EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 10, No. 3, 2017, 455-472
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

States on pseudo-BCI algebras

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Abstract. In this paper, we discuss the structure of pseudo-BCI algebras and get that any pseudo-BCI algebra is a union of it's branches. We introduce the notion of local bounded pseudo-BCI algebras and study some related properties. Moreover we define two operations $\wedge_1$, $\wedge_2$ in a local bounded pseudo-BCI algebra A and two local operations $\vee_1$ and $\vee_2$ in $V(a)$ for $a \in M(A)$. We show that in a $\wedge_1(\wedge_2)$-commutative local bounded pseudo-BCI algebra A, $(V(a), \wedge_1, \vee_1)((V(a), \wedge_2, \vee_2))$ forms a lattice for all $a \in M(A)$. We define a Bosbach state on a local bounded pseudo-BCI algebra. Then we give two examples of local bounded pseudo-BCI algebras to show that there is local bounded pseudo-BCI algebras having a Bosbach state but there is some one having no Bosbach states. Moreover we discuss some basic properties about Bosbach states. If s is a Bosbach state of a local bounded pseudo-BCI algebra A, we prove that $A/ker(s)$ is equivalent to an MV-algebra. We also introduce the notion of state-morphisms on local bounded pseudo-BCI algebras and discuss the relations between Bosbach states and state-morphisms. Finally we give some characterization of Bosbach states.

2010 Mathematics Subject Classifications: 03G25

Key Words and Phrases: Pseudo-BCI algebra, local bounded, state, state-morphism, MV-algebra

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1. Introduction

BCK/BCI algebras were introduced originally by Iséki in [17] and [18] with a binary operation $*$ modeling the set-theoretical difference. Another motivation is from classical and non-classical propositional calculi modeling logical implications. Such algebras contain as a special subfamily of a family of MV-algebras where some important fuzzy structures can be studied. For more about BCK algebras, see [22].

Pseudo-BCK algebras were originally introduced by Georgescu and Iorgulescu in [13] as algebras with "two differences", a left- and right-difference, instead of one $*$ and with a constant element 0 as the least element. In [12], a special subclass of pseudo-BCK algebras, called Łukasiewicz pseudo-BCK algebras, was introduced and it was shown that

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it is always a subalgebra of the positive cone of some ℓ-group (not necessarily abelian). The class of Łukasiewicz pseudo-BCKalgebras is a variety whereas the class of pseudo-BCKalgebras is not; it is only a quasivariety because it is not closed under homomorphic images. For a guide through the pseudo-BCK algebras realm, see the monograph [16]. In [8], W. A. Dudek and Y. B. Jun introduced the notion of pseudo-BCI algebras as an extension of BCI-algebras, and investigated some properties.

MV-algebras entered into mathematics just 50 years ago due to Chang [3], but the notion of a state for MV-algebras was introduced by Mundici [23] in 1995 as averaging of the truth-value in Łukasiewicz logic. BL-algebras were introduced in the 1990s by Hájek [14] as the equivalent algebraic semantics for its basic fuzzy logic. In [5], authors defined a state-operator and a strong state-operator for a BL-algebra and prove some of their basic properties. L. Z. Liu studied the existence of Bosbach states and Riečan states on finite monoidal t-norm based algebras in [21]. Some examples show that there exist MTL-algebras having no Bosbach states and Riečan states.

In [10], Dvurečenskij introduced measures and states on BCK-algebras, and showed that the set of elements of measure 0 is an ideal, and the corresponding quotient BCK-algebra is commutative with a lifted original measure. Ciungu and Dvurečenskij [4] extended the notions of measures and states presented in Dvurečen-skij and Pulmannová [9] to the case of pseudo-BCK algebras, studied similar properties, and prove that, under some conditions, the notion of a state in the sense of Dvurečenskij and Pulmann*ová* [9] coincides with the Bosbach state.

The aim of this paper is to introduce and study the state theory on local bounded pseudo-BCI algebras. This paper is organized as follows: in Section 2, we recall notions of BCI-algebras and the notion and some properties of pseudo-BCI algebras. In the same time, we discuss the structure of pseudo-BCI algebras and get that any pseudo-BCI algebra is a union of it's branches. In Section 3, we introduce the notion of local bounded pseudo-BCI algebras and study some related properties. In Section 4, we define a Bosbach state on a local bounded pseudo-BCI algebra. Then we give two examples of local bounded pseudo-BCI algebras to show that there is local bounded pseudo-BCI algebras having a Bosbach state but there is some one having no Bosbach states. Moreover we discuss some of their basic properties. We discuss the relation between local bounded pseudo-BCI algebras and MV-algebras. We also introduce the notion of state-morphisms on local bounded pseudo-BCI algebras and discuss the relations between Bosbach states and state-morphisms. Finally we give some characterization on Bosbach states.

2. Pseudo-BCI algebras

Recall that a BCI-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$, (1) $((x*y)*(x*z))*(z*y) = 0$, (2) $(x*(x*y))*y = 0$, (3) $x*x = 0$, (4) $x*y = 0$ and $y*x = 0$ imply $x = y$.

For any BCI-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x*y = 0$ is a partial order on X. A nonempty subset I of a BCI-algebra X is called a BCI- ideal of X if it satisfies (1) $0 \in I$, (2) For all $x, y \in X, x*y \in I, y \in I \Rightarrow x \in I$.

We recall the notion and some properties of pseudo-BCI algebras.

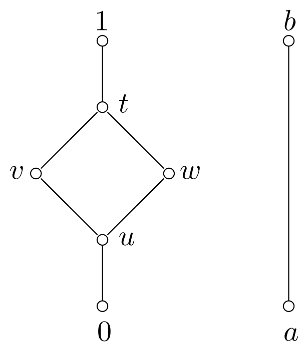
Definition 1. *[19] A pseudo-BCI algebras is a structure* $\mathbb{A} = (A, \leq, *, \circ, 0)$*, where* $\leq$ *is a binary relation on* A*,* $*$ *and* $\circ$ *are binary operations on* A *and "0" is an element of* A*, satisfying, for all* $x, y, z \in A$*,*
(I_1) $(x * y) \circ (x * z) \leq z * y$, $(x \circ y) * (x \circ z) \leq z \circ y$.
(I_2) $x * (x \circ y) \leq y$, $x \circ (x * y) \leq y$.
(I_3) $x \leq x$.
(I_4) $x \leq y$ *and* $y \leq x$ *imply* $x = y$.
(I_5) $x \leq y$ *iff* $x * y = 0$ *iff* $x \circ y = 0$.

Definition 2. *[13] A pseudo-BCK algebra is a structure* $\mathbb{A} = (A, \preceq, \rightarrow, \rightsquigarrow, 1)$ *where* $\preceq$ *is a binary relation on* A*, and* $\rightarrow$ *and* $\rightsquigarrow$ *are binary operations on* A *and* 1 *is an element of* A *satisfying, for all* $x, y, z \in A$*, the axioms:*
(K_1) $x \rightarrow y \preceq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \preceq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$.
(K_2) $x \preceq (x \rightarrow y) \rightsquigarrow y$, $x \preceq (x \rightsquigarrow y) \rightarrow y$.
(K_3) $x \preceq x$.
(K_4) $x \preceq 1$.
(K_5) *if* $x \preceq y$ *and* $y \preceq x$*, then* $x = y$.
(K_6) $x \preceq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$.

