EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 8, No. 4, 2015, 514-525
ISSN 1307-5543 – www.ejpam.com

On Locally Hurewicz Spaces

Clarice Aparecida Roika[1], Soraya R.T Kudri[1]*, Tomaz. K. Breuckmann[1]

[1]*Department of Mathematics, Federal University of Paraná, P. O. Box 019081, Curitiba, PR, 81531-990, Brazil.*

Abstract. In this paper we define locally Hurewicz spaces, weakly locally Hurewicz spaces and relatively locally Hurewicz spaces. We obtain some results and prove the equivalence of those definitions in Hausdorff C-spaces.

2010 Mathematics Subject Classifications: 54D20

Key Words and Phrases: Hurewicz Spaces, Locally Hurewicz Spaces.

---

1. Introduction

In [3] Hurewicz introduced the notion of Hurewicz topological spaces. Those spaces generalize compact spaces and are contained in the class of Lindelöf spaces. The principal purpose of this work is to localize the Hurewicz property, introducing here the locally Hurewicz spaces.

In general topology there are three ways to define local compactness, which here are called local compactness, weak local compactness and relative local compactness. In this paper we define locally Hurewicz spaces, weakly locally Hurewicz spaces and relatively locally Hurewicz spaces, prove their equivalence in Hausdorff C-spaces and study some of their properties.

2. Preliminaries

Throughout this paper we use the notation $F \subset_{<\infty} X$ as abbreviation for "F is a finite subset of X".

Definition 1. *[4] A topological space $\langle X, T \rangle$ is locally compact if and only if for each $x \in X$ and for every neighborhood V of x, there are $U \in T$ and a compact subset C of X such that $x \in U \subset C \subset V$.*

Definition 2. *[5] A topological space $\langle X, T \rangle$ is weakly locally compact if and only if for each $x \in X$ there are $U \in T$ and a compact subset C of X such that $x \in U \subset C$.*

---

*Corresponding author.

Email address: `soraya@onda.com.br` (S.R.T.Kudri)

Definition 3. *[2] A topological space $\langle X, T\rangle$ is relatively locally compact if and only if for each $x \in X$ there are $U \in T$ such that $x \in U$ and $\overline{U}$ is compact.*

Definition 4. *[1] In any nonempty set X we can define a topology T by considering as open sets the empty set and all subsets of X containing a particular point $p \in X$. We shall call it the particular point p topology.*

Definition 5. *[3] A topological space $\langle X, T\rangle$ is Hurewicz if and only if for each sequence $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ of open coverings of X, there exists a sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ such that:*

(i) for each $n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) $\forall x \in X, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$, there exists $V \in \mathbb{V}_n$ with $x \in V$.

Example 1. *The Real line $\mathbb{R}$ in its usual topology is a Hurewicz space. Let $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ be a sequence of open coverings of $\mathbb{R}$, where $\mathbb{U}_n = \{U_{nj}\}_{j\in J_n}$. Since for each $n \in \mathbb{N}$, $[-n, n] \subset \mathbb{R}$ is compact, $\exists I_n \subset_{<\infty} J_n$ such that $\mathbb{V}_n = \{U_{nj}\}_{j\in I_n}$ covers $[-n, n]$, then the sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ is such that:*

(i) $\forall n \in \mathbb{N}$, $I_n \subset_{<\infty} J_n$ and $\mathbb{V}_n = \{U_{nj}\}_{j\in I_n}$, then we have $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) For each $x \in \mathbb{R}$, we have that there is $n_0 \in \mathbb{N}$ such that $|x| \leq n_0$. For $n \in \mathbb{N}$, if $n \geq n_0$, then $|x| \leq n$, i.e., $x \in [-n, n]$. But $\mathbb{V}_n = \{U_{nj}\}_{j\in I_n}$ covers $[-n, n]$, so there exists $j \in I_n$ such that $x \in U_{nj} \in \mathbb{V}_n$.

Proposition 1. *Let $X = \bigcup_{i\in\mathbb{N}} K_i$, where $\forall i \in \mathbb{N}$, K_i is compact and $K_i \subset K_{i+1}$, then X is Hurewicz.*

Proof. Let $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ be a sequence of open coverings of X, where each $\mathbb{U}_n = \{U_{nj}\}_{j\in J_n}$. Since $\forall n \in \mathbb{N}$, $K_n \subset X$ we have that $\mathbb{U}_n$ covers K_n and by the compactness of K_n there is $I_n \subset_{<\infty} J_n$ such that $K_n \subset \bigcup_{j\in I_n} U_{nj}$.

For each $n \in \mathbb{N}$ let $\mathbb{V}_n = \{U_{nj}\}_{j\in I_n}$, then the sequence $\{\mathbb{V}_{nj}\}_{n\in\mathbb{N}}$ is such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) For each $x \in X$, since $X = \bigcup_{i\in\mathbb{N}} K_i$ there exists $n_0 \in \mathbb{N}$ such that $x \in K_{n_0} \subset \bigcup_{j\in I_{n_0}} U_{n_0 j}$, then $x \in U_{n_0 j_0}$ for some $j_0 \in I_{n_0}$, then $U_{n_0 j_0} \in \mathbb{V}_n$. From $K_i \subset K_{i+1}$ we wave that $\forall n \in \mathbb{N}$, $n \geq n_0$, $x \in K_{n_0} \subset K_n \subset \bigcup_{j\in I_n} U_{nj}$. Hence there is $j \in I_n$ such that $x \in U_{nj} \in \mathbb{V}_n$.

