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Frenet Apparatus of the Curves and Some Special Curves in the Euclidean 5-Space E^5

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Abstract. In this study, initially the geometric meanings of the curvatures of the curves parametrized with the arc length are given in E^5. This is followed by the calculation of the Frenet vectors and curvatures of any curve. After these, some results have been given for the state of evolute curve X being a W-curve and the Frenet vectors and curvatures of involute curve Y have been calculated in terms of Frenet vectors and curvatures of the curve X. At last, the differential equation of the spherical curves, the equation of the radius and the center of the osculating hyperspheres have been achieved in E^5.

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1. Introduction

The involute-evolute curves and helices can often be seen in our daily lives. For example, the idea of a string involute is due to C. Huygens, who is also known for his works in optics. He discovered involutes while trying to build a more accurate clock [2, 7]. In addition to this, standard screws, bolts and a double-stranded molecule of DNA are the most common examples for helices in the nature and structures [11].

A. R. Forsyth (1930) [4] has took the hypothesis of curves and surfaces in the four-dimensional Euclidean space E^4 [4], while H. Gluck (1966) [5] examined the curvatures of the curve in the n-dimensional Euclidean space E^n. Lately the studies in the four and five-dimensional spaces have been accelerated. For example, some characterizations for the spherical curves and helices have been obtained in the four-dimensional Euclidean space E^4, [8, 10, 13]. Also some characterizations related to the inclined curves have been defined in the 5-dimensional Euclidean space E^5 and 5-dimensional Lorentzian space L^5, [1, 11]. The Frenet vectors of any curve and involute-evolute curves in E^4 and E_1^4 have been given by [12, 15]. In addition

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to this, the curvatures and Frenet vectors of the curves parametrized with the arc length in E^5 and L^5 have been determined, [14, 16]. At last, Bertrand curves in E^5 and L^5 have been defined, [3, 8].

In this study, initially we have given the geometrical meanings of the curvatures of curves parametrized with arc length in the Euclidean 5-Space. Afterwards, we have calculated the Frenet vectors and curvatures of an arbitrary curve in E^5. Moreover, we have given the Frenet vectors and curvatures of the involute curve Y in the state of the evolute curve X as the W-curve. Finally, we have defined the differential equation of the spherical curves, the equation of the center of osculating hyperspheres and the equation of their radius in E^5.

2. Preliminaries

In this section, we recall some basic concepts on classical differential geometry of space curve in the Euclidean 5-space and the definitions of special curves. Let $X : I \subset R \to E^5$ be an arbitrary curve in the Euclidean 5-space. We call the curve X as unit speed curve if $\langle X'(s), X'(s)\rangle = 1$, where $\langle , \rangle$ is the standard scalar product of E^5 given by

$$\langle a, b\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5,$$

for each vectors $a = (a_1, a_2, a_3, a_4, a_5)$ and $b = (b_1, b_2, b_3, b_4, b_5)$ of E^5, [6]. The norm of a vector a of E^5 is given by $\|a\| = \sqrt{\langle a, a\rangle}$, [6].

Let $a = (a_1, a_2, a_3, a_4, a_5)$, $b = (b_1, b_2, b_3, b_4, b_5)$, $c = (c_1, c_2, c_3, c_4, c_5)$ and $d = (d_1, d_2, d_3, d_4, d_5)$ be vectors in E^5. The vectorial product of these vectors is defined by the determinant,[6]

$$a \wedge b \wedge c \wedge d = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{vmatrix}$$

where e_i for $1 \le i \le 5$ are the standard basis vectors of E^5 which satisfies $e_1 \wedge e_2 \wedge e_3 \wedge e_4 = e_5$, $e_2 \wedge e_3 \wedge e_4 \wedge e_5 = e_1$, $e_3 \wedge e_4 \wedge e_5 \wedge e_1 = e_2$, $e_4 \wedge e_5 \wedge e_1 \wedge e_2 = e_3$, $e_5 \wedge e_1 \wedge e_2 \wedge e_3 = e_4$.

Let $\{V_1, V_2, V_3, V_4, V_5\}$ denotes the moving Frenet Frame of the unit speed curve X. Then the Frenet formulas are given by

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \\ V_5' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & 0 \\ 0 & 0 & -k_3 & 0 & k_4 \\ 0 & 0 & 0 & -k_4 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

where V_i, $i = 1, 2, 3, 4, 5$ are called the i^{th} Frenet vectors of the curve X and the functions k_i, $i = 1, 2, 3, 4$ are called the i^{th} curvatures of the curve X,[6]. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curve. A regular curve is called a W-curve if it has constant Frenet curvatures. A unit speed curve X is called inclined curve in E^5 if its tangent vector V_1 makes a constant angle with a unit fixed direction U.

