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Regularity of the Rees and Associated Graded Modules

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Abstract. Let A be a Noetherian ring and $\mathfrak{b}$ be an ideal of A. Let E be a finitely generated A-module. It is shown that there is a close relationship between the cohomological invariants of the associated graded module of E with respect to $\mathfrak{b}$ and the Rees module of E associated to $\mathfrak{b}$. Also a formula for the regularity of the Rees module of E associated to $\mathfrak{b}$ will be given.

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1. Introduction

Let $S = \oplus_{n\geq 0} S_n$ be a finitely generated standard graded algebra over a Noetherian commutative ring S_0. We denote by $S_+ = \oplus_{n\geq 1} S_n$ the ideal generated by the homogeneous elements of positive degree of S. For a graded S-module L, the homogeneous part of degree n of L, is denoted by L_n, and $L(t)$ is the same module L shifted by t. The end of L is defined by $end(L) = \max\{n : L_n \neq 0\}$, and $end(0) = -\infty$ by convention. For each $i \geq 0$, the ith local cohomology module $H^i_{S_+}(L)$ of a graded S-module L supported in S_+ is also a graded S-module in a natural way and $H^i_{S_+}(L)_n$ is a finitely generated S_0-module for all $i \geq 0$ and all n, and it is zero for for large values of n (see [1, Chapter 15]). Following [3], we put $a_i(L) = end(H^i_{S_+}(L))$. Then the regularity of L is defined by $reg(L) = \max\{a_i(L) + i : i \geq 0\}$.

Let A be a Noetherian commutative ring and $\mathfrak{b}$ an ideal of A. Let E be a finitely generated A-module. We denote by $R_{\mathfrak{b}}(E) = \oplus_{n\geq 0}\mathfrak{b}^n E$ the Rees module of E associated to $\mathfrak{b}$ and by $G_{\mathfrak{b}}(E) = \oplus_{n\geq 0}\mathfrak{b}^n E/\mathfrak{b}^{n+1}E = R_{\mathfrak{b}}(E)/\mathfrak{b}R_{\mathfrak{b}}(E)$ the associated graded module of E with respect to $\mathfrak{b}$. In the case $E = A$, these modules are denoted by $R(\mathfrak{b})$ and $G(\mathfrak{b}) = R(\mathfrak{b})/\mathfrak{b}R(\mathfrak{b})$ respectively.

Recall from [2, Definition 4.6.4] that an ideal $\mathfrak{a} \subseteq \mathfrak{b}$ is called a reduction of $\mathfrak{b}$ with respect to E if $R_{\mathfrak{b}}(E)$ is a finitely generated $R(\mathfrak{a})$-module, or equivalently, if $\mathfrak{b}^{r+1}E = \mathfrak{a}\mathfrak{b}^r E$ for some

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$r \geq 0$. The least such r is denoted by $r_{\mathfrak{a}}(\mathfrak{b}, E)$.

This paper is divided into 3 sections. In section 2 we prepare some results related to the Castelnuovo regularity of a graded module, from which we prove in theorem 1 that $\mathrm{reg}(L)$ can be characterized in terms of a minimal reduction of S_+ with respect to L, which is generated by an S_+-filter regular sequence of homogeneous elements of degree 1, for L. In section 3, using the ideas of [5], we will show that there is a close relationship between the invariants $a_i(R_{\mathfrak{b}}(E))$ and $a_i(G_{\mathfrak{b}}(E))$, from which we can easily derive the formula $reg(R_{\mathfrak{b}}(E)) = reg(G_{\mathfrak{b}}(E))$. Also we give a formula for the number $\mathrm{reg}(R_{\mathfrak{b}}(E))$ in Corollary 4.

2. Preliminaries

From now on assume that L is finitely generated. Let $\mathbf{f} = f_1, \ldots, f_h$ be a sequence of homogeneous elements of S. We call $f_1, \ldots, f_h$ an *S_+-filter regular sequence for L* if for all $i = 1, \ldots, h$

$$f_i \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_S(L/(f_1,\ldots,f_{i-1})L) \setminus V(S_+)} \mathfrak{p},$$

where $V(S_+)$ is the set of all prime ideals of S containing S_+ and for an S-module X, $\mathrm{Ass}_S(X)$ denotes the set of all associated prime ideals of X. We define

$$e(\mathbf{f}, L) = \sup\{end(((f_1, \ldots, f_{i-1})L :_L f_i)/(f_1, \ldots, f_{i-1})L) : i = 1, \ldots, h\}.$$

Then by [1, 18.3.8], $f_1, \ldots, f_h$ is an S_+-filter regular sequence on L if and only if $e(\mathbf{f}, L) < \infty$.