Remark 1. *(1) A pseudo-BCK algebra* $\mathbb{A} = (A, \preceq, \rightarrow \rightsquigarrow, 1)$ *can be seen a pseudo-BCI algebra* $\mathbb{A} = (A, \leq, *, \circ, 0)$ *if* $x \rightarrow y = y * x$, $x \rightsquigarrow y = y \circ x$, $1 = 0$ *and* $x \preceq y$ *iff* $y \leq x$ *for all* $x, y \in A$.
(2) A pseudo-BCI algebra is a BCI algebra if $* = \circ$.
(3) The relation $\leq$ *is a partial order on a pseudo-BCI algebra* A.

Now we give two pseudo-BCI algebras which are not pseudo-BCK algebras.

Example 1. *Let* $A = \{0, u, v, w, t, 1, a, b\}$*. The order of the elements in* A *is as the following Hasse diagram:*



Now the operations $*$ *and* $\circ$ *are defined by Tables 2.1 and 2.2, respectively. Simple calculations show that* $(A, \leqslant, *, \circ, 0)$ *is a pseudo-BCI algebra.*

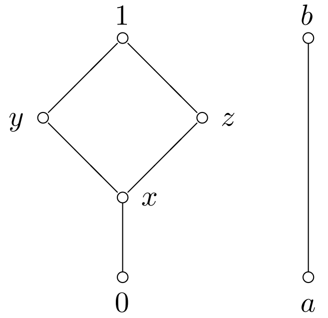
Example 2. *Let* $A = \{0, x, y, z, 1, a, b\}$ *in which the order of elements in* A *is as the following Hasse diagram:*

$*$	0	u	v	w	t	1	a	b
0	0	0	0	0	0	0	a	a
u	u	0	0	0	0	0	a	a
v	v	v	0	v	0	0	a	a
w	w	w	w	0	0	0	a	a
t	t	t	w	t	0	0	a	a
1	1	1	1	1	1	0	a	a
a	a	a	a	a	a	a	0	0
b	b	b	b	b	b	a	1	0

Tables 2.1

$\circ$	0	u	v	w	t	1	a	b
0	0	0	0	0	0	0	a	a
u	u	0	0	0	0	0	a	a
v	v	v	0	v	0	0	a	a
w	w	w	w	0	0	0	a	a
t	t	t	t	v	0	0	a	a
1	1	1	1	1	1	0	a	a
a	a	a	a	a	a	a	0	0
b	b	b	b	b	b	a	1	0

Tables 2.2



Let the operations $*, \circ$ be given by the following Tables 2.3 and 2.4.

$*$	0	x	y	z	1	a	b
0	0	0	0	0	0	a	a
x	x	0	0	0	0	a	a
y	y	y	0	y	0	a	a
z	z	z	z	0	0	a	a
1	1	1	1	y	0	a	a
a	a	a	a	a	a	0	0
b	b	a	a	a	a	y	0

Tables 2.3

$\circ$	0	x	y	z	1	a	b
0	0	0	0	0	0	a	a
x	x	0	0	0	0	a	a
y	y	y	0	y	0	a	a
z	z	z	z	0	0	a	a
1	1	1	z	1	0	a	a
a	a	a	a	a	a	0	0
b	b	b	a	b	a	x	0

Tables 2.4

Then $(A, \leqslant, *, \circ, 0)$ is a pseudo-BCI algebra.

Proposition 1. *[19] In a pseudo-BCI algebras A the following hold:*
(p_1) $x \leq 0 \Rightarrow x = 0$.
(p_2) $x \leq y \Rightarrow z * y \leq z * x$ *and* $z \circ y \leq z \circ x$.
(p_3) $x \leq y, y \leq z \Rightarrow x \leq z$.
(p_4) $(x * y) \circ z = (x \circ z) * y$.
(p_5) $x * y \leq z \Leftrightarrow x \circ z \leq y$.
(p_6) $(x * y) * (z * y) \leq x * z$, $(x \circ y) \circ (z \circ y) \leq x \circ z$.
(p_7) $x \leq y \Rightarrow x * z \leq y * z, x \circ z \leq y \circ z$.

(p_8) $x * 0 = x = x \circ 0$.
(p_9) $x * (x \circ (x * y)) = x * y$, $x \circ (x * (x \circ y)) = x \circ y$.

Proposition 2. *[19] In a pseudo-BCI algebra A the following holds for all $x, y, z \in A$:*
(i) $0 * (x \circ y) \leq y \circ x$.
(ii) $0 \circ (x * y) \leq y * x$.
(iii) $0 * (x * y) = (0 \circ x) \circ (0 * y)$.
(iv) $0 \circ (x \circ y) = (0 * x) * (0 \circ y)$.

Definition 3. *[19] An element a of a pseudo-BCI algebra A is called a pseudo-atom if for every $x \in A$, $x \leq a$ implies $x = a$.*

The set of all pseudo-atoms of a pseudo-BCI algebra A is denoted by $M(A)$. Obviously, $0 \in M(A)$.

Proposition 3. *Let A be a pseudo-BCI algebra and $a \in A$. The following conditions are equivalent:*
(1) a is a pseudo-atom of A;
(2) $y * (y \circ a) = a$ (or $y \circ (y * a) = a$) for all $y \in A$;
(3) $y * (y \circ (a * x)) = a * x$ (or $y \circ (y * (a \circ x)) = a \circ x$) for all $x, y \in A$.

Proof. (1) $\Rightarrow$ (2). By I_2, $y * (y \circ a) \leq a$. Since a is a pseudo-atom of A, we have $y * (y \circ a) = a$.

(2) $\Rightarrow$ (3) Obviously.

(3) $\Rightarrow$ (1) It follows from Proposition 3.6 of [19].

By Proposition 3, we have $x * (x \circ a) = x \circ (x * a) = a$ for all $a \in M(A)$ and $x \in A$.

Corollary 1. *Let A be a pseudo-BCI algebra. Then for all $a \in M(A)$ and $x \in A$, we have $a * x \in M(A)$ and $a \circ x \in M(A)$.*

Proof. Let $a \in M(A)$ and $x \in A$. By Proposition 3.8(3), we have $y * (y \circ (a * x)) = a * x$ for all $y \in A$. Using Proposition 3.8(2), we get that $a * x$ is a pseudo-atom of A, that is $a * x \in M(A)$. Similarly we can prove $a \circ x \in M(A)$.

Let A be a pseudo-BCI algebra. For $a \in M(A)$, define $V(a) = \{x \in A \mid a \leq x\}$. V(a) is called a branch of A. Obviously $a \in V(a)$.

Proposition 4. *Let A be a pseudo-BCI algebra, $a, b \in M(A)$ and $a \neq b$. Then $V(a) \cap V(b) = \emptyset$.*

Proof. Assume $V(a) \cap V(b) \neq \emptyset$, then there is $x \in V(a) \cap V(b)$. Hence $a \leq x$ and $b \leq x$. It follows that $(b * (b \circ a)) \circ (b * (b \circ x)) \leq (b \circ x) * (b \circ a) \leq a \circ x = 0$. So $(b * (b \circ a)) \circ (b * (b \circ x)) = 0$. Hence $b * (b \circ a) \leq (b * (b \circ x)) = b$. Since $b \in M(A)$, we have $b * (b \circ a) = b$. Note that $b = (b * (b \circ a)) \leq a$. Similarly $a \leq b$. By Definition 3.1, we have $a = b$. It is a contradiction, hence $V(a) \cap V(b) = \emptyset$.

Proposition 5. *Let A be a pseudo-BCI algebra and $x, y \in A$. If $x \leq y$, then x, y are in the same branch of A.*

Proof. Assume that $x \in V(a)$ and $y \in V(b)$ for some $a, b \in M(A)$ and $a \neq b$. Then $a \leq x \leq y$. Hence $y \in V(a)$ and so $y \in V(a) \cap V(b)$, a contradiction with Proposition 4.