So X is Hurewicz. □

Proposition 2. *Let $X = \bigcup_{i\in\mathbb{N}} K_i$, where for each $i \in \mathbb{N}$, K_i is compact, then X is Hurewicz.*

Proof. Let $X = \bigcup_{i\in\mathbb{N}} K_i$. Consider for each $i \in \mathbb{N}$, $K_i' = \bigcup_{j=1}^{i} K_j$. We have that K_i' is compact, $K_i' \subset K_{i+1}'$ and $X = \bigcup_{i\in\mathbb{N}} K_i'$ then by the previous proposition X is Hurewicz. □

Corollary 1. *If a topological space $\langle X, T\rangle$ is compact, then $\langle X, T\rangle$ is Hurewicz.*

Proof. It follows immediately from the Proposition 2. □

Proposition 3. *If a topological space $\langle X, T\rangle$ is Hurewicz, then $\langle X, T\rangle$ is Lindelöf.*

Proof. Let $\mathbb{U} = \{U_j\}_{j\in J}$ be an open covering of X. Consider the sequence $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$, where $\mathbb{U}_n = \mathbb{U}$. Since X is Hurewicz, there exists a sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n = \mathbb{U}$, i.e., $\exists I_n \subset_{<\infty} J$ such that $\mathbb{V}_n = \{U_j\}_{j\in I_n}$.

(ii) For each $x \in X, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$ then there exists $V \in \mathbb{V}_n$ with $x \in V$.

Considering $\mathbb{V} = \{U_j; j \in I_n, n \in \mathbb{N}\}$, we have that $\mathbb{V}$ is a countable subcovering of X, because for each $n \in \mathbb{N}$, I_n is finite and for each $x \in X$, by (ii) there exists $n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$, then there is $V \in \mathbb{V}_n$ such that $x \in V$, then there exists $V \in \mathbb{V}_{n_0}$, such that $x \in V$, but from $V \in \mathbb{V}_{n_0}$, we have that, $V = U_j$, for some $j \in I_{n_0}$, then $V \in \mathbb{V}$. So X is a Lindelöf space. □

Example 2. *The Real line $\mathbb{R}$ in the particular point p topology is not a Lindelöf space. In fact, since $\{\{x, p\}; x \in \mathbb{R}\}$ is an open covering of $\mathbb{R}$ which does not have a countable subcovering. By the previous proposition, the Real line in the particular point p topology is not Hurewicz.*

Proposition 4. *Let X and Y be topological spaces and let X be Hurewicz. If $f : X \to Y$ is a surjective continuous function, then Y is Hurewicz.*

Proof. Let $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ be a sequence of open coverings of Y, where each $\mathbb{U}_n = \{U_{nj}\}_{j\in J_n}$. Consider for each $n \in \mathbb{N}$, $\mathbb{W}_n = \{f^{-1}(U_{nj})\}_{j\in J_n}$. By the continuity of f we have that, $f^{-1}(U_{nj})$ is open in X and we also have that $\mathbb{W}_n$ is a covering of X, because if $x \in X$ we have that $f(x) \in Y$, hence $\exists j \in J_n$ such that $f(x) \in U_{nj}$, then $x \in f^{-1}(U_{nj})$. Therefore, $\{\mathbb{W}_n\}_{n\in\mathbb{N}}$ is a sequence of open coverings of X. Since X is Hurewicz, there is a sequence $\{\mathbb{H}_n\}_{n\in\mathbb{N}}$, such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{H}_n \subset_{<\infty} \mathbb{W}_n$, i. e., there is $I_n \subset_{<\infty} J_n$, such that $\mathbb{H}_n = \{f^{-1}(U_{nj})\}_{j\in I_n}$.

(ii) For each $x \in X, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$, then there exists $V \in \mathbb{H}_n$ with $x \in V$, i.e., there is $j \in I_n$ with $x \in f^{-1}(U_{nj})$.

Consider $\forall n \in \mathbb{N}$, $\mathbb{V}_n = \{U_{nj}\}_{j\in I_n}$. Then the sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ is such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) For each $y \in Y$, since f is surjective $\exists x \in X$ such that $f(x) = y$. By previous (ii) there exists $n_0 \in \mathbb{N}$, such that $\forall n \in \mathbb{N}$ if $n \geq n_0$, then there exists $j \in I_n$ with $x \in f^{-1}(U_{nj})$. Since $j \in I_n$ we have that $y = f(x) \in f(f^{-1}(U_{nj})) \subset U_{nj} \in \mathbb{V}_n$.

So Y is Hurewicz. □

Definition 6. *Let X be a topological space and Y a subset of X. We say that Y is Hurewicz if and only if Y is a Hurewicz subspace of X.*

Proposition 5. *Let Y be a subspace of X. Y is Hurewicz if and only if for each sequence $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ of coverings of Y by open sets in X, there is a sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ such that:*

(i) $\forall n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) $\forall y \in Y$, $\exists n_0 \in \mathbb{N}$, *such that* $\forall n \in \mathbb{N}$ *if* $n \geq n_0$, *then there exists* $V \in \mathbb{V}_n$ *with* $y \in V$.

Proof. ($\Rightarrow$) Considering Y Hurewicz, let $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ be a sequence of coverings of Y by open sets in X, where $\forall n \in \mathbb{N}$, $\mathbb{U}_n = \{U_{nj}\}_{j\in J_n}$.

