategory of indecomposable projectives in T- **FAct** is equivalent to $\mathbf{IP}_T$. It follows that $\mathbf{IP}_S$ is equivalent to $\mathbf{IP}_T$.

Let $\alpha : Se \to Sf$ be a left S-homomorphism. Put $a = e\alpha$. Define a functor F of $C(S)$ by:

$$F(e) = Se \text{ for all } e \in E(S),$$
$$F(e,a,f) = \rho_a : Se \to Sf,\ x\rho_a = xa \text{ for all } x \in Se.$$

Then F is a full and faithful functor and so $C(S)$ is equivalent to $\mathbf{IP}_S$. Hence $C(S)$ and $C(T)$ are equivalent.

From (2) to (3). Let S and T be semigroups with local units. If the categories $C(S)$ and $C(T)$ are equivalent, then we can construct a bipartite category $C = [C(S),C(T)]$ where $C(S)$ and $C(T)$ are strongly connected, and so C is strongly connected. For any $e \in E(S)$, we denote the identity (e,e,e) of $C(S)$ by $\bar{e}$. On $C(S)$ we define the consolidation p by $p_{e,f} = (e,ef,f)$. The function $\pi_1^\natural : C(S)^p \to S$ given by $(e,s,f) \to s$ is a surjective homomorphism. For any $i \in E(T)$, we denote the identity (i,i,i) of $C(S)$ by $\hat{i}$. On $C(T)$ we define the consolidation q by $q_{i,j} = (i,ij,j)$. The function $\pi_2^\natural : C(T)^q \to T$ given by $(i,t,j) \to t$ is a surjective homomorphism. Let π be the congruence on C^r generated by $\pi_1^\natural \cup \pi_2^\natural$. Then C^r/π is a semigroup with local units that is an enlargement of (isomorphic copies of) S and T.

From (3) to (4). Let S and T have a joint enlargement R. Put $P = SRT$ and $Q = TRS$. We define two maps

$$\langle -, - \rangle : P \otimes Q \to S \text{ and } [-, -] : Q \otimes P \to T$$

by $\langle p, q \rangle = pq$ and $[q, p] = qp$, where $p \in P$ and $q \in Q$. Then $\langle S, T, P, Q, \langle -, - \rangle, [-, -] \rangle$ forms a unitary Morita context with surjective maps.

From (4) to (1). Let $\langle S, T, P, Q, \langle -, - \rangle, [-, -] \rangle$ be a unitary Morita context with surjective maps. Then there exist two inverse functors

$$Q \otimes - : S-\mathbf{FAct} \to T- \mathbf{FAct}$$

and

$$P \otimes - : T-\mathbf{FAct} \to S- \mathbf{FAct},$$

and thus S and T are Morita equivalent.

Observe that to characterise two semigroups being Morita equivalent there exists another method using a consolidation and a local isomorphism as follows:

Theorem 10. *[25, Theorem 1.2] Let S and T be semigroups with local units. Then S and T are Morita equivalent if and only if there is a consolidation q on $C(S)$ and a local isomorphism $\psi : C(S)^q \to T$.*

The following is a list of Morita invariant properties. These go back to results obtained for enlargements [23], and they were known from the Morita framework to Talwar [43, 45].

Proposition 6. *[25, Proposition 5.1] Let S and T be semigroups with local units which are Morita equivalent. Then*

(1) each local submonoid of S is isomorphic to a local submonoid of T, and vice-versa;

(2) S is regular if and only if T is regular;

(3) the cardinalities of the sets of regular $\mathscr{D}$-classes in S and T are the same;

(4) the posets of two-sided ideals in S and T are order-isomorphic.

(5) the posets of principal two-sided ideals in S and T are order-isomorphic.

The following result was known to Talwar [44]. It shows how the theory simplifies radically when at least one of the semigroups is a monoid.

Proposition 7. *[25, Proposition 5.2] Let S be a monoid and T a semigroup with local units. Then S and T are Morita equivalent if and only if there is an idempotent f in T such that $T = TfT$ and fTf is isomorphic to S. Thus T is an enlargement of S.*

As a monoid is a semigroup with local units, we have the following corollaries.