Proposition 6. *Let A be a pseudo-BCI algebra and $x \in V(a)$ for some $a \in M(A)$. Then $0 * (0 \circ x) = a$ and $0 \circ (0 * x) = a$.*

Proof. Since $0 * (0 \circ x) \leq x$, we have $0 * (0 \circ x) \in V(a)$ by Proposition 5. Hence $a \leq 0 * (0 \circ x)$. On the other hand, we have $(0 * (0 \circ x)) \circ a = (0 \circ a) * (0 \circ x) = ((a * x) \circ a) * (0 \circ x) = ((a \circ a) * x) * (0 \circ x) = (0 * x) * (0 * x) = 0$. Therefore $0 * (0 \circ x) \leq a$. This shows that $0 * (0 \circ x) = a$. Similarly we can prove $0 \circ (0 * x) = a$.

Proposition 7. *Let A be a pseudo-BCI algebra. Then for any $x \in A$, $0 * (0 \circ x) \in M(A)$ and $0 \circ (0 * x) \in M(A)$.*

Proof. Let $x \in A$. In order to prove $0 \circ (0 * x) \in M(A)$, we assume $y \leq 0 \circ (0 * x)$. Then $y \circ (0 \circ (0 * x)) = 0$. By (p_4) and (p_9) of Proposition 3.3, we have

$(0 \circ (0 * x)) * y = (0 * y) \circ (0 * x)$

$= ((y \circ (0 \circ (0 * x))) * y) \circ (0 * x)$

$= ((y * y) \circ ((0 \circ (0 * x)))) \circ (0 * x)$

$= (0 \circ ((0 \circ (0 * x)))) \circ (0 * x)$.

By Proposition 2(iv), $0 \circ ((0 \circ (0 * x))) = (0 * 0) * (0 \circ (0 * x)) = 0 * (0 \circ (0 * x)) = 0 * x$. Hence $(0 \circ (0 * x)) * y = (0 \circ ((0 \circ (0 * x)))) \circ (0 * x) = (0 * x) \circ (0 * x) = 0$. This shows that $0 \circ (0 * x) \leq y$ and hence $y = 0 \circ (0 * x)$. Similarly we can prove $0 * (0 \circ x) \in M(A)$.

Corollary 2. *Let A be a pseudo-BCI algebra. Then for any $x \in A$, $(0 \circ x) \in M(A)$ and $(0 * x) \in M(A)$.*

Proof. Since $0 * x = 0 * (0 \circ (0 * x))$ and $0 \circ x = 0 \circ (0 * (0 \circ x))$, we have $0 * x \in M(A)$ and $0 \circ x \in M(A)$ by Proposition 7.

By Propositions 6 and 7, we have $0 * (0 \circ x) = 0 \circ (0 * x) \in M(A)$ for all $x \in A$. Denote $a_x = 0 * (0 \circ x) = 0 \circ (0 * x)$, for $x \in A$. Then $a_x \in M(A)$ and $x \in V(a_x)$.

Using above arguments we can get the structure of a pseudo-BCI algebra.

Theorem 1. *Let A be a pseudo-BCI algebra. Then $\{V(a) \mid a \in M(A)\}$ forms a partition of A, that is, $A = \cup_{a \in M(A)} V(a)$ and $V(a) \cap V(b) = \emptyset$ for all $a, b \in M(A)$ and $a \neq b$.*

3. Local bounded pseudo-BCI algebras

Let A be a pseudo-BCI algebra. For $a \in M(A)$, if there is an element $1_a \in V(a) \setminus \{a\}$ such that for all $x \in V(a)$, $x \leq 1_a$, then 1_a is called the local unit of $V(a)$. Note that 1_a is unique.

Definition 4. *Let A be a pseudo-BCI algebra. If for every $a \in M(A)$, $V(a)$ has a local unit, then A is called a local bounded pseudo-BCI algebra. For convenience we denote it by lbp-BCI algebra.*

Note that the pseudo-BCI algebras given in Examples 1 and 2 are local bounded pseudo-BCI algebras. In Examples 1, $M(A) = \{0, a\}$, $V(0) = \{0, u, v, w, t, 1\}$, $1_0 = 0$, $V(a) = \{a, b\}$, $1_a = b$. In Examples 2, $M(A) = \{0, a\}$, $V(0) = \{0, x, y, z, 1\}$, $1_0 = 0$, $V(a) = \{a, b\}$, $1_a = b$.

In the following, A shall mean a lbp-BCI algebra unless otherwise specified.

We define two negations, $^-$ and $^\sim$, as follows: for $a \in M(A)$ and $x \in V(a)$, $x^- \doteq 1_a * x$, $x^\sim \doteq 1_a \circ x$.

Proposition 8. *For all $x, y \in A$, we have*
(1) $x^{-\sim} \leq x$, $x^{\sim-} \leq x$.
(2) $x \leq y \Rightarrow y^- \leq x^-$, $y^\sim \leq x^\sim$.
(3) $x^- = x^{-\sim-}$, $x^\sim = x^{\sim-\sim}$.

Proof. (1) By (I_2) of Definition 1, we have $x^{-\sim} \leq x$ and $x^{\sim-} \leq x$.

(2) Let $x \leq y$, then $x, y \in V(a)$ for some $a \in M(A)$. Hence $(1_a \circ y) * (1_a \circ x) \leq x \circ y = 0$, and so $(1_a \circ y) * (1_a \circ x) = 0$. It follows that $1_a \circ y \leq 1_a \circ x$, or $y^\sim \leq x^\sim$. Similarly we can prove $y^- \leq x^-$.

(3) By (1), we have $x^{\sim-} \leq x$. Replace x by x^-, we get $x^{-\sim-} \leq x^-$. On the other hand, $x^{-\sim} \leq x$ implies $x^- \leq x^{-\sim-}$ by (2). So $x^- = x^{-\sim-}$. Similarly we can prove $x^\sim = x^{\sim-\sim}$.

Let A be a pseudo-BCI algebra. For any $x, y \in A$, define $x \wedge_1 y \doteq y \circ (y * x)$, $x \wedge_2 y \doteq y * (y \circ x)$.

Proposition 9. *In A the following properties hold:*
(1) $a_x \wedge_1 x = x \wedge_1 a_x = a_x$ and $a_x \wedge_2 x = x \wedge_2 a_x = a_x$ for all $x \in A$.
(2) $x \leq y$ implies $y \wedge_1 x = x$ and $y \wedge_2 x = x$.
(3) $x \wedge_1 x = x$ and $x \wedge_2 x = x$.
(4) If $x_1 \leq x_2$, then $x_1 \wedge_1 y \leq x_2 \wedge_1 y$ and $x_1 \wedge_2 y \leq x_2 \wedge_2 y$.

Proof. (1) By Proposition 3, we have $a_x \wedge_1 x = x \circ (x * a_x) = a_x$ since $a_x \in M(A)$. Note that for $x \in V(a_x)$, we get $x \wedge_1 a_x = a_x \circ (a_x * x) = a_x \circ 0 = a_x$. So we shows that $a_x \wedge_1 x = x \wedge_1 a_x = a_x$. Similarly we can prove $a_x \wedge_2 x = x \wedge_2 a_x = a_x$ for all $x \in A$.

(2) Let $x \leq y$. Then $y \wedge_1 x = x \circ (x * y) = x \circ 0 = x$ and $y \wedge_2 x = x * (x \circ y) = x * 0 = x$.

(3) We have $x \wedge_1 x = x \circ (x * x) = x$ and $x \wedge_2 x = x * (x \circ x) = x$.