Consider the sequence $\{\mathbb{W}_n\}_{n\in\mathbb{N}}$ where each $\mathbb{W}_n = \{W_{nj}\}_{j\in J_n}$ with $W_{nj} = U_{nj} \cap Y$. Then $\mathbb{W}_n$ is a covering of Y since for each $y \in Y$ by the fact that $\mathbb{U}_n$ be a covering of Y we have that there is $j \in J_n$ with $y \in U_{nj}$, then $y \in U_{nj} \cap Y = W_{nj}$ and $\mathbb{W}_n$ is formed by open sets in Y, but Y is Hurewicz then there exists a sequence $\{\mathbb{H}_n\}_{n\in\mathbb{N}}$, such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{H}_n \subset_{<\infty} \mathbb{W}_n$, i. e., $\exists I_n \subset_{<\infty} J_n$ such that $\mathbb{H}_n = \{W_{nj}\}_{j\in J_n}$.

(ii) For each $y \in Y, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$, then there exists $V \in \mathbb{H}_n$ with $y \in V$.

Consider $\forall n \in \mathbb{N}$, $\mathbb{V}_n = \{U_{nj}\}_{j\in I_n}$, then the sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ is such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) If $y \in Y$, from previous (ii) $\exists n_0 \in \mathbb{N}$, such that $\forall n \in \mathbb{N}$ if $n \geq n_0$ then there exists $V \in \mathbb{H}_n$ with $y \in V$. Therefore there is $j \in I_n$ with $V = W_{nj} = U_{nj} \cap Y$. Since $j \in I_n$ we have that $U_{nj} \in \mathbb{V}_n$ and $y \in U_{nj}$.

($\Leftarrow$) Let $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ be a sequence of coverings of Y by open sets in Y, where each $\mathbb{U}_n = \{U_{nj}\}_{j\in J_n}$. Since Y is subspace of X, for each $n \in \mathbb{N}$ and for each $j \in J_n$ there exists a open set V_{nj} in X such that $U_{nj} = V_{nj} \cap Y$. Consider $\forall n \in \mathbb{N}$, $\mathbb{W}_n = \{V_{nj}\}_{j\in I_n}$, then $\mathbb{W}_n$ is a covering of Y because, if $y \in Y$, $\exists j \in J_n$ such that $y \in U_{nj}$, then $y \in V_{nj}$ and the sequence $\{\mathbb{W}_{nj}\}_{j\in J_n}$ is a sequence of coverings of Y by open sets in X, then there exists a sequence $\{\mathbb{H}_n\}_{n\in\mathbb{N}}$, such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{H}_n \subset_{<\infty} \mathbb{W}_n$, i. e., $\exists I_n \subset_{<\infty} J_n$, such that $\mathbb{H}_n = \{V_{nj}\}_{j\in I_n}$.

(ii) $\forall y \in Y, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$ then there exists $V \in \mathbb{H}_n$ with $y \in V$.

Consider $\forall n \in \mathbb{N}$, $\mathbb{V}_n = \{U_{nj}\}_{j\in I_n}$, then the sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$ is such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) If $y \in Y$, from previous (ii) $\exists n_0 \in \mathbb{N}$, such that $\forall n \in \mathbb{N}$ if $n \geq n_0$ then there is $V \in \mathbb{H}_n$ with $y \in V$. Since $V \in \mathbb{H}_n$ we have that there exists $j \in I_n$ with $y \in V_{nj}$, but $y \in Y$, then $y \in V_{nj} \cap Y = U_{nj} \in \mathbb{V}_n$, because $j \in I_n$.

So Y is a Hurewicz space. $\square$

Proposition 6. *If F is a closed subspace of a Hurewicz space X, then F is Hurewicz.*

Proof. Let $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ be a sequence of coverings of F by open sets in X, where each $\mathbb{U}_n = \{U_{nj}\}_{j\in J_n}$. Since F is a closed set, F^c is an open set. Consider $\mathbb{W}_n = \{U_{nj}\}_{j\in I_n} \cup F^c$, then $\mathbb{W}_n$ is an open covering of X. Let $\{W_n\}_{n\in\mathbb{N}}$ be a sequence of open coverings of X. Since X is Hurewicz, there exists a sequence $\{\mathbb{H}_n\}_{n\in\mathbb{N}}$, such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{H}_n \subset_{<\infty} \mathbb{W}_n$, i. e., there exists a finite set I_n, such that $\mathbb{H}_n = \{V_{nj}\}_{j\in I_n}$ where $V_{nj} = U_{ni}$ for some $i \in J_n$ or $V_{nj} = F^c$.

(ii) For each $x \in X, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$ then, there exists $V \in \mathbb{H}_n$ with $x \in V$.

Consider $\mathbb{V}_n = \{U_{nj} \in \mathbb{U}_n; U_{nj} \in \mathbb{H}_n\}$, then the sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$, is such that:

(i) $\forall n \in \mathbb{N}$, we have that $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$.

(ii) Let $y \in F$. Then from previous (ii) $\exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$, there exists $V \in \mathbb{H}_n$ with $y \in V$. Since $V \in \mathbb{H}_n$, we have that $V = U_{nj} \in \mathbb{U}_n$ or $V = F^c$, but since $y \notin F$ we have $V = U_{ni} \in \mathbb{U}_n$, thus $V \in \mathbb{V}_n$, therefore $y \in V \in \mathbb{V}_n$.