It will be crucial to understand how the invariants $a_i(L)$ behave with respect to S_+-filter regular sequences for L. This relationship was illuminated by Trung in the following lemma. Because of its importance in our argument, we supply the proof along the statement.

Lemma 1 ([6, Lemma 2.3]). *Let $f \in S_1$ be a homogeneous S_+-filter regular element for L. Then for all $i \geq 0$,*

$$a_{i+1}(L) + 1 \leq a_i(L/fL) \leq \max\{a_i(L), a_{i+1}(L) + 1\}.$$

Proof. Note that by the statement after the definition of an S_+-filter regular sequence for L, $H^0_{S_+}(0 :_L f) = (0 :_L f)$ and hence $H^i_{S_+}(0 :_L f) = 0$ for all $i \geq 1$. Then from the exact sequence

$$0 \longrightarrow (0 :_L f) \longrightarrow L \longrightarrow L/(0 :_L f) \longrightarrow 0,$$

we see that $H^i_{S_+}(L) \cong H^i_{S_+}(L/(0 :_L f))$ for all $i \geq 1$. Now, from the exact sequence

$$0 \longrightarrow L/(0 :_L f) \xrightarrow{f} L(1) \longrightarrow L(1) \longrightarrow 0,$$

we obtain the exact sequence

$$H^i_{S_+}(L)_{n+1} \to H^i_{S_+}(L/fL)_{n+1} \to H^{i+1}_{S_+}(L)_n \to H^{i+1}_{S_+}(L)_{n+1},$$

for each $i \geq 0$ and $n \in \mathbb{Z}$. Analyzing these sequences easily yields the desired inequalities. □

Lemma 2. *Let* $\mathbf{f} = f_1, \ldots, f_h$ *be an* S_+*-filter regular sequence of homogeneous elements of degree* 1 *for* L. *Then:*

(i) $e(\mathbf{f}, L) = \max\{a_i(L) + i : i = 0, \ldots, h-1\}$,

(ii) for all $0 \le t \le h$,

$$\max\{a_i(L)+i : i = 0, \ldots, t\} = \max\{end(((f_1, \ldots, f_t)L :_L S_+)/(f_1, \ldots, f_t)L) : i = 0, \ldots, t\}.$$

Proof. (i) We prove by induction on $h \ge 1$. Since $(0 :_L f_1) \subseteq \cup_{n\ge1}(0 :_L {S_+}^n)$ and

$$f_1 H^0_{S_+}(L)_{a_0(L)} \subseteq H^0_{S_+}(L)_{a_0(L)+1} = 0,$$

thus $e(f_1, L) = a_0(L)$ and the case $h = 1$ is immediate. So let $h > 1$. Let $\bar{L} = L/f_1L$ and $\bar{\mathbf{f}} = \bar{f}_2, \ldots, \bar{f}_h$ in $\bar{S} = S/(f_1)$. By induction and using Lemma 1, we have

$$\begin{aligned}\max\{a_i(L) + i : i = 1, \ldots, h-1\} &\le e(\bar{\mathbf{f}}, \bar{L}) = \max\{a_i(L/f_1L) + i : i = 0, \ldots, h-2\}\\ &\le \max\{a_i(L) + i : i = 0, \ldots, h-1\}.\end{aligned}$$

Now since $e(\mathbf{f}, L) = \max\{e(f_1, L), e(\bar{\mathbf{f}}, \bar{L})\}$, the result follows.

(ii) Using Lemma 1 repeatedly, we deduce that

$$a_i(L) + i \le a_0(L/(f_1, \ldots, f_i)L) \le \max\{a_j(L) + j : j = 0, \ldots, i\}.$$

From this it follows that for $t \le h$,

$$\max\{a_i(L) + i : i = 0, \ldots, t\} = \max\{a_0(L/(f_1, \ldots, f_i)L) : i = 0, \ldots, t\}.$$

Set $a = a_0(L/(f_1, \ldots, f_i)L)$. We have

$$H^0_{S_+}(L/(f_1, \ldots, f_i)L) = \bigcup_{n\ge1}((f_1, \ldots, f_i)L :_L {S_+}^n)/(f_1, \ldots, f_i)L.$$

Therefore

$$H^0_{S_+}(L/(f_1, \ldots, f_i)L)_a \subseteq ((f_1, \ldots, f_i)L :_L S_+)/(f_1, \ldots, f_i)L \subseteq H^0_{S_+}(L/(f_1, \ldots, f_i)L).$$

Hence $a(((f_1, \ldots, f_i)L :_L S_+)/(f_1, \ldots, f_i)L) = a$, and the result follows. □

The following corollary generalizes [5, Corollary 2.3] to the module case.