(4) Let $x_1 \leq x_2$. Note that $(x_1 \wedge_1 y) * (x_2 \wedge_1 y) = (y \circ (y * x_1)) * (y \circ (y * x_2)) \leq (y * x_2) \circ (y * x_1) \leq x_1 * x_2 = 0$. We get $x_1 \wedge_1 y \leq x_2 \wedge_1 y$. Similarly we can prove $x_1 \wedge_2 y \leq x_2 \wedge_2 y$.

Proposition 10. *In A the following properties hold for all $a \in M(A)$ and $x, y \in V(a)$:*
(1) $x \wedge_1 y^{-\sim} = x^{-\sim} \wedge_1 y^{-\sim}$ and $x \wedge_2 y^{\sim-} = x^{\sim-} \wedge_2 y^{\sim-}$.
(2) $x \wedge_1 y^\sim = x^{-\sim} \wedge_1 y^\sim$ and $x \wedge_2 y^- = x^{\sim-} \wedge_2 y^-$.

Proof. (1) Using Proposition 1, we have $y^{-\backsim} * x = (1_a \circ (1_a * y)) * x = (1_a * x) \circ (1_a * y) = (1_a * (1_a \circ (1_a * x))) \circ (1_a * y) = (1_a \circ (1_a * y)) * (1_a \circ (1_a * x)) = y^{-\backsim} * x^{-\sim}$.
Thus $x \wedge_1 y^{-\sim} = y^{-\sim} \circ (y^{-\sim} * x) = y^{-\sim} \circ (y^{-\backsim} * x^{-\sim}) = x^{-\sim} \wedge_1 y^{-\backsim}$.

(2) By Proposition 8 and (1), we get
$x \wedge_1 y^{\sim} = x \wedge_1 (y^{\sim})^{-\sim} = x^{-\sim} \wedge_1 (y^{\sim})^{-\sim} = x^{-\sim} \wedge_1 y^{\sim}$.

Proposition 11. *In A the following properties hold for all $x, y \in A$:*
$y * (x \wedge_1 y) = y * x$ *and* $y \circ (x \wedge_2 y) = y \circ x$.

Proof. By Proposition 1, we have $y * (x \wedge_1 y) = y * (y \circ (y * x) = y * x$ and $y \circ (x \wedge_2 y) = y \circ (y * (y \circ x)) = y \circ x$.

Proposition 12. *Let $a \in M(A)$. If $x, y \in V(a)$, then $x * y \in V(0)$ and $x \circ y \in V(0)$.*

Proof. Using Proposition 2 and 6, we get $0 \circ (0 * (x * y)) = 0 \circ ((0 \circ x) \circ (0 * y)) = (0 * (0 \circ x)) * (0 \circ (0 * y)) = a * a = 0$. Since by (I_2) $0 \circ (0 * (x * y)) \leq x * y$, we have $0 \leq x * y$, and so $x * y \in V(0)$. Similarly we can prove $x \circ y \in V(0)$.

Proposition 13. *In A the following properties hold for all $a \in M(A)$, $x, y \in V(a)$:*
(1) $x \wedge_1 y$ $(y \wedge_1 x)$ is a lower bound of $\{x, y\}$.
(2) $x \wedge_2 y$ $(y \wedge_2 x)$ is a lower bound of $\{x, y\}$.

Proof. By Definition 3.1, we have $x \wedge_1 y = y \circ (y * x) \leq x$. Moreover by Proposition 12, $y * x \in V(0)$ and so $0 \circ (y * x) = 0$, and $(y \circ (y * x)) * y = (y * y) \circ (y * x) = 0 \circ (y * x) = 0$. It follows that $x \wedge_1 y = y \circ (y * x) \leq y$. Similarly we can get that $(y \wedge_1 x)$ is also a lower bound of $\{x, y\}$.

(2) Similar to the proof of (1).

Definition 5. *(1) If for all $a \in M(A)$ and $x, y \in V(a)$, $x \wedge_1 y = y \wedge_1 x$, we call A to be a local $\wedge_1$-commutative pseudo-BCI algebra.*
(2) If for all $a \in M(A)$ and $x, y \in V(a)$, $x \wedge_2 y = y \wedge_2 x$, we call A to be a local $\wedge_2$-commutative pseudo-BCI algebra.
(3) If A is local $\wedge_1$-commutative and local $\wedge_2$-commutative, we call A to be local commutative.

Proposition 14. *(1) If A is local $\wedge_1$-commutative, then $(V(a), \wedge_1)$ forms a lower similattice for all $a \in M(a)$.*
(2) If A is local $\wedge_2$-commutative, then $(V(a), \wedge_2)$ forms a lower similattice for all $a \in M(a)$.

Proof. (1) It needs only to prove that $x \wedge_1 y$ is the greatest lower bound of $\{x, y\}$ for all $a \in M(A)$ and $x, y \in V(a)$. Assume that m is a lower bound of $\{x, y\}$. We have
$m * (x \wedge_1 y) = (m \circ (m * y)) * (y \circ (y * x)) = (y \wedge_1 m) * (y \circ (y * x)) = (m \wedge_1 y) * (y \circ (y * x)) = (y \circ (y * m)) * (y \circ (y * x)) \leq (y * x) \circ (y * m) \leq m * x = 0$,
and so $m \leq (x \wedge_1 y)$.

(2) Similar to the proof of (1).

For a lbp-BCI algebra A, we can define the following operations in $V(a)$,
$x \vee_1 y = 1_a \circ ((1_a * x) \wedge_1 (1_a * y))$,
$x \vee_2 y = 1_a * ((1_a \circ x) \wedge_2 (1_a \circ y))$,
for all $a \in M(A)$ and for all $x, y \in V(a)$.

Proposition 15. *Let A be a lbp-BCI algebra.*
(1) If A is local $\wedge_1$-commutative, then $(V(a), \wedge_1, \vee_1)$ forms a lattice for all $a \in M(a)$.
(2) If A is local $\wedge_2$-commutative, then $(V(a), \wedge_2, \vee_2)$ forms a lattice for all $a \in M(a)$.

Proof. (1) Let $a \in M(A)$ and $x, y \in V(a)$. Since A is local $\wedge_1$-commutative, then $x = x \circ (x * 1_a) = 1_a \circ (1_a * x) \leq 1_a \circ ((1_a * x) \wedge_1 (1_a * y)) = x \vee_1 y$. Similarly we can prove $y \leq x \vee_1 y$.

If $z \geq x$ and $z \geq y$, then $z \in V(a)$, $1_a * x \geq 1_a * z$ and $1_a * y \geq 1_a * z$. By Proposition 14, we have $1_a * z \leq (1_a * x) \wedge_1 (1_a * y)$. Therefore $x \vee_1 y = 1_a \circ ((1_a * x) \wedge_1 (1_a * y)) \leq 1_a \circ (1_a * z) = z \circ (z * 1_a) = z$. It follows that $x \vee_1 y$ is the least upper bound of $\{x, y\}$.

Applying Proposition 14, we get $(V(a), \wedge_1, \vee_1)$ forms a lattice.

(2) Similar to the proof of (1).

Definition 6. *Let A be a pseudo-BCI algebra. (1) If for all $x, y \in A$, $x \wedge_1 y = y \wedge_1 x$, we call A to be $\wedge_1$-commutative.*
(2) If for all $x, y \in A$, $x \wedge_2 y = y \wedge_2 x$, we call A to be $\wedge_2$-commutative.
(3) If A is $\wedge_1$-commutative and $\wedge_2$-commutative, we call A to be sup-commutative.