By Proposition 5 F is a Hurewicz space. □

Proposition 7. *Let X be a topological space and let H and Y be subspaces of X, with H Hurewicz and Y closed, then $H \cap Y$ is Hurewicz.*

Proof. Let $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ be a sequence of coverings of $H \cap Y$ by open sets in X, where each $\mathbb{U}_n = \{U_{nj}\}_{j\in J_n}$.

Consider $\forall n \in \mathbb{N}$, $\mathbb{H}_n = \{U_{nj}\}_{j\in J_n} \cup Y^c$, then $\mathbb{H}_n$ is a covering of H by open sets in X, because Y is a closed set we have that Y^c is an open set in X and for each $x \in H$ if $x \in Y$ then $x \in H \cap Y$, then there exists $j \in J_n$ such that $x \in U_{nj}$, if $x \notin Y$ then $x \in Y^c$. Therefore we have that the sequence $\{\mathbb{H}_n\}_{n\in\mathbb{N}}$ is a sequence of open coverings of H by open sets in X, since H is Hurewicz in X by Proposition 5, there is a sequence $\{W_n\}_{n\in\mathbb{N}}$ such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{W}_n \subset_{<\infty} \mathbb{H}_n$, i. e., there exists a finite subset I_n such that $\mathbb{W}_n = \{V_{nj}\}_{j\in I_n}$ where $V_{nj} = U_{ni}$ for some $i \in J_n$ or $V_{nj} = Y^c$.

(ii) For each $x \in H$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$ then there exists $V \in \mathbb{W}_n$ with $x \in V$.

Consider $\mathbb{V}_n = \{U_{nj} \in \mathbb{H}_n; U_{nj} \in \mathbb{U}_n\}$, then the sequence $\{\mathbb{V}_n\}_{n\in\mathbb{N}}$, is such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{V}_n \subset_{<\infty} \mathbb{U}_n$, because $\mathbb{H}_n$ is formed by finite elements.

(ii) For each $x \in H \cap Y$, we have that $x \in H$ and $x \in Y$. If $x \in H$, by previous (ii) $\exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \geq n_0$ then there exists $V \in \mathbb{W}_n$ with $x \in V$. From $V \in \mathbb{H}_n$, we have that $V = U_{ni} \in \mathbb{U}_n$ or $V = Y^c$. Since $x \notin Y^c$, then $V = U_{ni} \in \mathbb{U}_n$, hence $V \in \mathbb{V}_n$, then $x \in V \in \mathbb{V}_n$.

By Proposition 5 $H \cap Y$ is a Hurewicz space. □

3. C-Space

Definition 7. *A topological space $\langle X, T\rangle$ is a C-space if and only if, for each $x \in X$ and for each sequence $\{\mathbb{A}_n\}_{n\in\mathbb{N}}$, where $\mathbb{A}_n = \{A_{nj} \in T; 1 \le j \le k_n\}$ with $x \in \bigcap_{j=1}^{k_n} A_{nj}$, there exists $V \in T$ such that $\forall n \in \mathbb{N}$, $x \in V \subset \bigcap_{j=1}^{k_n} A_{nj}$.*

Example 3. *Consider $X \neq \emptyset$, with the discrete topology. X is a C-space, because for each $x \in X$ and for each sequence $\{\mathbb{A}_n\}_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$, $\mathbb{A}_n = \{A_{nj}\}_{j=1}^{k_n}$ with A_{nj} open and $x \in \bigcap_{j=1}^{k_n} A_{nj}$, then $\{x\}$ is an open set such that $\forall n \in \mathbb{N}$, $x \in \{x\} \subset \bigcap_{j=1}^{k_n} A_{nj}$.*

Example 4. *The real line $\mathbb{R}$ in its usual topology is not a C-space, because $0 \in \mathbb{R}$ and considering the sequence $\{\mathbb{A}_n\}_{n\in\mathbb{N}}$, where each $\mathbb{A}_n = \{(-\frac{1}{k}, \frac{1}{k}); \forall k \in \mathbb{N}, 1 \le k \le n\}$, we have that for each $n \in \mathbb{N}$, $0 \in \bigcap_{k=1}^{n} (-\frac{1}{k}, \frac{1}{k})$, but there is not an open set U in $\mathbb{R}$ such that $\forall n \in N$, $0 \in U \subset \bigcap_{k=1}^{n} (-\frac{1}{k}, \frac{1}{k})$.*

Lemma 1. *Let X be a Hausdorff C-space, A a Hurewicz subspace of X and $x_0 \notin A$, then there are disjoint open sets V and U in X containing x_0 and A, respectively.*

Proof. Consider $a \in A$, with $x_0 \notin A$ we have that $x_0 \neq a$, since X is a Hausdorff space, there are V_a and U_a disjoint open sets in X containing x_0 and a, respectively.

By considering $\forall n \in N$, $\mathbb{U}_n = \{U_a\}_{a\in A}$, we have that $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ is a sequence of coverings of A by open sets in X. Since, A is Hurewicz by Proposition 5 there exists a sequence $\{\mathbb{W}_n\}_{n\in\mathbb{N}}$ such that:

(i) $\forall n \in \mathbb{N}$, $\mathbb{W}_n \subset_{<\infty} \mathbb{U}_n$,i. e., $\exists I_n \subset_{<\infty} A$, such that $\mathbb{W}_n = \{U_a\}_{a\in A}$.

