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An Extension of Kantorovich Inequality for Sesquilinear Maps

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Abstract. By using sesquilinear map we generalize some operator Kantorovich inequalities. Our results are more extensive than many previous results due to Mond and Pečarić.

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1. Introduction and Preliminaries

For every unit vector x and $MI \geq A \geq mI > 0$, the Kantorovich inequality [4] states

$$\langle x, Ax\rangle \langle x, A^{-1}x\rangle \leq \frac{(M+m)^2}{4Mm}. \tag{1}$$

In [3, Theorem 1.29], the authors obtained the following reverse of Hölder-McCarthy inequality by the Kantorovich inequality:

Theorem 1. *Let A be a positive operator on $\mathscr{H}$ satisfying $M1_{\mathscr{H}} \geq A \geq m1_{\mathscr{H}} > 0$ for some scalars $m < M$. Then*

$$\langle A^2x, x\rangle \leq \frac{(M+m)^2}{4Mm}\langle Ax, x\rangle^2, \tag{2}$$

for every unit vector $x \in \mathscr{H}$.

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Many authors have investigated on extensions of the Kantorovich one, such as Liu et al. [5], Furuta [2] and Ky Fan [1]. Among others, we pay our attentions to the long research series of Mond-Pečarić method [3].

As customary, we reserve M, m for scalars and $1_{\mathscr{H}}$ for identity operator. Other capital letters denote general elements of the C^*-algebra $\mathcal{B}(\mathscr{H})$ (with unit) of all bounded linear operators acting on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. Also, we identify a scalars with the unit multiplied by this scalar. We write $A \geq 0$ to mean that the operator A is positive and identify $A \geq B$ (the same as $B \leq A$) with $A - B \geq 0$. A positive invertible operator A is naturally denoted by $A > 0$. For $A, B > 0$, the geometric mean $A\#B$ is defined by

$$A\#B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

It is well known that

$$A\#B \leq \frac{A+B}{2}.$$

We use φ for sesquilinear map. A map $\varphi : \mathcal{B}(\mathscr{H}) \times \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ is a sesquilinear map, if satisfying the following conditions:

(a) $\varphi(\alpha A_1 + \beta A_2, B) = \alpha\varphi(A_1, B) + \beta\varphi(A_2, B)$;

(b) $\varphi(A, \alpha B_1 + \beta B_2) = \overline{\alpha}\varphi(A, B_1) + \overline{\beta}\varphi(A, B_2)$;

(c) $\varphi(A, A) \geq 0$;

(d) $\varphi(AX, Y) = \varphi(X, A^*Y)$;

for all $\alpha, \beta \in \mathbb{C}$ and $A_1, A_2, B_1, B_2, X, Y \in \mathcal{B}(\mathscr{H})$.

Note that, if $A \geq 0$ then $\varphi(AC, C) \geq 0$ for all $C \in \mathcal{B}(\mathscr{H})$. In fact, if $A \geq 0$ then $A = B^*B$ for some $B \in \mathcal{B}(\mathscr{H})$. Therefore,

$$\varphi(AC, C) = \varphi(B^*BC, C) = \varphi(BC, BC) \geq 0$$

It turn implies that, if $A \geq B$ then, $\varphi(AC, C) \geq \varphi(BC, C)$. Since $A - B \geq 0$.

We remark that if we define $\varphi(A, B) = B^*A$, then above definition coincides with the ordinal definition of positive operator. In fact, in this case $\varphi(AC, C) = C^*AC$ and $\varphi(BC, C) = C^*BC$, hence $A \geq B$ if and only if $C^*AC \geq C^*BC$ for any $C \in \mathcal{B}(\mathscr{H})$. We call $U \in \mathcal{B}(\mathscr{H})$ is φ-unitary if $\varphi(U, U) = 1_{\mathscr{H}}$.

The main results are given in the next section. In this paper, we will present some operator inequalities which are generalizations of (1) and (2).

2. Proofs of the inequalities

To prove our main results we need the following lemma.

Lemma 1. *[3, Lemma 1.24] Let $A \in \mathcal{B}(\mathscr{H})$ be positive and satisfying $M1_{\mathscr{H}} \geq A \geq m1_{\mathscr{H}} > 0$ for some scalars $m < M$. Then*

$$(M + m)1_{\mathscr{H}} \geq MmA^{-1} + A.$$

The following result is our first main result. It presents a generalization of the Kantorovich inequality.