Let X and Y be unit speed curves in E^5. Y is an involute of X if the tangent line V_1 at $X(s)$ and the tangent line V_1^* at $Y(s)$ are perpendicular for each s. X is an evolute of Y if Y is an involute of X. This curve couple is defined by [12]

$$Y = X + \mu V_1.$$

The Euclidean hypersphere with the center C and radius $r \in R^+$ in Euclidean 5-space E^5 is defined by [6]

$$S^4 = \left\{ X \in E^5 | \langle X - C, X - C \rangle = r^2 \right\}$$

If $X \subset S^4$ is a regular curve in E^5, then the curve X is called as a spherical curve in E^5. The hypersphere is called as osculating hypersphere if it has six common points with the curve X at the point $X(s)$,[6]

3. Geometric Meanings of the Curvatures in Euclidean 5-Space

Let $X = X(s)$ be a unit speed curve in Euclidean 5-space. The Frenet vectors and curvatures of X, are given by

$$\begin{aligned}
V_1 =& X', \\
V_2 =& \frac{X''}{k_1}, \\
V_3 =& \frac{\left\|X''\right\|^2 (X''' + \left\|X''\right\|^2 X') - \langle X'', X''' \rangle X''}{\left\| \|X''\|^2 (X''' + \|X''\|^2 X') - \langle X'', X''' \rangle X'' \right\|}, \\
V_4 =& \eta V_3 \wedge V_2 \wedge V_1 \wedge V_5, \\
V_5 =& \eta \frac{V_1 \wedge V_2 \wedge X''' \wedge X^{(4)}}{\left\| V_1 \wedge V_2 \wedge X''' \wedge X^{(4)} \right\|}, \\
k_1 =& \left\| X'' \right\|, \\
k_2 =& \frac{\langle X''', V_3 \rangle}{\|X''\|}, \\
k_3 =& \frac{\left\| V_1 \wedge V_2 \wedge X''' \wedge X^{(4)} \right\|}{\left[\langle X''', V_3 \rangle \right]^2}, \\
k_4 =& \frac{\langle X^{(4)}, V_5 \rangle \left[\langle X''', V_3 \rangle \right]^2}{\left\| V_1 \wedge V_2 \wedge X''' \wedge X^{(4)} \right\|}.
\end{aligned}$$

where V_1, V_2, V_3 , V_4, V_5 and k_1, k_2, k_3, k_4 denote the Frenet vectors and Frenet curvatures of the curve X, respectively.

Also, η number is selected as +1 or -1, in order to make the determinant of $\left[V_1, V_2, V_3, V_4, V_5 \right]$ matrix +1. Thus, the Frenet frame will be directed positively, [16].

The geometric meanings of the curvatures at the initial point $X(0)$ of the curve X can be given with respect to the Taylor expansion of the curve X at this point in the Euclidean 5-space E^5 as if in the Euclidean 3-space E^3.

Firstly, let us write Taylor expansion about the point $X(0)$ up to fifth order and take the terms including the lowest powers of s in every component. Thus the Taylor expansion can be given by

$$X(s) \cong X(0) + s\,X'(0) + \frac{s^2}{2}X''(0) + \frac{s^3}{6}X'''(0) + \frac{s^4}{4!}X^(4)(0) + \frac{s^5}{5!}X^{(5)}(0).$$

and considering the Frenet formulas, we obtain

$$\begin{aligned} X(s) \cong & X(0) + sV_1(0) + \frac{s^2}{2}k_1(0)V_2(0) + \frac{s^3}{3!}k_1(0)k_2(0)V_3(0) + \frac{s^4}{4!}k_1(0)k_2(0)k_3(0)V_4(0) \\ & + \frac{s^5}{5!}k_1(0)k_2(0)k_3(0)k_4(0)V_5(0). \end{aligned} \tag{1}$$

The first two terms of the equation (1)

$$X_1(s) = X(0) + sV_1(0)$$

gives us a tangent line which is the best linear approach of the curve X in the neighborhood of $X(0)$.

The first three terms of the equation (1)

$$X_2(s) = X(0) + sV_1(0) + \frac{s^2}{2}k_1(0)V_2(0)$$

is a parabola which is the best quadratic approach of the curve X in the neighborhood of $X(0)$. Thus parabola lies on the plane spanned by the vectors V_1 and V_2. thus the curvature $k_1(0)$ indicates how much V_2 changes in the direction that is tangent to the curve.

The first four terms of the equation (1)

$$X_3(s) = X(0) + sV_1(0) + \frac{s^2}{2}k_1(0)V_2(0) + \frac{s^3}{3!}k_1(0)k_2(0)V_3(0)$$

is cubic which is the best cubic approach of the curve X in the neighborhood of $X(0)$. This curve lies on $Sp\left\{V_1, V_2, V_3\right\}$-subspace. The torsion $k_2(0)$ indicates how much V_3 changes in the direction orthogonal to the V_1, V_2-plane of the curve. If $k_2(0)$ is zero, then the curve X lies on the $Sp\left\{V_1, V_2\right\}$-plane.

The first five terms of the equation (1)

$$X_4(s) = X(0) + sV_1(0) + \frac{s^2}{2}k_1(0)V_2(0) + \frac{s^3}{3!}k_1(0)k_2(0)V_3(0) + \frac{s^4}{4!}k_1(0)k_2(0)k_3(0)V_4(0)$$

is a curve which is the best quartic approach of the curve X in the neighborhood of $X(0)$. This curve lies on the $Sp\left\{V_1, V_2, V_3, V_4\right\}$-subspace. The curvature $k_3(0)$ is the scale of the curve

X separating from the $Sp\left\{V_1, V_2, V_3\right\}$-subspace. If $k_3(0)$ is zero, then the curve X lies on the $Sp\left\{V_1, V_2, V_3\right\}$-subspace.