(ii) For each $y \in A$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \in \mathbb{N}$ if $n \ge n_0$, there exists $a \in I_n$ with $y \in U_a$.

Consider now, $\forall n \in \mathbb{N}$, $\mathbb{V}_n = \{V_a\}_{a\in I_n}$, since for each $a \in A$, $x_0 \in V_a$ then $\forall n \in \mathbb{N}$, $x_0 \in \bigcap_{a\in I_n} V_a$, but since X is a C-space we have that there exists an open set V in X such that $\forall n \in \mathbb{N}$, $x_0 \in V \subset \bigcap_{a\in I_n} V_a$.

Consider $U = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{a\in I_n} U_a\right)$. We have that U is an open set in X and $A \subset U$, because for each $y \in A$ by (ii) $\exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ if $n \ge n_0$ there exists $a \in I_n$ such that $y \in U_a$. Hence $\bigcup_{a\in I_n} U_a \subset \bigcup_{n\in\mathbb{N}} \left(\bigcup_{a\in I_n} U_a\right) = U$.

Now we prove that V and U are disjoint sets. Suppose that there exists $y \in V \cap U$. Then $y \in U = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{a\in I_n} U_a\right)$ hence $\exists n_0 \in \mathbb{N}$ such that $y \in \bigcup_{a\in I_n} U_a$ then $\exists a_0 \in I_{n_0}$ such that $y \in U_{a_0}$, but since $y \in V$ and $\forall n \in \mathbb{N}$ $V \subset \bigcap_{a\in I_n} V_a$ then we have that $V \subset \bigcap_{a\in I_{n_0}} V_a$, hence $y \in V \subset \bigcap_{a\in I_{n_0}} V_a$ then $\forall a \in I_{n_0}$ $y \in V_a$ which implies that $y \in V_{a_0}$ and then $V_{a_0} \cap U_{a_0} \neq \emptyset$, contradiction. $\square$

Proposition 8. *Let $\langle X, T\rangle$ be a Hausdorff C-space and A be a Hurewicz subspace of X, then A is closed.*

Proof. We are going to show that if $x \in A^c$ there is $U \in T$ such that $x \in U \subset A^c$.

Since $x \notin A$, A is Hurewicz and X is a Hausdorff C-space then by Lemma 1, there are V and U disjoint open sets containing x and A, respectively. Hence we have that $x \in V \subset A^c$, because since $A \subset U$ and $U \cap V = \emptyset$ then $A \cap V = \emptyset$.

So A^c is open, hence A is closed. □

Proposition 9. *Let $\langle X, T_X\rangle$ be a topological C-space and let $\langle Y, T_Y\rangle$ be a subspace of X, then Y is a C-space.*

Proof. Let $x \in Y$, consider $\{\mathbb{A}_n\}_{n\in\mathbb{N}}$ a sequence such that for each $n \in \mathbb{N}$, $\mathbb{A}_n = \{A_{nj} \in T_Y; j = 1\ldots, k_n\}$ and $x \in \bigcap_{j=1}^{k_n} A_{nj}$ then, since Y is subspace of X, $\forall n \in N$ and $\forall j = 1,\ldots,k_n$, there exists $U_{nj} \in T_X$ such that $A_{nj} = U_{nj} \cap Y$. Hence $A_{nj} \subset U_{nj}$, then $\forall n \in \mathbb{N}$, $x \in \bigcap_{j=1}^{k_n} U_{nj}$. Since X is a C-space, there exists $U \in T_X$ such that for each $n \in \mathbb{N}$, $x \in U \subset \bigcap_{j=1}^{k_n} U_{nj}$, hence $V = U \cap Y$, $V \in T_Y$, $x \in V$ and for each $n \in \mathbb{N}$

$$V = U \cap Y \subset \left(\bigcap_{j=1}^{k_n} U_{nj}\right) \cap Y = \bigcap_{j=1}^{k_n}(U_{nj} \cap Y) = \bigcap_{j=1}^{k_n} A_{nj}.$$

So Y is a C-space. □

Proposition 10. *Let $\langle X, T_X\rangle$ be a topological C-space, $\langle Y, T_Y\rangle$ be a topological space and $f : X \to Y$ be an open, continuous and surjective function. Then Y is a C-space.*

Proof. Let $y \in Y$ and $\{\mathbb{A}_n\}_{n\in\mathbb{N}}$ a sequence such that for each $n \in \mathbb{N}$, $\mathbb{A}_n = \{A_{nj} \in T_Y; j = 1\ldots, k_n\}$ and $y \in \bigcap_{j=1}^{k_n} A_{nj}$. Since f is a surjection, $\exists x \in X$ such that $f(x) = y$. For each $n \in N$ and for each $j = 1,\ldots,k_n$, since f is continuous and $A_{nj} \in T_Y$, we have $f^{-1}(A_{nj}) \in T_X$. Hence for each $n \in \mathbb{N}, x \in f^{-1}(\{y\}) \subset f^{-1}(\bigcap_{j=1}^{k_n} A_{nj}) = \bigcap_{j=1}^{k_n} f^{-1}(A_{nj})$. Because X is a C-space, there exists $U \in T_X$ such that for each $n \in \mathbb{N}$, $x \in U \subset \bigcap_{j=1}^{k_n} f^{-1}(A_{nj})$, then for each $n \in \mathbb{N}$ $y = f(x) \in f(U) \subset f(\bigcap_{j=1}^{k_n} f^{-1}(A_{nj})) \subset \bigcap_{j=1}^{k_n} A_{nj}$, since f is an open function we have that $f(U)$ is an open set.