Corollary 4. *[43, Corollary 9.2] The monoids M and N are Morita equivalent if and only if there exists an idempotent $e \in M$ such that $M = MeM$ and $N \cong eMe$.*

If M is a monoid then M-**FAct** is in fact M-**Act** as studied by Knauer. Corollary 4 is therefore just Knauer's theorem on Morita equivalence. For Morita equivalent monoids we still have the following properties.

Corollary 5. *[43, Corollary 9.3] Let S and T be Morita equivalent monoids. Then S and T are isomorphic in any of the following cases:*

(1) S is a commutative monoid;

(2) there exists at most one element of S which has a finite order;

(3) the identity element e of S is externally adjoined, that is, if $ab = e$, for all $a, b \in S$, then we have $a = b = e$;

(4) S is a group.

The Morita theory of unital rings provides a framework for understanding the Wedderburn-Artin Theorem [22]. As an analogue of simple artinian rings in the range of semigroups is the class of completely simple semigroups. The following theorem gives a number of equivalent characterisations of completely simple semigroups. We first recall that a semigroup S is said to have a property *locally* if each local submonoid eSe has that property. By the local structure of a semigroup S, we mean the structure of the local submonoids eSe as e varies over the set of idempotents of S.

Theorem 11. *[25, Theorem 5.3]Let S be a semigroup with local units. Then the following statements are equivalent:*

(1) S is completely simple;

(2) S is regular and locally a group;

(3) there exists an idempotent e such that $S = SeS$ and eSe is a group;

(4) S is Morita equivalent to a group.

In 1940, Rees showed that every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup. We now proceed to recover the Rees theorem as follows:

Theorem 12. *[43, Theorem 9.8] A regular semigroup S with zero is completely 0-simple if and only if S is Morita equivalent to a group G with zero.*

Further, we have the following theorem.

Theorem 13. *[43, Theorem 9.11] A regular semigroup S with zero is bisimple if and only if S is Morita equivalent to a regular bisimple monoid with zero.*

From Theorem 13, we are able to deduce that a regular semigroup S is bisimple if and only if S is Morita equivalent to a regular bisimple monoid.

McAlister [27–31] investigated regular semigroups which are locally groups. Such results are generalisations of the theorem of completely simple semigroups. For instance, McAlister [27, 28] studied locally inverse semigroups which are analogous to both completely simple semigroups and inverse semigroups. We will interpret McAlister's results in terms of Morita equivalence as follows:

Theorem 14. *[25, Theorem 5.5] Let S be a semigroup with local units. Then*

(1) S is Morita equivalent to a group if and only if it is completely simple;

(2) S is Morita equivalent to an inverse semigroup if and only if it is regular and locally inverse;

(3) S is Morita equivalent to a semilattice if and only if it is regular, locally a semilattice, and $S/\mathscr{J}$ is a meet semilattice under subset inclusion;

(4) S is Morita equivalent to an orthodox semigroup if and only if it is regular and locally orthodox;

(5) S is Morita equivalent to an L-unipotent semigroup if and only if it is regular and locally L-unipotent;

(6) S is Morita equivalent to an E-solid semigroup if and only if it is regular and locally E-solid;

(7) S is Morita equivalent to a union of groups if and only if it is regular, locally a union of groups, and $S/\mathscr{J}$ is a meet semilattice under subset inclusion.

Let S be a semigroup having locally commuting idempotents. A function $p : E(S) \times E(S) \to S$ given by $p(u,v) = p_{u,v}$ for all $u, v \in S$, is called a *McAlister sandwich function* if it satisfies the following conditions:

(1) $p_{u,v} \in uSv$ and $p_{u,u} = u$;

(2) $p_{u,v} \in V(p_{v,u})$;

(3) $p_{u,v}p_{v,f} \leq p_{u,f}$.