The first six terms of the equation (1)

$$X_5(s) = X(0) + sV_1(0) + \frac{s^2}{2}k_1(0)V_2(0) + \frac{s^3}{3!}k_1(0)k_2(0)V_3(0) + \frac{s^4}{4!}k_1(0)k_2(0)k_3(0)V_4(0) + \frac{s^5}{5!}k_1(0)k_2(0)k_3(0)k_4(0)V_5(0)$$

is a curve which is the best quintic approach of the curve X in the neighborhood of $X(0)$. This curve lies on the $Sp\left\{V_1, V_2, V_3, V_4, V_5\right\}$-subspace. The curvature $k_4(0)$ is the scale of the curve X separating from the $Sp\left\{V_1, V_2, V_3, V_4\right\}$-subspace. If $k_4(0)$ is zero, then the curve X lies on the $Sp\left\{V_1, V_2, V_3, V_4\right\}$-subspace.

Therefore, the following theorem can be given.

Theorem 1.

(i) A unit speed curve is a line if and only if the first curvature is zero.

(ii) A unit speed curve is a quadratic (to be on the $Sp\left\{V_1, V_2\right\}$-plane) if and only if the second curvature is zero.

(iii) A unit speed curve is a cubic (to be on the $Sp\left\{V_1, V_2, V_3\right\}$-subspace) if and only if the third curvature is zero.

(iv) A unit speed curve is a quartic (to be on the $Sp\left\{V_1, V_2, V_3, V_4\right\}$-subspace) if and only if the fourth curvature is zero.

(v) A unit speed curve is a quintic (to be on the $Sp\left\{V_1, V_2, V_3, V_4, V_5\right\}$-subspace) if and only if the all curvatures are different from zero.

4. Calculation of the Frenet Apparatus of the curves in the Euclidean 5-Space

The Frenet apparatus of a curve with respect to any parameter in the Euclidean 5-space can be calculated via the same method in the Euclidean 3-space.

Let X be an arbitrary curve and a function is of class C^5 in E^5. If the derivatives of the curve X up to the fifth order are calculated with respect to parameter t in terms of the parameter s, the following equations are obtained

$$\dot{X} = v\, V_1, \quad v = \frac{ds}{dt} \neq 0 \tag{2}$$

$$\ddot{X} = \dot{v}\, V_1 + v^2\, k_1\, V_2 \tag{3}$$

$$\dddot{X} = (\ddot{v} - v^3 k_1^2)V_1 + (3v\dot{v}k_1 + v^2\dot{k}_1)V_2 + (v^3 k_1 k_2)V_3 \tag{4}$$

$$X^{(4)} = (\dddot{v} - 6v^2\dot{v}k_1^2 - 3v^3 k_1\dot{k}_1)V_1 + (4v\ddot{v}k_1 - v^4 k_1^3 + 3\dot{v}^2 k_1 + 5v\dot{v}\dot{k}_1 + v^2\ddot{k}_1 - v^4 k_1 k_2^2)V_2 + (6v^2\dot{v}k_1 k_2 + 2v^3\dot{k}_1 k_2 + v^3 k_1\dot{k}_2)V_3 + (v^4 k_1 k_2 k_3)V_4 \tag{5}$$

$$\begin{aligned}
X^{(5)} =& (v^{(4)} - 15v\dot{v}^2k_1^2 - 10v^2\ddot{v}k_1^2 - 26v^2\dot{v}k_1\dot{k}_1 - 3v^3\dot{k}_1^2 - 4v^3k_1\ddot{k}_1 + v^5k_1^4 + v^5k_1^2k_2^2)V_1 \\
&+ (5\dddot{v}vk_1 - 10v^3\dot{v}k_1^3 - 6v^4k_1^2\dot{k}_1 + 10\dot{v}\ddot{v}k_1 + 9v\ddot{v}\dot{k}_1 + 8\dot{v}^2\dot{k}_1 + 7v\dot{v}\ddot{k}_1 + v^2\dddot{k}_1 \\
&- 10v^3\dot{v}k_1k_2^2 - 3v^4\dot{k}_1k_2^2 - 3v^4k_1k_2\dot{k}_2)V_2 \\
&+ (10v^2\ddot{v}k_1k_2 - v^5k_1^3k_2 + 15v\dot{v}^2k_1k_2 + 17v^2\dot{v}\dot{k}_1k_2 + 3v^3\ddot{k}_1k_2 - v^5k_1k_2^3 \\
&+ 9v^2\dot{v}k_1\dot{k}_2 + 3v^3\dot{k}_1\dot{k}_2 + v^3k_1\ddot{k}_2 - v^5k_1k_2k_3^2)V_3 \\
&+ (10v^3\dot{v}k_1k_2k_3 + 3v^4\dot{k}_1k_2k_3 + 2v^4k_1\dot{k}_2k_3 + v^4k_1k_2\dot{k}_3)V_4 \\
&+ (v^5k_1k_2k_3k_4)V_5
\end{aligned} \tag{6}$$