So Y is a C-space. □

4. Locally Hurewicz Spaces

Definition 8. *A topological space $\langle X, T\rangle$ is locally Hurewicz if and only if, for each $x \in X$ and for each $V \in T$ with $x \in V$, there exists $U \in T$ and H Hurewicz, where $x \in U \subset H \subset V$.*

Definition 9. *A topological space $\langle X, T\rangle$ is weakly locally Hurewicz if and only if, for each $x \in X$, there exists $U \in T$ and H Hurewicz, such that $x \in U$.*

Definition 10. *A topological space $\langle X, T\rangle$ is relatively locally Hurewicz if and only if, for each $x \in X$, there exists $U \in T$ with $\overline{U}$ Hurewicz, such that $x \in U$.*

Proposition 11. *If $\langle X, T\rangle$ is a Hurewicz topological space then $\langle X, T\rangle$ is weakly locally Hurewicz.*

Proof. For each $x \in X$, consider $U = X$ and $H = X$, then $U \in T$, H is Hurewicz and $x \in U \subset H$. Hence X is weakly locally Hurewicz. □

Proposition 12. *If $\langle X, T\rangle$ is a Hurewicz topological space then $\langle X, T\rangle$ is relatively locally Hurewicz.*

Proof. For each $x \in X$, consider $U = X$. Then $x \in U$ and $\overline{U} = X$, hence $\overline{U}$ is Hurewicz. Therefore X is relatively locally Hurewicz. □

Proposition 13. *If $\langle X, T\rangle$ is a locally Hurewicz topological space then $\langle X, T\rangle$ is weakly locally Hurewicz.*

Proof. For each $x \in X$, since X is locally Hurewicz, for $V = X$ there are U open set in X and H Hurewicz such that $x \in U \subset H \subset X$. Hence X is a weakly locally Hurewicz space. □

Proposition 14. *If $\langle X, T\rangle$ is a relatively locally Hurewicz topological space then $\langle X, T\rangle$ is weakly locally Hurewicz.*

Proof. For each $x \in X$, because X is locally Hurewicz, there exists U open set in X, with $\overline{U}$ Hurewicz such that $x \in U$. Then $x \in U \subset \overline{U}$ with $\overline{U}$ Hurewicz. Hence X is a weakly locally Hurewicz space. □

Proposition 15. *A Hausdorff C-space $\langle X, T\rangle$ is locally Hurewicz if and only if, X is weakly locally Hurewicz.*

Proof. ($\Rightarrow$) Proposition 13.

($\Leftarrow$) Consider $x \in X$ and V a neighborhood of x. Since X is weakly locally Hurewicz there exist $U \in T$ and H Hurewicz with $x \in U \subset H$.

Consider $A = H \cap V^c$, since $x \in V$ we have that $x \notin A$ and A is Hurewicz because H is Hurewicz and X is a Hausdorff C-space by Proposition 8 H is closed, then $A = H \cap V^c$, is a closed set. Since $A \subset H$ by Proposition 6 A is Hurewicz, then by Lemma 1 there are W_x and W_A disjoint open sets containing x and A, respectively.

Consider $U = W_x \cap int(H)$. Then U is an open set containing x and $U \subset int(H) \subset H$, then $\overline{U} \subset \overline{H} = H$. Therefore $\overline{U}$ is a closed subspace of a Hurewicz space hence by Proposition 6 $\overline{U}$ is Hurewicz.

Now we prove, that $\overline{U} \subset V$. We have $\overline{U} \cap A = \emptyset$. In fact, supposing by contradiction that $\exists y \in \overline{U} \cap A$, then $y \in \overline{U} \cap A$, then $y \in \overline{U}$ hence any neighborhood of y intersects U and $y \in A \subset W_A$ then $W_A \cap U \neq \emptyset$, but $U \subset W_x$ he have that $W_A \cap W_x \neq \emptyset$, contradiction. Hence $\overline{U} \subset H$ and $\overline{U} \cap A = \emptyset$, which implies that $\overline{U} \subset V$, then we have that $x \in U \subset \overline{U} \subset V$.

Therefore, X is a locally Hurewicz space. □

Proposition 16. *A Hausdorff C-space $\langle X, T \rangle$ is relatively locally Hurewicz if and only if, X is weakly locally Hurewicz.*

Proof. (⇒) Proposition 14.

(⇐) Consider $x \in X$. Because X is weakly locally Hurewicz, there are $U \in T$ and H Hurewicz such that $x \in U \subset H$. So $\overline{U} \subset \overline{H}$ and since H is a Hurewicz subspace of a Hausdorff C-space by Proposition 8 H is closed. Then $\overline{U} \subset H$, but $\overline{U}$ being closed by Proposition 6 $\overline{U}$ is Hurewicz, then $x \in U$, with $\overline{U}$ Hurewicz.