Theorem 15. *[2, Theorem 3.10] A semigroup S with local unitsis Morita equivalent to a semigroup with local units having commuting idempotents if and only if it has locally commuting idempotents and is equipped with a McAlister sandwich function.*

Notice that if S is a semigroup with local units and with locally commuting idempotents, then S has a McAlister sandwich function if and only if the inverse category $I(S)$ is equipped with a McAlister consolidation. If $I(S)$ is strongly connected and is equipped with a McAlister consolidation, then $I(S)$ is Morita equivalent to the Cauchy completion of an inverse semigroup. Thus, we have the following theorem.

Theorem 16. *[2, Theorem 3.11] The semigroup with local units S is Morita equivalent to a semigroup with local units and with commuting idempotents if and only if the following two conditions hold:*

(1) S has locally commuting idempotents;

(2) the inverse category $I(S)$ is equivalent to a category of the form C(T), where T is an inverse semigroup.

Theorem 17. *[2, Theorem 3.12] Suppose that the set of regular elements of a semigroup S with local units forms a regular subsemigroup. Then S is Morita equivalent to a semigroup with commuting idempotents if and only if S has locally commuting idempotents.*

To end this subsection we discuss the properties of strongly$_1$ Morita equivalent semigroups, which are cited from [21].

Proposition 8. *Let S and T be strongly$_1$ Morita equivalent semigroups. Then*

(1) if S and T are semigroups with weak local units, then there exists an isomorphism $\Phi : \boldsymbol{Id}(S) \to \boldsymbol{Id}(T)$ between their lattices of ideals which takes finitely generated ideals to finitely generated ideals and principal ideals to principal ideals;

(2) if S and T are semigroups with common two-sided weak local units then their greatest commutative images are isomorphic semigroups;

(3) if S and T are commutative semigroups with common two-sided weak local units then S and T are isomorphic;

(4) if S and T are semigroups with common two-sided local units then S and T satisfy the same identities.

From (3) in Proposition 8, we observe that if S and T are strongly$_1$ Morita equivalent semigroups with common two-sided weak local units then their greatest semilattice images are isomorphic.

The following theorem shows that Rees congruences correspond to Rees congruences under strongly$_1$ Morita equivalence and also implies that the ideal lattices of strongly$_1$ Morita equivalent semigroups with common joint weak local units are isomorphic.

Theorem 18. *[21, Theorem 6] If S and T are strongly$_1$ Morita equivalent semigroups with common joint weak local units then there exists an isomorphism $\prod : Con(S) \to Con(T)$ between their congruence lattices such that if $\rho \in Con(S)$ then the semigroups S/ρ and $T/\prod(\rho)$ are strongly$_1$ Morita equivalent, and $\prod$ takes*

(1) Rees congruences to Rees congruences;

(2) finitely generated congruences to finitely generated congruences;

(3) principal congruences to principal congruences.

4.2. Inverse semigroups

Let S be an inverse semigroups with semilattice of idempotents $E(S)$. We define an action of S on $E(S)$ by the rule that for any $s \in S$ and $e \in E(S)$, $s \cdot e = ses^{-1}$, where s^{-1} is the unique inverse of s. Then $E(S)$ together with the action is called the *Munn module*. A left S-act X paired with an S-homomorphism p to $E(S)$ the Munn module, such that $p(x) \cdot x = x$, is what we call an *étale left S-act*. We denote the category of étale left S-acts by **Étale**. **Étale** can be taken as the definition of the classifying topos of S, denoted $\mathscr{B}(S)$. **Étale** (or $\mathscr{B}(S)$) is equivalent to the category $PSh(L(S))$ of presheaves on $L(S)$.

Theorem 19. *[11, Theorem 1.1] Let S and T be inverse semigroups. Then the following are equivalent:*

(1) S and T are strongly$_2$ Morita equivalent;

(2) the classifying toposes of S and T are equivalent;

(3) the inductive groupoids S and T have an ordered groupoid joint enlargement, which can be chosen to be bipartite;

(4) the categories $C(S)$ and $C(T)$ are equivalent;

(5) S and Thave a regular joint enlargement;

(6) S and T are Morita equivalent.