where "$\cdot$" denotes the derivative with respect to t.
From the equation (2), we find

$$v = \left\|\dot{X}\right\| \tag{7}$$

and

$$V_1 = \frac{\dot{X}}{\left\|\dot{X}\right\|}. \tag{8}$$

Since $v^2 = \left\langle \dot{X}, \dot{X} \right\rangle$, if the derivative of this term is taken consecutively, we have

$$\dot{v} = \frac{\left\langle \dot{X}, \ddot{X} \right\rangle}{\left\|\dot{X}\right\|} \tag{9}$$

and

$$\ddot{v} = \frac{\left\|\ddot{X}\right\|^2 \left\|\dot{X}\right\|^2 + \left\langle \dot{X}, \dddot{X} \right\rangle \left\|\dot{X}\right\|^2 - \left\langle \dot{X}, \ddot{X} \right\rangle^2}{\left\|\dot{X}\right\|^3} \tag{10}$$

If the first curvature is calculated from the equation (3), the following equations are obtained

$$k_1 = \frac{\left\| \left\|\dot{X}\right\|^2 \ddot{X} - \langle \dot{X}, \ddot{X} \rangle \dot{X} \right\|}{\left\|\dot{X}\right\|^4} \tag{11}$$

and

$$k_1^2 = \frac{\left\|\ddot{X}\right\|^2 \left\|\dot{X}\right\|^2 - \langle \dot{X}, \ddot{X} \rangle^2}{\left\|\dot{X}\right\|^6} \tag{12}$$

If we take the derivative of both sides of the equation (11) with respect to t, we get

$$\dot{k}_1 = \frac{\left\|\dot{X}\right\|^4 \langle \ddot{X}, \dddot{X} \rangle + 3\langle \dot{X}, \ddot{X} \rangle^3 - \left\|\dot{X}\right\|^2 \langle \dot{X}, \ddot{X} \rangle \langle \dot{X}, \dddot{X} \rangle - 3\left\|\dot{X}\right\|^2 \left\|\ddot{X}\right\|^2 \langle \dot{X}, \ddot{X} \rangle}{\left\|\dot{X}\right\|^4 \left\| \left\|\dot{X}\right\|^2 \ddot{X} - \langle \dot{X}, \ddot{X} \rangle \dot{X} \right\|} \tag{13}$$

In addition to this, the second Frenet vector from the equation (3) is

$$V_2 = \frac{\ddot{X} \left\|\dot{X}\right\|^2 - \langle \dot{X}, \ddot{X} \rangle \dot{X}}{\left\| \left\|\dot{X}\right\|^2 \ddot{X} - \langle \dot{X}, \ddot{X} \rangle \dot{X} \right\|}. \tag{14}$$

By using the equation (4), we can write

$$\langle \dddot{X}, V_3 \rangle = v^3 k_1 k_2.$$

Substituting the equations (7) and (11) in the above equation, we obtain the second curvature k_2 as follows

$$k_2 = \frac{\langle \dddot{X}, V_3 \rangle \left\| \dot{X} \right\|}{\left\| \left\| \dot{X} \right\|^2 \ddot{X} - \langle \dot{X}, \ddot{X} \rangle \dot{X} \right\|}. \tag{15}$$

Again, considering the equation (4), the third Frenet vector V_3 of X is given by

$$V_3 = \frac{\dddot{X} - aV_1 - bV_2}{\left\| \dddot{X} - aV_1 - bV_2 \right\|}$$

such that

$$\begin{aligned} a &= \ddot{v} - v^3 k_1^2 \\ b &= 3v\dot{v}k_1 + v^2 \dot{k}_1 \end{aligned}. \tag{16}$$

Substituting the equations (7), (9), (10), (11) and (13) in the equation (16), we have

$$a = \frac{\langle \dot{X}, \dddot{X} \rangle}{\left\| \dot{X} \right\|}.$$

and

$$b = \frac{\left\| \dot{X} \right\|^2 \langle \ddot{X}, \dddot{X} \rangle - \langle \dot{X}, \ddot{X} \rangle \langle \dot{X}, \dddot{X} \rangle}{\left\| \left\| \dot{X} \right\|^2 \ddot{X} - \langle \dot{X}, \ddot{X} \rangle \dot{X} \right\|}.$$

Now, we can compute the vector form $V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)}$ as the follows;

$$V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)} = v^7 k_1^2 k_2^2 k_3 V_5. \tag{17}$$

then from the above equation

$$V_5 = \eta \frac{V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)}}{\left\| V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)} \right\|}. \tag{18}$$

and η is taken ± 1 to make $\det(V_1, V_2, V_3, V_4, V_5) = +1$.