The following result shows that $\wedge_1$-commutative ($\wedge_2$-commutative) pseudo-BCI algebras must be pseudo-BCK algebras.

Proposition 16. *Let A be a pseudo-BCI algebra. Then the following are equivalent:*
(1) A is $\wedge_1$-commutative ($\wedge_2$-commutative).
(2) A is a $\wedge_1$-commutative ($\wedge_2$-commutative) pseudo-BCK algebra.

Proof. (1) $\Rightarrow$ (2). Let A be $\wedge_1$-commutative. Then for any $a \in M(A)$, we have $a \wedge_1 0 = 0 \wedge_1 a$. Note that $a \wedge_1 0 = 0 \circ (0 * a) = a$ by Proposition 6 and $0 \wedge_1 a = a \circ (a * 0) = 0$. This shows that $a = 0$, that is $A = V(0)$. Thus A is a $\wedge_1$-commutative pseudo-BCK algebra. Similarly we can prove the result for case of $\wedge_2$-commutative.

(2) $\Rightarrow$ (1). It is straightforward.

Proposition 17. *[15] If A is a sup-commutative pseudo-BCK algebra, then $\wedge_1 = \wedge_2$.*

By Proposition 16 and 17, we can get a characterization of sup-commutative pseudo-BCI algebras.

Proposition 18. *Let A be a pseudo-BCI algebra. Then the following are equivalent:*
(1) A is a sup-commutative pseudo-BCI algebra.
(2) A is a sup-commutative pseudo-BCK algebra.

4. States on local bounded pseudo-BCI algebras

Definition 7. *Let A be a lbp-BCI algebra. A Bosbach state on A is a function $s : A \to [0,1]$ such that the following conditions hold:*
(1) $s(x) + s(y * x) = s(y) + s(x * y)$, for all $x, y \in A$,
(2) $s(x) + s(y \circ x) = s(y) + s(x \circ y)$, for all $x, y \in A$,
(3) $s(a) = 1$ and $s(1_a) = 0$ where $a \in M(A)$ and 1_a is the local unit of $V(a)$.

Example 3. *Consider the local bounded pseudo-BCI algebra A given in Example 1. Define the function $s : A \to [0,1]$ by $s(0) = 1, s(u) = 1, s(v) = 1, s(w) = 1, s(t) = 1, s(1) = 0, s(a) = 1, s(b) = 0$. Then s is a unique Bosbach state on A.*

Example 4. *Consider the local bounded pseudo-BCI algebra A given in Example 2. Define a function $s : A \to [0,1]$ as follows: $s(0) = 1, s(x) = \alpha, s(y) = \beta, s(z) = \gamma, s(1) = 0, s(a) = 1, s(b) = 0$. Using $s(u) + s(v * u) = s(v) + s(u * v)$, taking $u = x, v = 1, u = y, v = 1$ and $u = z, v = 1$, respectively, we get $\alpha = 1, \beta = 1, \gamma = 0$. On the other hand, taking $u = z$, $v = 1$ in $s(u) + s(v \circ u) = s(v) + s(u \circ v)$, we get $\gamma + 0 = 0 + 1$, so $0 = 1$ which is a contradiction. Hence A does not admit a Bosbach state.*

Proposition 19. *Let A be a lbp-BCI algebra and s a Bosbach state on A. Then the following properties hold for all $x, y \in A$:*
(1) If $x \leq y$, then $s(y * x) = 1 + s(y) - s(x) = s(y \circ x)$ and $s(y) \leq s(x)$.
(2) If x, y are in same branch, then $s(x \wedge_1 y) = s(y \wedge_1 x)$, $s(x \wedge_2 y) = s(y \wedge_2 x)$.
(3) If x, y are in same branch, then $s(x \wedge_1 y^{-\sim}) = s(x^{-\sim} \wedge_1 y^{-\sim})$, $s(x \wedge_2 y^{\sim-}) = s(x^{\sim-} \wedge_2 y^{\sim-})$.
(4) If x, y are in same branch, then $s(x^{-\sim} \wedge_1 y) = s(x \wedge_1 y^{-\sim})$, $s(x^{\sim-} \wedge_2 y) = s(x \wedge_2 y^{\sim-})$.
(5) $s(x^{-\sim}) = s(x) = s(x^{\sim-})$.
(6) $s(x^{-}) = 1 - s(x) = s(x^{\sim})$.

Proof. (1) Let $x \leq y$. It follows from Definition 5.1 that $s(y * x) = 1 + s(y) - s(x) = s(y \circ x)$. Moreover $s(x) - s(y) = 1 - s(y * x) \geq 0$ and hence $s(y) \leq s(x)$.

(2) By Proposition 1, we have $y * x = y * (x \wedge_1 y)$. Since x, y are in same branch, then $x \wedge_1 y \leq x, y$ by proposition 13. By property (1), we have $s(y * x) = s(y * (x \wedge_1 y)) = 1 + s(y) - s(x \wedge_1 y)$ and $s(x * y) = s(x * (y \wedge_1 x)) = 1 + s(x) - s(y \wedge_1 x)$. Using condition (1) from Definition 7 we get $s(x \wedge_1 y) = s(y \wedge_1 x)$. Similarly we can prove $s(x \wedge_2 y) = s(y \wedge_2 x)$.

(3) It follows from Proposition 10.

(4) It follows from (2) and (3).

(5) For $x \in A$, there is $a \in M(A)$ such that $x \in V(a)$. Note that $x^{-\sim} = x \wedge_1 1_a$. By (2), we have $s(x^{-\sim}) = s(x \wedge_1 1_a) = s(1_a \wedge_1 x) = s(x \circ (x * 1_a)) = s(x)$. In a similar way, we can prove $s(x) = s(x^{\sim-})$.

(6) By (1), we have $s(x^{-}) = s(1_a * x) = 1 + s(1_a) - s(x) = 1 - s(x)$. In a similar way we can get $s(x^{\sim}) = 1 - s(x)$.

Proposition 20. *Let A be a lbp-BCI algebra and s be a Bosbach state on A. Then the following properties hold for all $a \in M(A)$ and $x, y \in V(a)$:*
(1) $s(y * x^{-\sim}) = s(y^{-\sim} * x)$, $s(y \circ x^{\sim-}) = s(y^{\sim-} \circ x)$.
(2) $s(y^{-\sim} * x) = s(x^{-} \circ y^{-}) = s(y^{-\sim} * x^{-\sim}) = s(y * x^{-\sim})$,
$s(y^{\sim-} \circ x) = s(x^{\sim} * y^{\sim}) = s(y^{\sim-} \circ x^{\sim-}) = s(y \circ x^{\sim-})$.
(3) $s(y^{-\sim} * x^{\sim}) = s(y * x^{\sim})$, $s(y^{\sim-} \circ x^{-}) = s(y \circ x^{-})$.

Proof. (1) Note that $s(y * x^{-\sim}) + s(y \circ (y * x^{-\sim})) = s(y) + s((y * x^{-\sim}) \circ y)$, or $s(y * x^{-\sim}) + s(x^{-\sim} \wedge_1 y) = s(y) + s((y * x^{-\sim}) \circ y)$. By Proposition 19(4), we have $s(y*x^{-\sim})+s(x\wedge_1 y^{-\sim}) = s(y)+s((y*x^{-\sim})\circ y) = s(y)+s((y\circ y)*x^{-\sim}) = s(y)+s(0*x^{-\sim})$. Using Corollary 2, we get $0 * x^{-\sim} \in M(A)$, and so $s(0 * x^{-\sim}) = 1$. Thus $s(y * x^{-\sim}) = s(y)+1-s(x\wedge_1 y^{-\sim}) = 1-s(x\wedge_1 y^{-\sim})+s(y^{-\sim}) = s((y^{-\sim}*x)\circ y^{-\sim})-s(x\wedge_1 y^{-\sim})+s(y^{-\sim}) = s(y^{-\sim} * x)$.
Similarly we can prove $s(y \circ x^{\sim-}) = s(y^{\sim-} \circ x)$.