Theorem 2. *Let $A, C \in \mathcal{B}(\mathscr{H})$ and A be a positive satisfying $M1_{\mathscr{H}} \geq A \geq m1_{\mathscr{H}} > 0$ for some scalars $m < M$. Then*

$$\varphi(AC, C) \# \varphi\left(A^{-1}C, C\right) \leq \frac{M+m}{2\sqrt{Mm}} \varphi(C, C). \tag{3}$$

Proof. By Lemma 1, we have

$$(M+m) 1_{\mathscr{H}} \geq MmA^{-1} + A.$$

Since φ is sesquilinear map, we obtain

$$\begin{aligned}(M+m)\varphi(C, C) &\geq Mm\varphi\left(A^{-1}C, C\right) + \varphi(AC, C) \\ &\geq 2\sqrt{Mm}\varphi\left(A^{-1}C, C\right) \# \varphi(AC, C).\end{aligned}$$

Which is exactly desired result (3).

Example 1. *By taking $\varphi(A, B) = B^*A$ in Theorem 2 we infer that*

$$C^*AC \# C^*A^{-1}C \leq \frac{M+m}{2\sqrt{Mm}} C^*C.$$

In addition, if C is unitary then

$$C^*AC \# C^*A^{-1}C \leq \frac{M+m}{2\sqrt{Mm}}.$$

Theorem 3. *Let $A_i, C_i \in \mathcal{B}(\mathscr{H})$ and A_i be a positive satisfying $M1_{\mathscr{H}} \geq A_i \geq m1_{\mathscr{H}} > 0$ for some scalars $m < M$ $(i = 1, \ldots, n)$. Then*

$$\left(\sum_{i=1}^{n} \varphi(A_iC_i, C_i)\right) \# \left(\sum_{i=1}^{n} \varphi\left(A_i^{-1}C_i, C_i\right)\right) \leq \frac{M+m}{2\sqrt{Mm}} \sum_{i=1}^{n} \varphi(C_i, C_i).$$

Proof. Putting

$$\widetilde{A} = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix}, \quad \widetilde{C} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$$

then we have $sp\left(\widetilde{A}\right) \subset [m, M]$. Next we define

$$\begin{cases} \widetilde{\varphi} : \oplus\mathcal{B}(\mathscr{H}) \times \oplus\mathcal{B}(\mathscr{H}) \to \oplus\mathcal{B}(\mathscr{H}) \\ \widetilde{\varphi}\left(\begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}\right) = \sum_{i=1}^{n} \varphi(A_i, A_i) \end{cases}.$$

In particular, we have

$$\begin{aligned}\widetilde{\varphi}\left(\widetilde{A}\widetilde{C},\widetilde{C}\right)&=\widetilde{\varphi}\left(\begin{pmatrix}A_1&\dots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&A_n\end{pmatrix}\begin{pmatrix}C_1\\ \vdots\\ C_n\end{pmatrix},\begin{pmatrix}C_1\\ \vdots\\ C_n\end{pmatrix}\right)\\&=\widetilde{\varphi}\left(\begin{pmatrix}A_1C_1\\ \vdots\\ A_nC_n\end{pmatrix},\begin{pmatrix}C_1\\ \vdots\\ C_n\end{pmatrix}\right)\\&=\sum_{i=1}^{n}\varphi\left(A_iC_i,C_i\right).\end{aligned}$$

It can be deduced from Theorem 2 that

$$\widetilde{\varphi}\left(\widetilde{A}\widetilde{C},\widetilde{C}\right)\#\widetilde{\varphi}\left(\widetilde{A}^{-1}\widetilde{C},\widetilde{C}\right)\leq\frac{M+m}{2\sqrt{Mm}}\widetilde{\varphi}\left(\widetilde{C},\widetilde{C}\right).$$

This completes the proof.

The following corollary follows immediately.

Corollary 1. *If in Theorem 3,* $\widetilde{C}=\begin{pmatrix}C_1\\ \vdots\\ C_n\end{pmatrix}$ *is a* $\widetilde{\varphi}$*-unitary, then*

$$\left(\sum_{i=1}^{n}\varphi\left(A_iC_i,C_i\right)\right)\#\left(\sum_{i=1}^{n}\varphi\left(A_i^{-1}C_i,C_i\right)\right)\leq\frac{M+m}{2\sqrt{Mm}}.$$

Theorem 4. *Let A be a positive operator on $\mathscr{H}$ satisfying $M1_{\mathscr{H}}\geq A\geq m1_{\mathscr{H}}>0$ for some scalars $m<M$. Then*

$$\varphi\left(A^{-1}C,C\right)-\varphi(AC,C)^{-1}\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}\varphi\left(C,C\right),$$

for every $C\in\mathcal{B}\left(\mathscr{H}\right)$.