Substituting the equations (7), (12) and (15) in the equation (17), the third curvature is found

$$k_3 = \frac{\left\| V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)} \right\|}{\langle \dddot{X}, V_3 \rangle^2 \left\| \dot{X} \right\|}. \tag{19}$$

The inner product $\langle X^{(5)}, V_5 \rangle$ gives us the fourth curvature k_4 as

$$k_4 = \frac{\langle X^{(5)}, V_5 \rangle}{v^5 k_1 k_2 k_3}. \tag{20}$$

Then, if we substitute the equations (7), (11), (15) and (19) in the above equation, we immediately arrive to

$$k_4 = \frac{\langle X^{(5)}, V_5\rangle \langle \dddot{X}, V_3\rangle}{\left\|V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)}\right\| \left\|\dot{X}\right\|}. \tag{21}$$

Finally, the fourth Frenet vector is

$$V_3 \wedge V_2 \wedge V_1 \wedge V_5 = V_4 \tag{22}$$

Therefore, the following theorem can be given.

Theorem 2. *Let X be an arbitrary curve of class C^5 in the Euclidean 5-space E^5. In this regard, the Frenet vectors and curvatures of the curve X are*

$$\begin{aligned}
V_1 =& \frac{\dot{X}}{\left\|\dot{X}\right\|},\\
V_2 =& \frac{\ddot{X}\left\|\dot{X}\right\|^2 - \dot{X}\left\langle \dot{X}\,,\ddot{X}\right\rangle}{\left\|\left\|\dot{X}\right\|^2 \ddot{X} - \dot{X}\left\langle \dot{X}\,,\ddot{X}\right\rangle\right\|},\\
V_3 =& \frac{\dddot{X} - aV_1 - bV_2}{\left\|\dddot{X} - aV_1 - bV_2\right\|}, \quad a = \frac{\left\langle \dot{X}\,,\dddot{X}\right\rangle}{\left\|\dot{X}\right\|}, b = \frac{\left\|\dot{X}\right\|^2 \langle \ddot{X}\,,\dddot{X}\rangle - \left\langle \dot{X}\,,\ddot{X}\right\rangle\left\langle \dot{X}\,,\dddot{X}\right\rangle}{\left\|\left\|\dot{X}\right\|^2 \ddot{X} - \left\langle \dot{X}\,,\ddot{X}\right\rangle \dot{X}\right\|},\\
V_4 =& \eta V_3 \wedge V_2 \wedge V_1 \wedge V_5,\\
V_5 =& \eta \frac{V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)}}{\left\|V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)}\right\|},\\
k_1 =& \frac{\left\|\left\|\dot{X}\right\|^2 \ddot{X} - \left\langle \dot{X}\,,\ddot{X}\right\rangle \dot{X}\right\|}{\left\|\dot{X}\right\|^4},\\
k_2 =& \frac{\left\langle \dddot{X}, V_3\right\rangle \left\|\dot{X}\right\|}{\left\|\left\|\dot{X}\right\|^2 \ddot{X} - \left\langle \dot{X}\,,\ddot{X}\right\rangle \dot{X}\right\|},\\
k_3 =& \frac{\left\|V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)}\right\|}{\left\langle \dddot{X}, V_3\right\rangle \left\|\dot{X}\right\|},\\
k_4 =& \frac{\left\langle X^{(5)}, V_5\right\rangle \left\langle \dddot{X}, V_3\right\rangle}{\left\|V_1 \wedge V_2 \wedge \dddot{X} \wedge X^{(4)}\right\| \left\|\dot{X}\right\|},
\end{aligned}$$

respectively.

5. Involute-Evolute Curve Couples in the Euclidean 5-Space

Let X be a W-curve and Y be the involute of X in E^5. While the Frenet apparatus of X is $\{V_1, V_2, V_3, V_4, V_5, k_1, k_2, k_3, k_4\}$, we will denote the Frenet apparatus of Y with

$\left\{V_1^Y, V_2^Y, V_3^Y, V_4^Y, V_5^Y, k_1^Y, k_2^Y, k_3^Y, k_4^Y\right\}$. So, from the definition of involute-evolute curve, we may express

$$Y = X + \mu V_1 \tag{23}$$

where s and s_Y denote the arc-parameters of the curves X and Y, respectively.

Differentiating the both sides of the equation (23) with respect to s, one can obtain

$$\frac{dY}{ds_Y}\frac{ds_Y}{ds} = \frac{dX}{ds} + \frac{d\mu}{ds}V_1 + \mu k_1 V_2. \tag{24}$$

Since the tangent vector V_1 of the curve X orthogonal to the tangent vector V_1^Y of the curve Y, it is easily seen that

$$1 + \frac{d\mu}{ds} = 0. \tag{25}$$

We know that $\mu = c - s$ from the equation (25). So, we can write

$$Y = X + (c-s)V_1 \tag{26}$$

and

$$V_1^Y \frac{ds_Y}{ds} = (c-s)k_1 V_2. \tag{27}$$

Also the equation (27) yields

$$\dot{Y} = (c-s)k_1 V_2. \tag{28}$$

If we take the norm of $\dot{Y}$, we have

$$\left\|\dot{Y}\right\| = (c-s)k_1. \tag{29}$$

where the subscript dot "$\cdot$" denotes the derivative of Y with respect to s.