Observe that the usual Morita equivalence and the strongly$_2$ Morita equivalence coincide with each other for inverse semigroups.

Inverse semigroups form a special subclass of semigroups with local units so we have the following theorem as an analogue of Theorem 10.

Theorem 20. *[2, Theorem 2.15] Let S and T be inverse semigroups. Then S and T are Morita equivalent if and only if there is a local isomorphism $\theta : C(S)^p \to T$ for some consolidation p defined on $C(S)$.*

Corollary 6. *[42, Corollary 5.5] Suppose S and T are inverse semigroups such that T is a monoid. Then the following are equivalent:*

(1) there exists an idempotent $e \in E(S)$ such that $S = SeS$ and $T \cong eSe$;

(2) S and T are strongly$_2$ Morita equivalent;

(3) the categories $C(S)$ and $C(T)$ are equivalent.

We remark that if an inverse semigroup S is strongly$_2$ Morita equivalent to an inverse monoid T, then S is an enlargement of T. In particular, if S and T are inverse monoids, then S is strongly$_2$ Morita equivalent to T if and only if there exist $e \in E(S)$ and $f \in E(T)$ so that $S = SeS$, $eSe \cong T$ and $T = TfT$ and $fTf \cong S$.

In the following we give a list of properties of strongly$_2$ Morita equivalent inverse semigroups which was taken as a corollary in [42].

Proposition 9. *[42, Corollary 5.2] Let S and T be strongly$_2$ Morita equivalent inverse semigroups. Then the following statements hold:*

(1) the categories $C(S)$ and $C(T)$ are equivalent;

(2) the categories $L(S)$ and $L(T)$ are equivalent;

(3) for each $e \in E(S)$, there exists an idempotent $f \in E(T)$ such that $eSe \cong fSf$ and conversely;

(4) the underlying groupoids of S and T are naturally equivalent;

(5) there is a bijection $F : E(S)/\mathscr{D} \to E(T)/\mathscr{D}$ such that if D is a $\mathscr{D}$-class of S with maximal subgroup G, then $F(D)$ is a $\mathscr{D}$-class of T with maximal subgroup isomorphic to G;

(6) the posets $E(S)/\mathscr{J}$ and $E(T)/\mathscr{J}$ are isomorphic;

(7) S and T have isomorphic lattices of two-sided ideals;

(8) the classifying toposes $\mathscr{B}(S)$ and $\mathscr{B}(T)$ are equivalent;

(9) S and T have the same cohomology groups;

(10) S has a zero if and only if T has a zero;

(11) they have isomorphic maximal group images.

If E is a semilattice, then $(E, \leq) \cong (E/\mathscr{J}, \leq_{\mathscr{J}})$ and so we have the following corollaries.

Corollary 7. *[42, Corollary 5.3] Let E and F be strongly$_2$ Morita equivalent semilattices. Then E is isomorphic to F.*

Corollary 8. *[42, Corollary 4.8] Let S and T be strongly$_2$ Morita equivalent inverse semigroups. Then the universal and reduced C^*-algebras of S and T are strongly$_2$ Morita equivalent.*

For strongly Morita equivalent modules, we have the following theorems.

Theorem 21. *[42, Theorem 4.13] Let S and T be strongly Morita equivalent inverse semigroups and let K be a commutative unital ring. Then the semigroup algebras KS and KT are Morita equivalent.*

Every proper image of the bicyclic monoid is a group [7], so no residually finite inverse semigroup can contain a copy of the bicyclic monoid. (Actually it is known that the bicyclic monoid cannot embed in any compact semigroup since compact semigroups are stable and the bicyclic monoid cannot embed in any stable semigroup [40].) Also no semigroup with central idempotents contains a copy of the bicyclic monoid since its idempotents are not central. Hence we have the following corollary.

Corollary 9. *[42, Corollary 5.7] Suppose that S and T are strongly$_2$ Morita equivalent monoids such that S is either:*

(1) a group;

(2) commutative;

(3) has central idempotents;

(4) is residually finite.