So X is relatively locally Hurewicz space. □

Proposition 17. *If $\langle X, T \rangle$ is a locally compact topological space then $\langle X, T \rangle$ is locally Hurewicz.*

Proof. For each $x \in X$ and V a neighborhood of x, since X is locally compact there are $U \in T$ and a compact set C with $x \in U \subset C \subset V$, but by Corollary 1, C is Hurewicz then X is locally Hurewicz. □

Proposition 18. *If $\langle X, T \rangle$ is a weakly locally compact topological space then $\langle X, T \rangle$ is weakly locally Hurewicz.*

Proof. Consider $x \in X$. Because X is weakly locally compact there are $U \in T$ and a compact set C such that $x \in U \subset C$. Since by Corollary 1 C is Hurewicz we have X weakly locally Hurewicz. □

Proposition 19. *If $\langle X, T \rangle$ is a relatively locally compact topological space then $\langle X, T \rangle$ is relatively locally Hurewicz.*

Proof. Consider $x \in X$. Because X is relatively locally compact there is $U \in T$ with $\overline{U}$ compact with $x \in U$. By Corollary 1, $\overline{U}$ is Hurewicz. Therefore X relatively locally Hurewicz. □

Example 5. *Consider $\mathbb{R}$ in its topology, we have that $\mathbb{R}$ is Hurewicz, then by Proposition 11 and by Proposition 12 we have that $\mathbb{R}$ is weakly locally Hurewicz and relatively locally Hurewicz. We have that $\mathbb{R}$ is not a C-space, but it is a locally Hurewicz space because for each $x \in \mathbb{R}$ and V open in $\mathbb{R}$, there exists $\varepsilon > 0$ such that $x \in (x-\varepsilon, x+\varepsilon) \subset V$, but $(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2})$ is such that $x \in (x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}) \subset [x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}] \subset (x-\varepsilon, x+\varepsilon) \subset V$, where $(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2})$ is an open set and $[x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}]$ is compact, hence by Corollary 1 is Hurewicz.*

Example 6. *Consider $\mathbb{R}$ with the discrete topology. We have that $\mathbb{R}$ is not Lindelöf, because $\{\{x\}/x \in \mathbb{R}\}$ is an open covering of $\mathbb{R}$ which does not have a countable subcovering, hence by Proposition 3 $\mathbb{R}$ with the discrete topology is not Hurewicz, but for each $x \in \mathbb{R}$, $\{x\}$ is a compact set, then $\{x\}$ is Hurewicz. Then we have that $\mathbb{R}$ with the discrete topology is locally Hurewicz, because for each $x \in \mathbb{R}$ and a neighborhood V of x, we have that $x \in \{x\} \subset V$, where $\{x\}$ is an open set and Hurewicz. Since $\mathbb{R}$ with the discrete topological is a Hausdorff C-space, we have that $\mathbb{R}$ is weakly locally Hurewicz and relatively locally Hurewicz.*

Example 7. *Consider $\mathbb{R}$ with the particular point p topology (Definition 4), by Example 2 we have that $\mathbb{R}$ is not Hurewicz. Given a nonempty open set A in $\mathbb{R}$, we have that $\overline{A} = \mathbb{R}$, because for $y \in \mathbb{R}$, we have that any neighborhood of y being nonempty contains p, then intersects A. Hence $\mathbb{R}$ is not relatively locally Hurewicz, because for each $x \in \mathbb{R}$ any neighborhood V of x is a nonempty set, then $\overline{V} = \mathbb{R}$, but since $\mathbb{R}$ is not Hurewicz, then we can not obtain a neighborhood of x where the closure is Hurewicz. But $\mathbb{R}$ is weakly locally compact, because for each $x \in \mathbb{R}$ we have that $x \in \{x, p\}$ and $\{x, p\}$ is an open and compact set, hence by Proposition 18, $\mathbb{R}$ is weakly locally Hurewicz.*

Proposition 20. *Let $\langle X, T_X\rangle$ and $\langle Y, T_Y\rangle$ be topological spaces, where X is locally Hurewicz and let $f : X \to Y$ be a continuous, open and surjective function, then Y is locally Hurewicz.*

Proof. Consider $y \in Y$ and V a neighborhood of y, then $\exists x \in X$ such that $f(x) = y$. By the continuity of f, we have that $f^{-1}(V) \in T_X$. Because X is locally Hurewicz, there are U open set in X and H Hurewicz such that $x \in U \subset H \subset f^{-1}(V)$, then

$$y = f(x) \in f(U) \subset f(H) \subset f(f^{-1}(V)) \subset V,$$

since f is an open function $f(U)$ is open in Y and since f is continuous $f(H)$ is Hurewicz by Proposition 4.

So Y is a locally Hurewicz space. □

Proposition 21. *Let $\langle X, T_X\rangle$ and $\langle Y, T_Y\rangle$ be topological spaces, where X is weakly locally Hurewicz and let $f : X \to Y$ be a continuous, open and surjective function, then Y is weakly locally Hurewicz.*

Proof. By consider $y \in Y$, $\exists x \in X$ such that $f(x) = y$. Because X is weakly locally Hurewicz, there are $U \in T_X$ and H Hurewicz with $x \in U \subset H$, then $y = f(x) \in f(U) \subset f(H)$, since f is an open we have that $f(U) \in T_Y$ and by the continuity of f and Proposition 4 $f(H)$ is Hurewicz.