(2) By (p_4) we have $s(y^{-\sim} * x) = s((1_a \circ (1_a * y)) * x) = s((1_a * x) \circ (1_a * y)) = s(x^{-} \circ y^{-})$. Moreover we have $s(y^{-\sim} * x^{-\sim}) = s((1_a \circ (1_a * y)) * ((1_a \circ (1_a * x))) = s((1_a \circ ((1_a \circ (1_a * x))) \circ (1_a * y)) = s(x^{-\sim-} \circ y^{-}) = s(x^{-} \circ y^{-})$ by Proposition 8. Using (1) we can get $s(y^{-\sim} * x) = s(x^{-} \circ y^{-}) = s(y^{-\sim} * x^{-\sim}) = s(y * x^{-\sim})$. Similarly we have $s(y^{\sim-} \circ x) = s(x^{\sim} * y^{\sim}) = s(y^{\sim-} \circ x^{\sim-}) = s(y \circ x^{\sim-})$.

(3) By Proposition 5.4(4) we get
$s(y^{-\sim} * x^{\sim}) = s(y^{-\sim}) + s((y^{-\sim} * x^{\sim}) \circ y^{-\sim}) - s(y^{-\sim} \circ (y^{-\sim} * x^{\sim})) = s(y) + 1 - s(x^{\sim} \wedge_1 y^{-\sim}) = s(y) + 1 - s(x^{\sim-\sim} \wedge_1 y) = s(y) + 1 - s(x^{\sim} \wedge_1 y) = s(y \circ x^{\sim})$.
Similarly we can get $s(y^{\sim-} \circ x^{-}) = s(y \circ x^{-})$.

Proposition 21. *Let A be a lbp-BCI algebra and s be a Bosbach state on A. Then for all $a \in M(A)$ and $x, y \in V(a)$, $s(y*x) = 1-s(x\wedge_1 y)+s(y)$ and $s(y\circ x) = 1-s(x\wedge_2 y)+s(y)$.*

Proof. Let $a \in M(A)$ and $x, y \in V(a)$. Note that $x \wedge_1 y \leq x, y$ and $x \wedge_2 y \leq x, y$. By 19(1), we have $s(y*x) = s(y*(x\wedge_1 y)) = 1-s(x\wedge_1 y)+s(y)$ and $s(y\circ x) = s(y\circ(x\wedge_2 y)) = 1 - s(x \wedge_2 y) + s(y)$.

The following results are important for our study.

Proposition 22. *Let A be a lbp-BCI algebra and s be a Bosbach state on A. Then for all $a \in M(A)$ and $x, y \in V(a)$, we have*
(1) $s(x \wedge_1 y) = s(x \wedge_2 y)$.
(2) $s(x * y) = s(x \circ y)$.

Proof. (1) First we prove the equality for $x \leq y$.

By Propositions 19(2) and 9(2), we have $s(x \wedge_1 y) = s(y \wedge_1 x) = s(x)$ and $s(x \wedge_2 y) = s(y \wedge_2 x) = s(x)$, that is $s(x \wedge_1 y) = s(x \wedge_2 y)$.

Now assume that x and y are arbitrary elements of $V(a)$, where $a \in M(A)$. Using Propositions 19(2) again and first part of the proof, we have $s(x \wedge_1 y) = s(x \wedge_1 (x \wedge_1 y)) = s((x \wedge_1 y) \wedge_1 x) = s((x \wedge_1 y) \wedge_2 x) \leq s(y \wedge_2 x) = s(x \wedge_2 y)$.
Dually, we can prove $s(x \wedge_2 y) \leq s(x \wedge_1 y)$. Hence $s(x \wedge_1 y) = s(x \wedge_2 y)$.

(2) It follows from Proposition 21 and the first equation.

Consider the real interval [0,1] of reals equipped with the Łukasiewicz implication $\to_{Ł}$ defined by

$x \to_{Ł} y = min\{1 - x + y, 1\}$, for all $x, y \in [0, 1]$.

Definition 8. *Let A be a lbp-BCI algebra. A state-morphism on A is a function $m : A \to [0, 1]$ such that:*
(SM1) $m(a) = 0, m(1_a) = 1$ for all $a \in M(A)$.
(SM2) $m(y * x) = m(y \circ x) = m(x) \to_{Ł} m(y)$, for all $x, y \in A$.

Proposition 23. *Let A be a lbp-BCI algebra. Then every state-morphism on A is a Bosbach state on A.*

Proof. It is similar to the proof of [[4], Proposition 3.9].

Proposition 24. *Let A be a lbp-BCI algebra. A Bosbach state m on A is a state-morphism if and only if $m(x \wedge_1 y) = min\{m(x), m(y)\}$ for all $x, y \in A$, or equivalently, $m(x \wedge_2 y) = min\{m(x), m(y)\}$ for all $x, y \in A$.*

Proof. It is similar to the proof of [[4], Proposition 3.10].

Let A be a lbp-BCI algebra and s be a Bosbach state on A. Define a set $Ker(s) := \{x \in A \mid s(x) = 1\}$. $Ker(s)$ is called the kernel of s on A.

Definition 9. *Let A be a pseudo BCI algebra and I be a nonempty subset of A. If I satisfies the following conditions:*
(1) $0 \in I$,
(2) $x \in I$ and $y * x \in I$ (or $y \circ x \in I$) imply $y \in I$ for all $x, y \in A$,
I is called a pseudo ideal of A, simply called an ideal of A.

Let I be a pseudo ideal of a pseudo BCI algebra A. If I satisfies $0 * x \in I$ and $0 \circ x \in I$, we call I a closed pseudo ideal of A. If I satisfies $x * y \in I$ if and only if $x \circ y \in I$, we call I a normal pseudo ideal of A. If I satisfies $x * y \in I$ if and only if $x \circ y \in I$ for all $a \in M(A), x, y \in V(a)$, we call I a local normal pseudo ideal of A.

Proposition 25. *Let A be a lbp-BCI algebra and s be a Bosbach state on A. Then $Ker(s)$ is a closed and local normal proper ideal of A.*

Proof. Obviously, $0 \in Ker(s)$ and $1 \notin Ker(s)$.

Assume that $x, y * x \in Ker(s)$. Then we have $1 = s(x)$ and $s(y * x) = 1$. It follows from Definition 5.1 that $s(y) = s(x) + s(y * x) - s(x * y) = 2 - s(x * y) \geq 1$ and thus $s(y) = 1$. Hence $y \in Ker(s)$. This shows that $Ker(s)$ is a proper ideal of A. For any $x \in A$, we have $0 * x \in M(A)$ and $0 * x \in M(A)$ by Corollary 2. Hence $s(0 * x) = 1$ and $s(0 \circ x) = 1$. It follows that $0 * x \in Ker(s)$ and $0 \circ x \in Ker(s)$. This shows that I is a closed pseudo ideal of A. By Proposition 22, we can get that A is local normal.