Then S and T are isomorphic.

4.3. Factorisable Semigroups

We recall that a semigroup S is *factorisable* if $S = S^2$, that is, every element of S can be written as a product of two elements.

To proceed the following, we first recall a construction from [44]. Let R be a semigroup, let X and Y be finite non-empty sets, and let

$$\langle , \rangle : Y \times Y \to R$$

be a function (with values denoted $\langle y, x\rangle$). Then the set

$$M = X \times R \times Y$$

equipped with the associative product

$$(x, s, y)(x', s', y') = (x, s\langle y, x'\rangle)s', y')$$

is a semigroup, known as the *Rees matrix semigroup over R* defined by $\langle , \rangle$.

We formulate the following: Let R be a semigroup, let ${}_RP$ and Q_R be respectively left and right R-acts. Also, let

$$\langle , \rangle :_R P \times Q_R \to R$$

be an R-R-bilinear, that is, $\langle rp, q\rangle = r\langle p, q\rangle$ and $\langle p, qs\rangle = \langle p, q\rangle s$. Then the set $Q \otimes_R P$ becomes a semigroup with product

$$(q \otimes p)(q' \otimes p') = q \otimes \langle p, q'\rangle p'.$$

The multiplication is well defined since $\langle , \rangle$ is an $R-R$-bilinear map. It is easy to verify that it is associative. We shall refer to $Q \otimes_R P$ as the *Morita semigroup over R* defined by $\langle , \rangle$.

Theorem 22. *[44] Let R be a factorisable semigroup, and let $_RP$ and Q_R be respectively unitary left and right R-acts. Also, let*

$$\langle,\rangle :_R P \times Q_R \to R$$

be a surjective R-R-bilinear map. The Morita semigroup $Q \otimes_R P$ is strongly$_1$ Morita equivalent to R.

Let S be an arbitrary semigroup and $M \in S$-**Act**. We set

$$\zeta_M = \{(m_1, m_2) | sm_1 = sm_2, \forall s \in S\}.$$

Then it is clear that ζ_M is an S-congruence on $_SM$ and ζ_S is a two-sided congruence on S. We can denote the quotient semigroup S/ζ_S by S'. It is clear that S' is an S-S-biact in a natural way: for any $s, t \in S$,

$$t\bar{s} = \overline{ts}, \quad \bar{s}t = \overline{st}.$$

Theorem 23. *[6, Theorem 3] Let R, S be factorisable semigroups. Then the category R-**UFAct** is equivalent to the category S-**UFAct** if and only if there exists a unitary Morita context $(R', S',_{R'} P_{S'},_{S'} Q_{R'}, \langle\rangle, \lceil\rceil)$ with $\langle\rangle$ and $\lceil\rceil$ surjective, where R' and S' constructed as above.*

Moreover, if this is the case, then we have the following category inverse equivalence:

$$R - \textbf{\textit{UFAct}} \underset{G}{\overset{F}{\rightleftarrows}} S - \textbf{\textit{UFAct}}, \text{ where } F = SHom_R({}_RP, -) \text{ and } G = RHom_S({}_SQ, -).$$

Let S be a monoid with identity 1. Then it is clear that for any unitary S-act $_SM$, $1m = m$ for any $m \in M$. Therefore, S-**UFAct** = S-**UAct** if S is a monoid.

Corollary 10. *[6, 16] Let R, S be monoids. Then the category R-**UAct** is equivalent to the category S-**UAct** if and only if there exists a unitary Morita context $(R, S,_R P_S,_S Q_R, \langle\rangle, \lceil\rceil)$ with $\langle\rangle$ and $\lceil\rceil$ surjective.*

Moreover, if this is the case, then we have the following category inverse equivalence:

$$R - \textbf{\textit{UAct}} \underset{G}{\overset{F}{\rightleftarrows}} S - \textbf{\textit{UAct}}, \text{ where } F = Hom_R({}_RP, -) \text{ and } G = Hom_S({}_SQ, -).$$

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