Moreover, the derivatives of the curve Y up to the fifth order are given by

$$\ddot{Y} = -(c-s)k_1^2 V_1 - k_1 V_2 + (c-s)k_1 k_2 V_3, \tag{30}$$

$$\dddot{Y} = 2k_1^2 V_1 - (c-s)k_1(k_1^2 + k_2^2)V_2 - 2k_1 k_2 V_3 + (c-s)k_1 k_2 k_3 V_4, \tag{31}$$

$$\begin{aligned} Y^{(4)} =& (c-s)k_1^2(k_1^2 + k_2^2)V_1 + 3k_1(k_1^2 + k_2^2)V_2 - (c-s)k_1 k_2(k_1^2 + k_2^2 + k_3^2)V_3 \\ & - 3k_1 k_2 k_3 V_4 + (c-s)k_1 k_2 k_3 k_4 V_5, \end{aligned} \tag{32}$$

$$\begin{aligned} Y^{(5)} =& -4k_1^2(k_1^2 + k_2^2)V_1 + (c-s)k_1\left[k_1^2(k_1^2 + k_2^2) + k_2^2(k_1^2 + k_2^2 + k_3^2)\right]V_2 \\ & + 4k_1 k_2(k_1^2 + k_2^2 + k_3^2)V_3 - (c-s)k_1 k_2 k_3(k_1^2 + k_2^2 + k_3^2 + k_4^2)V_4 \\ & - 4k_1 k_2 k_3 k_4 V_5. \end{aligned} \tag{33}$$

From the equation (8), the first Frenet vector of the curve Y can be written as

$$V_1^Y = \frac{\dot{Y}}{\left\|\dot{Y}\right\|}.$$

Considering the equations (28) and (29), we find

$$V_1^Y = V_2. \tag{34}$$

From the equations (28) and (30), we get

$$\left\|\dot{Y}\right\|^2 \ddot{Y} - \left\langle \dot{Y}, \ddot{Y} \right\rangle \dot{Y} = -(c-s)^3 k_1^4 V_1 + (c-s)^3 k_1^3 k_2 V_3 \tag{35}$$

and

$$\left\| \left\|\dot{Y}\right\|^2 \ddot{Y} - \left\langle \dot{Y}, \ddot{Y} \right\rangle \dot{Y} \right\| = (c-s)^3 k_1^3 \sqrt{k_1^2 + k_2^2}. \tag{36}$$

If we use the equations (11) and (14), then we will obtain the second Frenet vector and the first curvature of curve Y as follows;

$$V_2^Y = -\frac{k_1}{\sqrt{k_1^2 + k_2^2}} V_1 + \frac{k_2}{\sqrt{k_1^2 + k_2^2}} V_3 \tag{37}$$

and

$$k_1^Y = \frac{\sqrt{k_1^2 + k_2^2}}{(c-s)k_1} \tag{38}$$

respectively. Besides, considering the equations (28), (29), (30), (31) and (36), one can calculate

$$\dddot{Y} - \frac{\langle \dot{Y}, \dddot{Y} \rangle}{\left\|\dot{Y}\right\|} V_1^Y - \frac{\left\|\dot{Y}\right\|^2 \langle \ddot{Y}, \dddot{Y} \rangle - \left\langle \dot{Y}, \ddot{Y} \right\rangle \langle \dot{Y}, \dddot{Y} \rangle}{\left\| \left\|\dot{Y}\right\|^2 \ddot{Y} - \left\langle \dot{Y}, \ddot{Y} \right\rangle \dot{Y} \right\|} V_2^Y = (c-s) k_1 k_2 k_3 V_4. \tag{39}$$

The third Frenet vector of Y is obtained from the equation (16) as

$$V_3^Y = V_4. \tag{40}$$

If the equations (31) and (40) are taken into consideration, we get

$$\left\langle \dddot{Y}, V_3^Y \right\rangle = (c-s) k_1 k_2 k_3. \tag{41}$$

From the equations (15), (36) and (41), the second curvature of Y is found as

$$k_2^Y = \frac{k_2 k_3}{(c-s) k_1 \sqrt{k_1^2 + k_2^2}}. \tag{42}$$

Moreover, from the equations (31), (32), (34) and (37), we have

$$V_1^Y \wedge V_2^Y \wedge \dddot{Y} \wedge Y^{(4)} = \frac{(c-s)^2 k_1^2 k_2^3 k_3^2 k_4}{\sqrt{k_1^2 + k_2^2}} V_1 + \frac{(c-s)^2 k_1^3 k_2^2 k_3^2 k_4}{\sqrt{k_1^2 + k_2^2}} V_3 + \frac{(c-s)^2 k_1^3 k_2^2 k_3^3}{\sqrt{k_1^2 + k_2^2}} V_5 \tag{43}$$

and

$$\left\| V_1^Y \wedge V_2^Y \wedge \dddot{Y} \wedge Y^{(4)} \right\| = \frac{(c-s)^2 k_1^2 k_2^2 k_3^2}{\sqrt{k_1^2 + k_2^2}} \sqrt{k_1^2 k_3^2 + k_2^2 k_4^2 + k_1^2 k_4^2}. \tag{44}$$

Thus, if we take the equations (18), (43) and (44), the fifth Frenet vector of the curve Y is

$$\begin{aligned} V_5^Y =& k_2k_4\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}V_1+k_1k_4\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}V_3 \\ &+k_1k_3\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}V_5. \end{aligned} \tag{45}$$