Theorem 2. *Let A be a pseudo BCI algebra and I be a pseudo ideal of A. Define a binary relation "$\sim$" on A by $x \sim y$ if and only if $x*y, y*x \in I$ if and only if $x \circ y, y \circ x \in I$. Then $\sim$ is a congruence relation on A. Denote $C_x = \{y \in A \mid x \sim y\}$. Define $C_x * C_y = C_{x*y}$ and $C_x \circ C_y = C_{x \circ y}$. Denote $A/I = \{C_x \mid x \in A\}$. Then $(A/I, *, \circ, C_0)$ is a pseudo BCI algebra. If I is a closed pseudo ideal of A, then $C_0 = I$.*

Proof. Obviously $\sim$ is reflexive and symmetric. Now we prove that it is transitive. Let $x \sim y$ and $y \sim z$. Then $x*y, y*z \in I$. By (I_1), $(x*z) \circ (x*y) \leq y*z$, thus $x*z \in I$. Similarly we can prove $z*x \in I$. This shows that $x \sim z$ and hence $\sim$ is transitive. Thus it is an equivalent relation on A. We also can show that $\sim$ is a congruence relation on A and omit it. Denote $A/I = \{C_x | x \in A\}$. Then binary operations "$*$" and "$\circ$" on A/I are well-defined. Moreover we can show that $(A/I, *, \circ)$ satisfies $I_1 - I_5$ in Definition 3.1. It follows that $(A/I, *, \circ, C_0)$ is a pseudo BCI algebra.

Finally we assume that I is a closed pseudo ideal of A. Then for $x \in I$, we have $0*x \in I$ and $x*0 = x \in I$. Hence $x \sim 0$, that is, $x \in C_0$. Therefore $C_0 = I$.

Proposition 26. *Let s be a Bosbach state on a lbp-BCI algebra A and $K = ker(s)$. Then we have the following.*
(1) $x/K \leq y/K$ iff $s(x*y) = 1$ iff $s(x \circ y) = 1$, where $x/K = \{y \in A | y \sim x\}$ for all $x \in A$.
(2) For all $a \in M(A)$ and all $x, y \in V(a)$, we have that $x/K \leq y/K$ iff $s(y \wedge_1 x) = s(x)$ iff $s(y \wedge_2 x) = s(x)$.
(3) $x/K = y/K$ iff $s(x*y) = s(y*x) = 1$ iff $s(x \circ y) = s(y \circ x) = 1$.
(4) For all $a \in M(A)$ and all $x, y \in V(a)$, $x/K = y/K$ iff $s(x) = s(y) = s(x \wedge_1 y)$ iff $s(x) = s(y) = s(x \wedge_2 y)$.
(5) $(A/K, \leq, *, \circ, 0/K, 1_0/K)$ is a bounded pseudo-BCK algebra where 1_0 is the unit of $V(0)$.
(6) The mapping $\tilde{s} : A/K \to [0,1]$ defined by $\tilde{s}(x/K) := s(x)(x \in A)$ is a Bosbach state on A/K.

Proof. (1) By Theorem 2, we know that $(A/K, \leq, *, \circ, 0/K)$ is a pseudo-BCI algebra. Note that $x/K \leq y/K$ iff $x/K * y/K = (x*y)/K = 0/K$ iff $x*y \in K$ iff $s(x*y) = 1$. Similarly, $x/K \leq y/K$ iff $x/K \circ y/K = (x \circ y)/K = 0/K$ iff $x \circ y \in K$ iff $s(x \circ y) = 1$.

(2) Let $a \in M(A)$ and $x, y \in V(a)$. As $s(x*y) = 1 - s(y \wedge_1 x) + s(x)$ by Proposition 21, we get $x/K \leq y/K$ iff $s(y \wedge_1 x) = s(x)$. Similarly, we have $x/K \leq y/K$ iff $s(y \wedge_2 x) = s(x)$.

(3) It follows easily from (1).

(4) It follows easily from (2).

(5) First we prove $M(A/K) = \{0/K\}$. Let $x/K \leq 0/K$. By (1), $s(x*0) = 1$. Note that $0*x \in M(A)$, then we have $s(0*x) = 1$. By (3), $x/K = 0/K$. Thus $0/K \in M(A/K)$.

Conversely let $x/K \in M(A/K)$. Obviously $(0*(0*x))/K \leq x/K$. Hence $(0*(0*x))/K = x/K$. Since for any $a \in M(A)$, $s(a*0) = s(0*a) = 1$, we have $0/K = a/K$. Thus $x/K = (0*(0*x))/K = 0/K$. This shows that $M(A/K) = \{0/K\}$, and hence $(A/K, \leq, *, \circ, 0/K)$ is a pseudo-BCK algebra.

Now we prove that $1_0/K$ is the greatest element of A/K. First we claim $1_0/K = 1_a/K$ for all $a \in M(A)$. Note that $s(1_0) + s(1_a * 1_0) = s(1_a) + s(1_0 * 1_a)$ and $s(1_0) = s(1_a) = 0$

by Definition 7, we have $s(1_a * 1_0) = s(1_0 * 1_a)$. Moreover $s(1_a * 1_0) + s(a \circ (1_a * 1_0)) = s(a) + s((1_a * 1_0) \circ a)$ by Definition 7. By Corollary 1, $a \circ (1_a * 1_0) \in M(A)$, and so $s(a \circ (1_a * 1_0)) = 1$. Since $(1_a * 1_0) \circ a = (1_a \circ a) * 1_0$ and $1_a \circ a \in V(0)$ by Proposition 12, we have $s((1_a * 1_0) \circ a) = s((1_a \circ a) * 1_0) = s(0) = 1$. Hence $s(1_a * 1_0) = 1$. By (3), $1_0/K = 1_a/K$ for all $a \in M(A)$. Let $x/K \in A/K$. Then $x/K \leq 1_{(0*(0\circ x))}/K = 1_0/K$. This shows that $1_0/K$ is the greatest element of A/K. It follows that $(A/K, \leq, *, \circ, 0/K, 1_0/K)$ is a bounded pseudo BCK algebra.

(6) The fact that $\tilde{s}$ is a well-defined Bosbach state on A/K is now straightforward.

Definition 10. *Let A be a lbp-BCI algebra. Then*
(1) A is called good if $x^{-\sim} = x^{\sim -}$ for all $x \in A$.
(2) A is with the condition (pDN) if $x^{-\sim} = x^{\sim -} = x$ for all $x \in A$.

Proposition 27. *Let s be a Bosbach state on a bounded pseudo-BCI algebra A and let $K = ker(s)$. For every element $x \in A$, we have $x^{-\sim}/K = x/K = x^{\sim -}/K$, that is, A/K satisfies the (pDN) condition.*

Proof. It is similar to the proof of [[4], Proposition 3.14].

Remark 2. *Let s be a Bosbach state on a pseudo-BCI algebra A. According to the proof of Proposition 27, we have $s(x * x^{-\sim}) = 1 = s(x * x^{\sim -})$ and $s(x \circ x^{-\sim}) = 1 = s(x \circ x^{\sim -})$.*

Theorem 3. *Let A be a lbp-BCI algebra, s be a Bosbach state on A and $K = ker(s)$. Then A/K is $\wedge_1$-commutative as well as $\wedge_2$-commutative. In addition, A/K is a $\wedge$-semilattice and good.*

Proof. It is similar to the proof of [[4], Proposition 3.16].

Proposition 28. *([4]) Let A be a good pseudo-BCK algebra. We define a binary operation $\otimes$ on A by $x \otimes y := y^{-\sim} * x^{\sim}$. For all $x, y \in A$, the following hold:*
(1) $x \otimes y = x^{\sim -} \circ y^{-}$.
(2) $x \otimes y \leq x, y$.
(3) $x \otimes 1 = 1 \otimes x = x^{\sim -}$.
(4) $x \otimes 0 = 0 \otimes x = 0$.
(5) $(x \otimes y)^{-\sim} = x \otimes y = x^{-\sim} \otimes y^{-\sim}$.
(6) $\otimes$ is associative.