So Y is a weakly locally Hurewicz space. □

Proposition 22. *Let $\langle X, T_X\rangle$ and $\langle Y, T_Y\rangle$ be topological spaces, with X is relatively locally Hurewicz and Y a Hausdorff C-space and let $f : X \to Y$ be a continuous, open surjection, then Y is relatively locally Hurewicz.*

Proof. Consider $y \in Y$, then $\exists x \in X$ with $f(x) = y$. From the fact that X is relatively locally Hurewicz, there is $U \in T_X$, with $\overline{U}$ Hurewicz with $x \in U$, then $y = f(x) \in f(U)$ and since $\overline{U}$ Hurewicz and f is continuous by Proposition 4 $f(\overline{U})$ is Hurewicz, but $\overline{f(U)} \subset f(\overline{U})$. Since Y is a Hausdorff C-space and $f(\overline{U})$ is Hurewicz, we have that $f(\overline{U})$ is closed, then $\overline{f(U)} \subset f(\overline{U})$, then $\overline{f(U)}$ is a closed subspace of a Hausdorff space, then $\overline{f(U)}$ is Hurewicz, so $y \in f(U) \subset \overline{f(U)}$, with $f(U)$ is open and $\overline{f(U)}$ Hurewicz.

So Y is a relatively locally Hurewicz space. □

Proposition 23. *Let $\langle X, T_X \rangle$ be a locally Hurewicz topological space and let $\langle Y, T_Y \rangle$ be a closed subspace of X, then Y is locally Hurewicz.*

Proof. Consider $y \in Y$ and V an open set in Y such that $y \in V$. Then $y \in X$ and there is V' opens in X, such that $V = V' \cap X$. Since X is locally Hurewicz, there are U' open in X and H Hurewicz such that $y \in U' \subset H \subset V'$, but since $y \in Y$ we have that
$y \in U' \cap Y \subset H \cap Y \subset V' \cap Y = V$, where $U' \cap Y$ is open in Y and $H \cap Y$ is Hurewicz by Proposition 7. Therefore Y is locally Hurewicz. □

Proposition 24. *Let $\langle X, T_X \rangle$ be a weakly locally Hurewicz topological space and let $\langle Y, T_Y \rangle$ be a closed subspace of X, then Y is weakly locally Hurewicz.*

Proof. Consider $y \in Y$. Then there is $U \in T_X$ and H Hurewicz in X such that $y \in U \subset H$. So we have that $y \in U \cap Y \subset H \cap Y$, where $U \cap Y$ is open in Y and $H \cap Y$ is Hurewicz by Proposition 7, then Y is weakly locally Hurewicz. □

Proposition 25. *Let $\langle X, T_X \rangle$ be a relatively locally Hurewicz topological space and let $\langle Y, T_Y \rangle$ be a closed subspace of X, then Y is relatively locally Hurewicz.*

Proof. Consider $y \in Y$. Then there exists $U \in T_X$ with $\overline{U}$ Hurewicz and $y \in U$. Then $y \in U \cap Y$ and by Proposition 7 $\overline{U} \cap Y$ is Hurewicz, but $\overline{U \cap Y} \subset \overline{U} \cap \overline{Y} = \overline{U} \cap Y$. By Proposition 6 we have that $\overline{U \cap Y}$ is Hurewicz, hence Y is relatively locally Hurewicz. □

Proposition 26. *Let X be a weakly locally Hurewicz topological space then $A \subset X$ is open in X if and only if, $A \cap H$ is open in H for each H Hurewicz.*

Proof. ($\Rightarrow$) If A is open in X then $A \cap H$ is open in H.

($\Leftarrow$) Consider $a \in A$. We have that $a \in X$ and since X is weakly locally Hurewicz there are $U \in T_X$ and H Hurewicz in X such that $a \in U \subset H$. Since H is Hurewicz, $A \cap H$ is open in H, then since $A \cap U = (A \cap H) \cap U$ we have that $A \cap U$ is open in U, then there is $V \in T_X$ such that $A \cap U = U \cap V$, but since U and V are open in X we have that $U \cap V$ is open in X and $U \cap A$ is open in X, hence $a \in A \cap U \subset A$, whit $A \cap U$ open in X.

Then A is open. □

Proposition 27. *Let X be a locally Hurewicz topological space, then a subset A of X is open in X if and only if, $A \cap H$ is open in H for each H Hurewicz.*

Proof. ($\Rightarrow$) If A is open in X then $A \cap H$ is open in H.

($\Leftarrow$) Let $A \subset X$ such that $A \cap H$ is open in H for each H Hurewicz, then by Proposition 13 we have that X is weakly locally Hurewicz and by Proposition 26 we have that A is open in X. □

Proposition 28. *Let X be a relatively locally Hurewicz topological space and $A \subset X$, then A is open in X if and only if, $A \cap H$ is open in H for each H Hurewicz.*

Proof. ($\Rightarrow$) If A is open in X then $A \cap H$ is open in H.

($\Leftarrow$) If $A \cap H$ is open in H for each H Hurewicz, by Proposition 14 we have that X is weakly locally Hurewicz and by Proposition 26 we have that A is open in X. □

References

[1] F. Cammaroto and G. Santoro. *Some counterexamples and properties on generalizations of Lindelöf spaces*, International Journal of Mathematical Sciences, 19(4), 737–746. 1996.

[2] J. Dugundji. *Topology*, Allyn an Bacon, Boston, 1988.

[3] W. Hurewicz. *Über eine verallgeneinerung des borelschen Theorems*, Mathematische Zeitschrift, 24, 401–421. 1925.

[4] E.L. Lima. *Elementos de topologia geral*, Editora da Universidade de São Paulo, 1970.

[5] J. Munkres. *Topology: A First Course*, Prentice-Hall, Englewood Cliffs, New Jersey, 1999.