By (41) and (44), we obtain

$$\frac{\left\|V_1^Y\wedge V_2^Y\wedge \dddot{Y}\wedge Y^{(4)}\right\|}{\left(\left\langle \dddot{Y},V_3^Y\right\rangle\right)^2\left\|\dot{Y}\right\|}=\frac{\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}{(c-s)k_1\sqrt{k_1^2+k_2^2}}.$$

Therefore, from the equation (19) the third curvature of the curve Y can be found as follows

$$k_3^Y=\frac{\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}{(c-s)k_1\sqrt{k_1^2+k_2^2}}. \tag{46}$$

If the vectorial product $V_3^Y\wedge V_2^Y\wedge V_1^Y\wedge V_5^Y$ is calculated by using the equations (34), (37), (40) and (45), we find the fourth Frenet vector of the curve Y, considering the equation (22) as follows:

$$\begin{aligned} V_4^Y =& \frac{-k_1k_2k_3}{\sqrt{k_1^2+k_2^2}\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}V_1 \\ &-\frac{k_1^2k_3}{\sqrt{k_1^2+k_2^2}\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}V_3 \\ &+\frac{k_4(k_1^2+k_2^2)}{\sqrt{k_1^2+k_2^2}\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}V_5 \end{aligned} \tag{47}$$

From the equations (33) and (45), the inner product of the vectors $Y^{(5)}$ and V_5^Y is

$$\left\langle Y^{(5)},V_5^Y\right\rangle=\frac{4k_1^2k_2^2k_4(1-k_2)}{\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}. \tag{48}$$

Finally, considering the equations (21), (41), (44) and (48) the fourth curvature of Y is found as

$$k_4^Y=\frac{4k_2k_4(1-k_2)\sqrt{k_1^2+k_2^2}}{k_3(c-s)^2(k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2)}. \tag{49}$$

Therefore, the following theorem and results can be given

Theorem 3. *Let X be a W-curve and Y be the involute of X in E^5. $\{V_1,V_2,V_3,V_4,V_5,k_1,k_2,k_3,k_4\}$ and $\{V_1^Y,V_2^Y,V_3^Y,V_4^Y,V_5^Y,k_1^Y,k_2^Y,k_3^Y,k_4^Y\}$ denote the Frenet apparatus of the curves X and Y, respectively. The relation can be expressed as*

$$V_1^Y = V_2,$$

$$
\begin{aligned}
V_2^Y =& -\frac{k_1}{\sqrt{k_1^2+k_2^2}}V_1+\frac{k_2}{\sqrt{k_1^2+k_2^2}}V_3,\\
V_3^Y =& V_4,\\
V_4^Y =& -\frac{k_1k_2k_3}{\sqrt{k_1^2+k_2^2}\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}V_1-\frac{k_1^2k_3}{\sqrt{k_1^2+k_2^2}\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}V_3\\
&+\frac{k_4\left(k_1^2+k_2^2\right)}{\sqrt{k_1^2+k_2^2}\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}V_5,\\
V_5^Y =& k_2k_4\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}V_1+k_1k_4\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}V_3\\
&+k_1k_3\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}V_5,\\
k_1^Y =& \frac{\sqrt{k_1^2+k_2^2}}{(c-s)\,k_1},\\
k_2^Y =& \frac{k_2k_3}{(c-s)\,k_1\sqrt{k_1^2+k_2^2}},\\
k_3^Y =& \frac{\sqrt{k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2}}{(c-s)\,k_1\sqrt{k_1^2+k_2^2}},\\
k_4^Y =& \frac{4k_2k_4\left(1-k_2\right)\sqrt{k_1^2+k_2^2}}{(c-s)^2\,k_3\left(k_1^2k_3^2+k_2^2k_4^2+k_1^2k_4^2\right)}.
\end{aligned}
$$

Corollary 1. $\left\{V_1^Y, V_2^Y, V_3^Y, V_4^Y, V_5^Y\right\}$ *is an orthonormal frame in* E^5.

Corollary 2. *While* X *is a W-curve,* Y *can not be a W-curve.*

Corollary 3. *The involute curve* Y *can't be an inclined curve.*

6. The Spherical Curves in Euclidean 5-Space

Let $X \subset R^5$ curve be given with coordinate neighborhood (I,X) and $s \in I$ be arc-length parameter of X. Also, assume that S^4 is a hypersphere which has six common coalescent points with the curve X. If $X(s)$ is a point on this hypersphere, C is the center of this hypersphere and r is the radius of it, then the equation of the hypersphere S^4 is

$$\langle X(s)-C, X(s)-C\rangle = r^2. \tag{50}$$

On the other hand, for the base $\left\{V_1, V_2, V_3, V_4, V_5\right\}$ and $m_i(s) \in R$

$$C-X(s) = m_1(s)V_1(s)+m_2(s)V_2(s)+m_3(s)V_3(s)+m_4(s)V_4(s)+m_5(s)V_5(s), \tag{51}$$

can be written. Hence,

$$m_i(s) = \left\langle C-X(s),\, V_i(s)\right\rangle, \quad 1 \le i \le 5 \tag{52}$$