Corollary 1. *Let* $g = grade(S_+, L)$. *Then:*

(i) $a_i(L) = -\infty$ *for* $i < g$.

(ii) $a_g(L) \ge -g$.

(iii) If $H^1_{S_+}(L) \ne 0$, *then* $a_1(L) \ge -1$.

Proof. We may assume that the base ring S_0 is local with infinite residue field. Then from the graded version of prime avoidance theorem (see for example [2, Proposition 1.5.12]) there exists an L-sequence $f_1, \ldots, f_g$ of homogeneous elements of S_1. Since

$$(f_1, \ldots, f_g)L :_L S_+ = (f_1, \ldots, f_g)L$$

for $i = 1, \ldots, g$; hence Lemma 2(ii) implies that $\max\{a_j(L)+j : j = 1, \ldots, i-1\} = -\infty$. Hence $a_i(L) = -\infty$ for $i = 0, \ldots, g-1$. As a consequence,

$$a_g(L) + g = \max\{a_i(L) + i : i = 0, \ldots, g\} = a(((f_1, \ldots, f_g)L :_L S_+)/(f_1, \ldots, f_g)L) \geq 0.$$

Therefore, $a_g(L) \geq -g$ and (i) and (ii) have been proved.

To prove (iii), set $\bar{S} = S/H^0_{S_+}(S)$ and $\bar{L} = L/H^0_{S_+}(L)$. Then it is easy to see that $grade(\bar{S}_+, \bar{L}) \geq 1$ and $H^1_{\bar{S}_+}(\bar{L}) \cong H^1_{S_+}(L) \neq 0$. Therefore $a_1(L) = a_1(\bar{L}) \geq -1$ by (ii). □

Theorem 1. *Let* $\mathbf{f} = f_1 \in S_1, \ldots, f_h \in S_1$ *be an* S_+*-filter regular sequence for* L*. Let* $\mathfrak{b} = (f_1, \ldots, f_h)$ *be a reduction of* S_+ *with respect to* L*. Then*

$$reg(L) = \max\{e(\mathbf{f}, L), r_{\mathfrak{b}}(S_+, L)\}.$$

Proof. By Lemma 2 we have

$$e(\mathbf{f}, L) = \max\{end(((f_1, \ldots, f_i)L :_L S_+)/(f_1, \ldots, f_i)L) : i = 0, \ldots, h-1\}.$$

Furthermore, $r_{\mathfrak{b}}(S_+, L) = end(L/\mathfrak{b}L) = end(((f_1, \ldots, f_h)L :_L S_+)/(f_1, \ldots, f_h)L)$. Therefore

$$\begin{aligned}\max\{e(\mathbf{f}, L), r_{\mathfrak{b}}(S_+, L)\} &= \max\{end(((f_1, \ldots, f_i)L :_L S_+)/(f_1, \ldots, f_i)L) : i = 0, \ldots, h\}\\ &= \max\{a_i(L) + i : i = 0, \ldots, h\}.\end{aligned}$$

Since $reg(L) = \max\{a_i(L) + i : i \geq 0\}$, it is enough to show that $H^i_{S_+}(L) = 0$ for all $i > h$. If $h = 0$, then L is annihilated by some power of S_+ and so $H^i_{S_+}(L) = 0$ for all $i > 0$. So let $h \geq 1$. By induction, we have $H^i_{S_+}(L/f_1L) = 0$ for all $i > h-1$. Hence $a_i(L/f_1L) = -\infty$ for all $i > h-1$. By Lemma 1, this implies $a_{i+1}(L) = -\infty$ and $H^{i+1}_{S_+}(L) = 0$ for all $i > h$. □

3. Regularity results

In this section, using the ideas of [5], we will show that there is a close relationship between the invariants $a_i(R_{\mathfrak{b}}(E))$ and $a_i(G_{\mathfrak{b}}(E))$, from which we can easily derive the formula $reg(R(E)) = reg(G(E))$ which is a generalization of that of Ooishi [4] and [5, Theorem 3.1]. For simplicity we shall denote $R_{\mathfrak{b}}(E)$ by $R(E)$, $G_{\mathfrak{b}}(E)$ by $G(E)$, $R(\mathfrak{b})_+$ by R_+ and $G(\mathfrak{b})_+$ by G_+.