An MV-algebra is an algebra $(A, \oplus, ^-, 0)$ of type $(2, 1, 0)$ such that (i) $\oplus$ is commutative and associative, (ii) $x \oplus 0 = x$, (iii) $x \oplus 0^- = 0^-$,(iv) $x^{--} = x$,(v) $y \oplus (y \oplus x^-)^- = x \oplus (x \oplus y^-)^-$. If we define $x * y = x \circ y = y^- \oplus x$, then $(A, *, \circ, 1, 0)$ is a bounded pseudo-BCK algebra.

An MV-state on an MV-algebra A is a mapping $s : A \to [0, 1]$ such that $s(1) = 1$ and $s(a \oplus b) = s(a) + s(b)$ whenever $a \odot b = 0$. Every MV-algebra admits at least one MV-state, and due to [17], every MV-state on A coincides with a Bosbach state on the BCK algebra A and vice versa.

We note that the radical, $Rad(A)$, of an MV-algebra A is the intersection of all maximal ideals of A([7]).

Proposition 29. *([9]). In any MV-algebra A the following conditions are equivalent:*
(a) $Rad(A) = 0$.
(b) $nx \leq x^-$ for all $n \in N$ implies $x = 0$.
(c) $nx \leq y^-$ for all $n \in N$ implies $x \wedge y = 0$.
(d) $nx \leq y$ for all $n \in N$ implies $x \odot y = x$, where $nx = x_1 \oplus \cdots x_n$ with $x_1 = \cdots = x_n = x$.

Remark 3. *An MV-algebra A is archimedean in the sense of [9] if it satisfies the condition (b) of Proposition 29 and A is archimedean in Belluces sense [1] if it satisfies the condition (d) of Proposition 29. By Proposition 29 the two definitions of archimedean MV-algebras are equivalent.*

Theorem 4. *Let s be a Bosbach state on a lbp-BCI algebra A and let $K = Ker(s)$. Then $(A/K, \oplus, ^-, 0/K)$, where $a/K \oplus b/K = (b * a^-)/K$ and $(a/K)^- = a^-/K$, is an archimedean MV-algebra and the map $\hat{s}(a/K) := s(a)$ is an MV-state on this MV-algebra.*

Proof. It is similar to the proof of [[4], Theorem 3.20].

By Theorem 3, A/K is a good pseudo-BCK algebra that is a $\wedge$-semilattice and $\tilde{s}$ on A/K is a Bosbach state such that $Ker(\tilde{s}) = \{0/K\}$. Due to [[20], Proposition 3.4.7], $(A/K)/Ker(\tilde{s})$ is term-equivalent to an MV-algebra that is archimedean and $\tilde{s}$ is an MV-state on it. Since $A/K = (A/K)/Ker(\tilde{s})$, the same is true also for A/K, and this proves the theorem.

In the following, we give properties of state-morphisms on lbp-BCI algebras.

Lemma 1. *Let A be a lbp-BCI algebra and m be a state-morphism on A. Then we have the following.*
(1) $m(y^{-\sim} * x^{\sim}) = min\{m(x) + m(y), 1\}$, for all $a \in M(A)$ and $x, y \in V(a)$.
(2) $m(x^{\sim-} \circ y^{-}) = min\{m(x) + m(y), 1\}$, for all $a \in M(A)$ and $x, y \in V(a)$.

Proof. Assume that m is a state-morphism on A, so it is a Bosbach state on A. By Propositions 19 and 20, for for all $a \in M(A)$ and $x, y \in V(a)$, we have $m(y^{-\sim} * x^{\sim}) = m(y * x^{\sim}) = m(x^{\sim}) \to_{Ł} m(y) = m(x)^{\sim} \to_{Ł} m(y) = min\{1 - m(x)^{\sim} + m(y), 1\} = min\{m(x) + m(y), 1\}$. Similarly we can prove $m(x^{\sim-} \circ y^{-}) = min\{m(x) + m(y), 1\}$, for all $a \in M(A)$ and $x, y \in V(a)$.

Proposition 30. *Let A be a lbp-BCI algebra and s be a Bosbach state on A. Then the following are equivalent:*
(1) s is a state-morphism.
(2) $ker(s)$ is a maximal ideal of A.

Proof. It is similar to the proof of [[4], Proposition 3.22].

Lemma 2. *Let m be a state-morphism on a lbp-BCI algebra A and $K = ker(m)$. Then*
(1) $a/K \leq b/K$ if and only if $m(a) \leq m(b)$,
(2) $a/K = b/K$ if and only if $m(a) = m(b)$.

Proof. It is similar to the proof of [[4], Lemma 3.23].

Proposition 31. *Let A be a lbp-BCI algebra and m_1, m_2 be two state-morphisms on A such that $ker(m_1) = ker(m_2)$. Then $m_1 = m_2$.*

Proof. By Proposition 23, m_1 and m_1 are two Bosbach states on A. Since $ker(m_1) = ker(m_2)$, we have $A/ker(m_1) = A/ker(m_2)$. By the proof of Proposition 30, we have that $A/ker(m_1)$ is in fact an MV-subalgebra of the MV-algebra of the real interval $[0, 1]$. But $ker(\hat{m_1}) = 0/K = ker(\hat{m_2})$. Hence, by [[11], Proposition 4.5], $\hat{m_1} = \hat{m_2}$, consequently, $m_1 = m_2$.

Let A be a lbp-BCI algebra. We say that a Bosbach state s is extremal if for any $0 < \lambda < 1$ and for any two Bosbach states s_1, s_2 on A, $s = \lambda s_1 + (1 - \lambda)s_2$ implies $s_1 = s_2$. Summarizing previous characterizations of state-morphisms, we have the following result.

Theorem 5. *Let s be a Bosbach state on a lbp-BCI algebra A. Then the following are equivalent:*
(1) s is an extremal Bosbach state.
(2) $s(x \wedge_1 y) = max\{s(x), s(y)\}$ for all $x, y \in A$.
(3) $s(x \wedge_2 y) = max\{s(x), s(y)\}$ for all $x, y \in A$.
(4) s is a state-morphism.
(5) $ker(s)$ is a maximal ideal.

Proof. It is similar to the proof of [[4], Theorem 3.26].

5. Conclusions

Until now, the states on unbounded algebraic structures have been studied for Hilbert algebras and integral residuated lattices in [2] and [6], respectively.

In this paper, we first study state theory on non-bounded algebraic structures, and introduce a notion of state on pseudo-BCI algebras. In order to adapt a state to pseudo-BCI algebras, we first discuss the structure of pseudo-BCI algebras, which can be decomposed in to the union of it's branches. Note that for all $a \in M(A)$ and $a \neq 0$, $V(a)$ is not a BCK-algebra, hence the structure of pseudo-BCI algebras is different from the structure of pseudo-BCK algebras. Therefore it is valuable to study state theory on pseudo-BCI algebras. Moreover we introduce a notion of local bounded pseudo-BCI algebras and set up the theory of states on such algebraic structure. We also introduce a notion of state-morphisms on local bounded pseudo-BCI algebras and discuss the relations between Bosbach states and state-morphisms. By use of state's theory, we discuss the relation between pseudo-BCI algebras and MV-algebras. In the next work, we will consider the following problem: satisfying what apposite conditions a local bounded pseudo-BCI algebra admits a Bosbach state?

Acknowledgements

This research is partially supported by a grant of National Natural Science Foundation of China (11571281,61602359), China Postdoctoral Science Foundation (2015M582618), China 111 Project (B16037).

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