Proof. According to Lemma 1, we have

$$\left(M+m\right)1_{\mathscr{H}}\geq MmA^{-1}+A$$

and hence

$$\varphi\left(A^{-1}C,C\right)\leq\frac{M+m}{Mm}\varphi\left(C,C\right)-\frac{1}{Mm}\varphi\left(AC,C\right),$$

for every $C \in \mathcal{B}(\mathscr{H})$. Then it follows that

$$\begin{aligned}
&\varphi\left(A^{-1}C,C\right)-\varphi(AC,C)^{-1}\\
&\le\left(\frac{1}{m}+\frac{1}{M}\right)\varphi\left(C,C\right)-\frac{1}{Mm}\varphi\left(AC,C\right)-\varphi(AC,C)^{-1}\\
&=\left(\frac{1}{\sqrt{m}}-\frac{1}{\sqrt{M}}\right)^{2}\varphi\left(C,C\right)-\left(\frac{1}{\sqrt{Mm}}\varphi(AC,C)^{\frac{1}{2}}-\varphi(AC,C)^{-\frac{1}{2}}\right)^{2}\\
&\le\left(\frac{1}{\sqrt{m}}-\frac{1}{\sqrt{M}}\right)^{2}\varphi\left(C,C\right).
\end{aligned}$$

Based on the discussion above, we conclude that

$$\varphi\left(A^{-1}C,C\right)-\varphi(AC,C)^{-1}\le\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{Mm}\varphi\left(C,C\right).$$

We have completed the proof of Theorem 4.

Proposition 1. *Let A be a positive operator on $\mathscr{H}$ satisfying $M1_{\mathscr{H}}\ge A\ge m1_{\mathscr{H}}>0$ for some scalars $m<M$. Then*

$$\varphi\left(A^{2}C,C\right)\#\varphi\left(C,C\right)\le\frac{M+m}{2\sqrt{Mm}}\varphi\left(AC,C\right),$$

for every $C\in\mathcal{B}(\mathscr{H})$.

Proof. Replacing C with $A^{\frac{1}{2}}C$ in the (3), we have

$$\varphi\left(AA^{\frac{1}{2}}C,A^{\frac{1}{2}}C\right)\#\varphi\left(A^{-1}A^{\frac{1}{2}}C,A^{\frac{1}{2}}C\right)\le\frac{M+m}{2\sqrt{Mm}}\varphi\left(A^{\frac{1}{2}}C,A^{\frac{1}{2}}C\right)$$

therefore

$$\varphi\left(A^{2}C,C\right)\#\varphi\left(C,C\right)\le\frac{M+m}{2\sqrt{Mm}}\varphi\left(AC,C\right).$$

Which completes the proof.

To prove the Theorem 5, we need the following basic lemma.

Lemma 2. *Let A be a self-adjoint operator on $\mathscr{H}$ satisfying $M1_{\mathscr{H}}\ge A\ge m1_{\mathscr{H}}$ for some scalars $m<M$, then*

$$\left(M1_{\mathscr{H}}-A\right)\left(A-m1_{\mathscr{H}}\right)\le\left(\frac{M-m}{2}\right)^{2}.$$

Proof. A simple computation yields

$$\begin{aligned}
&(M1_{\mathscr{H}} - A)(A - m1_{\mathscr{H}}) \\
&= (M+m)A - Mm1_{\mathscr{H}} - A^2 \\
&= \frac{(M-m)^2}{4}1_{\mathscr{H}} - \left(A - \frac{M+m}{2}1_{\mathscr{H}}\right)^2 \\
&\le \left(\frac{M-m}{2}\right)^2 1_{\mathscr{H}},
\end{aligned}$$

as desired.

Theorem 5. *Let A be a self-adjoint operator on $\mathscr{H}$ satisfying $M1_{\mathscr{H}} \ge A \ge m1_{\mathscr{H}}$ for some scalars $m < M$ and $\varphi(C,C) = 1_{\mathscr{H}}$. Then*

$$\varphi\left(A^2C, C\right) - \varphi(AC,C)^2 \le \frac{(M-m)^2}{4}.$$

Proof. By Lemma 2 we have

$$\begin{aligned}
&\varphi\left(A^2C, C\right) - \varphi(AC,C)^2 \\
&\quad = (M1_{\mathscr{H}} - \varphi(AC,C))(\varphi(AC,C) - m1_{\mathscr{H}}) - \varphi((M1_{\mathscr{H}} - A)(A - m1_{\mathscr{H}})C, C) \\
&\quad \le (M1_{\mathscr{H}} - \varphi(AC,C))(\varphi(AC,C) - m1_{\mathscr{H}}) \\
&\quad \le \frac{(M-m)^2}{4}1_{\mathscr{H}},
\end{aligned}$$

which is exactly what we needed to prove.

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