Theorem 2. *Let the notation be as in above. Then:*

(i) *For each* $i \neq 1$, $a_i(R(E)) \leq a_i(G(E))$.

(ii) $a_i(R(E)) = a_i(G(E))$ *if* $a_{i+1}(G(E)) \leq a_i(G(E))$, $i \neq 1$.

(iii) *If* $H^1_{G_+}(G(E)) \neq 0$ *or if* $\mathfrak{b} \subseteq \sqrt{(0:_A E)}$, *the statements (i) and (ii) hold for* $i = 1$.

(iv) *If* $H^1_{G_+}(G(E)) = 0$ *and* $\mathfrak{b} \nsubseteq \sqrt{(0:_A E)}$ *then* $a_1(R(E)) = -1$.

Proof. We consider the exact sequence

$$0 \longrightarrow R(E)_+ \longrightarrow R(E) \longrightarrow E \longrightarrow 0, \tag{1}$$

where E is considered as a graded R-module concentrated in degree zero.

Since $H^0_{R_+}(E)_n = 0$ for $n \neq 0$ and $H^i_{R_+}(E) = 0$ for $i \geq 1$, so from the exact sequence (1) we deduce that $H^i_{R_+}(R(E)_+)_n \cong H^i_{R_+}(R(E))_n$ for $n = 0$, $i \geq 2$, and for $n \neq 0$, $i \geq 0$. Since $H^i_{G_+}(G(E)) = H^i_{R_+}(G(E))$, the exact sequence

$$0 \longrightarrow R(E)_+(1) \longrightarrow R(E) \longrightarrow G(E) \longrightarrow 0, \tag{2}$$

induces the exact sequence

$$H^i_{R_+}(R(E)_+)_{n+1} \to H^i_{R_+}(R(E))_n \to H^i_{R_+}(G(E))_n \to H^{i+1}_{R_+}(R(E)_+)_{n+1}. \tag{3}$$

Replacing $H^i_{R_+}(R(E)_+)_{n+1}$ by $H^i_{R_+}(R(E))_{n+1}$ and setting $H^i_{R_+}(G(E)) = 0$ whenever that is possible, we get an epimorphism $H^i_{R_+}(R(E))_{n+1} \to H^i_{R_+}(R(E))_n$ for all $n \geq \max\{0, a_i(G(E)) + 1\}$ if $i = 0, 1$, and for $n \geq a_i(G(E)) + 1$ if $i \geq 2$. Since $H^i_{R_+}(R(E))_n = 0$ for large values of n, so we deduce that

$$H^i_{R_+}(R(E))_n = 0$$

for $n \geq \max\{0, a_i(G(E)) + 1\}$ if $i = 0, 1$ and for $n \geq a_i(G(E)) + 1$ if $i \geq 2$. From the above formula immediately we have $a_i(R(E)) \leq a_i(G(E))$ for $i \geq 2$.

For $i = 0$ we consider two cases. If $H^0_{R_+}(G(E)) = 0$, then $a_0(G(E)) = -\infty$. Therefore by (4) $H^0_{R_+}(R(E))_n = 0$ for all $n \geq 0$. From this it follows that $H^0_{R_+}(R(E)) = 0$. Hence $a_0(R(E)) = -\infty = a_0(G(E))$. If $H^0_{R_+}(G(E)) \neq 0$, $a_0(G(E)) \geq 0$. Hence $H^0_{R_+}(R(E))_n = 0$ for $n \geq a_0(G(E)) + 1$ by (4), which implies $a_0(R(E)) \leq a_0(G(E))$. So (i) is proved.

If $H^1_{R_+}(G(E)) \neq 0$, then $a_1(G(E)) \geq -1$ by Corollary 1(iii). Hence by (4) $H^1_{R_+}(R(E))_n = 0$ for $n \geq a_1(G(E))$ which implies $a_1(R(E)) \leq a_1(G(E))$. If $\mathfrak{b} \subseteq \sqrt{(0:_A E)}$, then $H^i_{R_+}(R(E)) = 0$ and $H^i_{R_+}(G(E)) = 0$ for all $i \geq 1$. Hence $a_1(R(E)) = a_1(G(E)) = -\infty$. So the first part of (iii) is proved.