In accordance with this, let us consider

$$f: I \to R$$
$$s \to f(s) = \langle X - C, X - C \rangle - r^2.$$

If we have the following equations

$$f(s) = f'(s) = f''(s) = f'''(s) = f^{(4)}(s) = f^{(5)}(s) = 0$$

then we say that the hypersphere touches to X at the fifth order to the curve at $X(s)$. Therefore,

$$f(s) = \langle X - C, X - C \rangle - r^2 = 0, \tag{53}$$
$$f'(s) = \left\langle V_1, X - C \right\rangle = 0 \Rightarrow m_1 = 0, \tag{54}$$
$$f''(s) = 0 \Rightarrow \left\langle V_2, X - C \right\rangle = \frac{1}{k_1} \Rightarrow m_2 = \frac{1}{k_1}, \tag{55}$$
$$f'''(s) = 0 \Rightarrow \left\langle V_3, X - C \right\rangle = \frac{m_2'}{k_2} \Rightarrow m_3 = \frac{m_2'}{k_2}, \tag{56}$$
$$f^{(4)}(s) = 0 \Rightarrow \left\langle V_4, X - C \right\rangle = \frac{m_3' + k_2 m_2}{k_3} \Rightarrow m_4 = \frac{m_3' + k_2 m_2}{k_3}, \tag{57}$$
$$f^{(5)}(s) = 0 \Rightarrow \left\langle V_5, X - C \right\rangle = \frac{m_4' + k_3 m_3}{k_4} \Rightarrow m_5 = \frac{m_4' + k_3 m_3}{k_4}. \tag{58}$$

are obtained. Thus, the center of the hypersphere is

$$C = X + m_2 V_2 + \left(\frac{m_2'}{k_2}\right) V_3 + \left(\frac{\left(\frac{m_2'}{k_2}\right)' + k_2 m_2}{k_3}\right) V_4 + \left(\left(\frac{\left(\frac{m_2'}{k_2}\right)' + k_2 m_2}{k_3}\right)' + k_3 \left(\frac{m_2'}{k_2}\right)\right) \frac{1}{k_4} V_5 \tag{59}$$

and for the square of the radius

$$r^2 = m_2^2 + \left(\frac{m_2'}{k_2}\right)^2 + \left(\frac{\left(\frac{m_2'}{k_2}\right)' + k_2 m_2}{k_3}\right)^2 + \left(\left(\frac{\left(\frac{m_2'}{k_2}\right)' + k_2 m_2}{k_3} + k_3 \left(\frac{m_2'}{k_2}\right)\right)'\right)^2 \frac{1}{k_4^2} \tag{60}$$

is attained. Differentiating the equation (59), we obtain the derivative of the center as follows

$$C' = (m_4 k_4 + m_5') V_5. \tag{61}$$

Considering the equality (61), it can be said that the centers of the osculating hyperspheres of a spherical curve are in the direction of V_5. In addition, all spherical curves satisfy the following differential equation:

$$m_2^2+\left(\frac{m_2'}{k_2}\right)^2+\left(\left(\frac{m_2'}{k_2}\right)'+k_2m_2\right)^2\frac{1}{k_3^2}+\left(\left(\frac{\left(\frac{m_2'}{k_2}\right)'+k_2m_2}{k_3}+k_3\left(\frac{m_2'}{k_2}\right)\right)'\right)^2\frac{1}{k_4^2}=a^2. \quad (62)$$

If the curve is spherical, then the hypersphere is also an osculating hypersphere. Here a will be the radius of the hypersphere. Conversely, if the equation (62) is provided, the radius of the osculating hypersphere is constant.

If the derivative of the equation (62) is taken,

$$m_5(m_4k_4+m_5')=0 \quad (63)$$

is found. Therefore, if we consider the equation (63) with (61), then $C'=0$. This means that the center of the osculating hypersphere is constant. From the equation (63), all of the differential equations of the spherical curves are

$$m_4k_4+m_5'=0 \quad (64)$$

or

$$\left(\left(\frac{m_2'}{k_2}\right)'+k_2m_2\right)\frac{k_4}{k_3}+\left(\left(\left(\frac{\left(\frac{m_2'}{k_2}\right)'+k_2m_2}{k_3}\right)'+k_3\left(\frac{m_2'}{k_2}\right)\right)\frac{1}{k_4}\right)'=0.$$

However, the following theorem can be given:

Theorem 4. *Let X be a unit speed curve in E^5.*

(i) The curve X is a spherical curve if and only if the differential equation

$$m_4k_4+m_5'=0$$

is satisfied.

(ii) If X is a spherical curve, then the center of the hypersphere is

$$C=X+m_2V_2+m_3V_3+m_4V_4+m_5V_5$$

and the radius is

$$r=\sqrt{m_2^2+m_3^2+m_4^2+m_5^2}$$

such that

$$m_2=\frac{1}{k_1},\quad m_3=\frac{m_2'}{k_2},\quad m_4=\frac{m_3'+k_2m_2}{k_3},\quad m_5=\frac{m_4'+k_3m_3}{k_4}.$$

(iii) The radius of the osculating hypersphere is constant at the point $X(s)$ if and only if the centers of the osculating hyperspheres are the same[9].

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