Now we prove (ii) and the second part of (iii). It is sufficient to show that $a_i(G(E)) \le a_i(R(E))$ for $i \ge 0$. We may assume that $a_i(G(E)) \ne -\infty$. For $i = 0$, we have either $a_1(R(E)) \le -1$ or $a_1(R(E)) \le a_1(G(E))$ by (4). For $i \ge 1$, we have $a_{i+1}(R(E)) \le a_{i+1}(G(E))$ by (i). Hence the assumption $a_{i+1}(G(E)) \le a_i(G(E))$ implies that $a_{i+1}(R(E)) \le a_i(G(E))$. Put $n = a_i(G(E))$. Then $H^{i+1}_{R_+}(R(E)_+)_{n+1} \cong H^{i+1}_{R_+}(R(E))_{n+1} = 0$. Using this in the exact sequence (3), we get an epimorphism

$$H^i_{R_+}(R(E))_n \longrightarrow H^i_{R_+}(G(E))_n.$$

Since $H^i_{R_+}(G(E))_n \ne 0$, so $H^i_{R_+}(R(E))_n \ne 0$. Therefore, $a_i(G(E)) \le a_i(R(E))$.

To prove (iv) we assume that $H^1_{R_+}(G(E)) = 0$. Then $a_1(G(E)) = -\infty$. Hence $a_1(R(E)) \le -1$ by (4). If $a_1(R(E)) < -1$, $H^1_{R_+}(R(E))_{-1} = 0$. Since $H^0_{R_+}(G(E))_{-1} = 0$, from the exact sequence (2) we can deduce that $H^1_{R_+}(R(E)_+)_0 = 0$. Now, using the exact sequence (1) we get the exact sequence

$$H^0_{R_+}(R(E)_+)_0 \longrightarrow H^0_{R_+}(R(E))_0 \longrightarrow H^0_{R_+}(E) \longrightarrow 0.$$

But since $(R(E)_+)_0 = 0$, so $H^0_{R_+}(R(E)_+)_0 = 0$. Furthermore, $H^0_{R_+}(R(E))_0 = H^0_{\mathfrak{b}}(E)$ and $H^0_{R_+}(E) = E$. Therefore, $H^0_{\mathfrak{b}}(E) = E$ which is equivalent to the condition $\mathfrak{b}^t E = 0$ for some $t \ge 1$. Thus if, $\mathfrak{b} \nsubseteq \sqrt{(0 :_A E)}$, we must have $a_1(R(E)) = -1$. Now, the proof of the theorem is complete. □

Corollary 2. *Let $\ell := \max\{i : H^i_{G_+}(G(E)) \ne 0\}$. Then:*

(i) $a_\ell(R(E)) = a_\ell(G(E))$,

(ii) If $\mathfrak{b} \subseteq \sqrt{(0 :_A E)}$ *or* $\ell \ge 1$, *then* $\ell = \max\{i : H^i_{R_+}(R(E)) \ne 0\}$.

Proof. For $i \ge \ell$, we have $a_i(G(E)) \ge a_{i+1}(G(E)) = -\infty$. Therefore, $a_i(R(E)) = a_i(G(E))$ if $i \ne 1$ by Theorem 2(ii). Hence (i) and(ii) are obvious if $\ell > 1$. It remains to show that $a_1(R(E)) = a_1(G(E))$ if $\ell = 1$ or if $\ell = 0$ and $\mathfrak{b} \subseteq \sqrt{(0 :_A E)}$. But this follows from Theorem 2(iii). □

Corollary 3. *With the notation as in above we have*

$$reg(R(E)) = reg(G(E)).$$

Proof. By Theorem 2(i) we have $a_i(R(E)) + i \le a_i(G(E)) + i$ for $i \ne 1$. By Theorem 2(iii) and (iv), either $a_1(R(E)) + 1 \le a_1(G(E)) + 1$ or $a_1(R(E)) + 1 = 0 \le reg(G(E))$. Therefore,

$$reg(R(E)) = \max\{a_i(R(E)) + i : i \ge 0\} \le \max\{a_i(G(E)) + i : i \ge 0\} = reg(G(E)).$$

To prove $reg(G(E)) \le reg(R(E))$, let i be maximal such that $reg(G(E)) = a_i(G(E)) + i$. Then $H^i_{G_+}(G(E)) \ne 0$ and $a_{i+1}(G(E)) < a_i(G(E))$. Now, using Theorem 2(ii),(iii), we get $a_i(R(E)) = a_i(G(E))$. Hence $reg(G(E)) = a_i(R(E)) + i \le reg(R(E))$. □

In the following we consider $R(\mathfrak{b})$ as a subring of the polynomial ring $A[t]$.

Proposition 1. *Let $f_1, \ldots, f_h$ be a sequence of elements of $\mathfrak{b}$. Then $\mathbf{f} := f_1 t, \ldots, f_h t$ is an $R(\mathfrak{b})_+$-filter regular sequence for $R(E)$ if and only if for all large $n \geq 1$,*

$$[(f_1, \ldots, f_{i-1})\mathfrak{b}^n E :_E f_i] \cap \mathfrak{b}^n E = (f_1, \ldots, f_{i-1})\mathfrak{b}^{n-1} E \text{ for } i = 1, \ldots, h. \tag{4}$$

If this is the case, then $e(\mathbf{f}, R(E))$ is the least integer r such that (4) holds for all $n \geq r+1$.

Proof. The sequence $\mathbf{f} = f_1 t, \ldots, f_h t$ is an $R(\mathfrak{b})_+$-filter regular sequence for $R(E)$ if and only if $[(f_1 t, \ldots, f_{i-1} t)R(E) :_{R(E)} f_i t]_n$ is equal to $[(f_1 t, \ldots, f_{i-1} t)R(E)]_n$ for all large $n \geq 1$ and all $i = 1, \ldots, h$. But the first module is equal to $[(f_1, \ldots, f_{i-1})\mathfrak{b}^n E :_E f_i] \cap \mathfrak{b}^n E$ and the second is equal to $(f_1, \ldots, f_{i-1})\mathfrak{b}^{n-1} E$. We note that $e(\mathbf{f}, R(E))$ is the least integer r such that the equality $[(f_1 t, \ldots, f_{i-1} t)R(E) :_{R(E)} f_i t]_n = [(f_1 t, \ldots, f_{i-1} t)R(E)]_n$ holds for all $n \geq r+1$. □

Corollary 4. *Let $\mathfrak{a} = (f_1, \ldots, f_h)$ be a reduction of $\mathfrak{b}$ with respect to E. Suppose that $\mathbf{f} = f_1 t, \ldots, f_h t$ is an $R(\mathfrak{b})_+$-filter regular sequence for $R(E)$. Then*

$$reg(R(E)) = \min\{r \geq 0 : r \geq r_{\mathfrak{a}}(\mathfrak{b}, E) \text{ and (4) holds for all } n \geq r+1\}.$$

Proof. Let $Q = (f_1 t, \ldots, f_h t)$. Since $\mathfrak{a}$ is a reduction of $\mathfrak{b}$ relative to E, then Q is a reduction of $R(\mathfrak{b})_+$ relative to $R(E)$. Moreover if $\mathfrak{a}\mathfrak{b}^n E = \mathfrak{b}^{n+1}$, then $QR(\mathfrak{b})_+^n R(E) = R(\mathfrak{b})_+^{n+1} R(E)$ and $r_{\mathfrak{a}}(\mathfrak{b}, E) = r_Q(R(\mathfrak{b})_+, R(E))$. By Theorem 1,

$$reg(R(E)) = \max\{e(\mathbf{f}, R(E)), r_{\mathfrak{a}}(\mathfrak{b}, E)\}.$$

Therefore, the result follows from Proposition 1. □

Similarly as for Proposition 1, we can prove the following characterization of a homogeneous $G(\mathfrak{b})_+$ filter regular sequence of degree 1 for $G(E)$. If $x \in A$ then x^* denotes the initial form of x in $G(\mathfrak{b})$.

Proposition 2. *Let $f_1, \ldots, f_h$ be elements of $\mathfrak{b}$. Then $\mathbf{f}^* = f_1^*, \ldots, f_h^*$ is an $G(\mathfrak{b})_+$-filter regular sequence for $G(E)$) if and only if for large values of n,*

$$[(f_1, \ldots, f_{i-1})\mathfrak{b}^n E + \mathfrak{b}^{n+2} E] :_E f_i \cap \mathfrak{b}^n E = ((f_1, \ldots, f_{i-1})\mathfrak{b}^{n-1} E + \mathfrak{b}^{n+1} E)$$

for $i = 1, \ldots, s$. If this is the case, $e(\mathbf{f}^*, G(E))$ is the least number r such that the above equality holds for $n \geq r+1$.

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