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n-tupled fixed point results with rational type contraction in b-metric spaces

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Abstract. In this manuscript, using rational type contractive conditions the existence and uniqueness of common n-tupled fixed point for a pair of mappings in complete b-metric spaces are studied. Using the derived results some fixed theorems can be deduced in $b-$metric spaces.

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Key Words and Phrases: Complete b-metric space, n-tupled fixed point, n-tupled coincidence point, rational type contractive mappings

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1. Introduction and preliminaries

The Bananch contraction theorem is the most important technique for solving non-linear integral equations, differential equations and functional equations etc. It has fruitful applications within as well as outside mathematics. Many authors have extended this theorem employing relatively more general contractive conditions ensuring the existence and uniqueness of a fixed point.

To solve the problem of the convergence of measurable functions with respect to a measure, Bakhtin [2] and Czerwik [7] introduced the concept of b-metric spaces also called metric type space [18]. Using this concept Czerwik, generalized the Banach contraction principle in b-metric spaces (see [7, 8, 20, 21]). Yamaod and Sintunavarat [27] introduced the concept of (α, β)-(ψ, ϕ)-contractive mapping in b-metric spaces, and established some fixed point results for such mappings in b-metric spaces. Yamaod et al. [19] studied the existence of a common solution for a system of nonlinear integral equations via fixed point methods in b-metric space.

The concept of coupled fixed point was introduced by Gou and Lakshmikantham [9] for partially ordered set. Bhaskar and Lakshmikantham [5] studied the existence and uniqueness of a coupled fixed point results in partially ordered metric space. Lakshmikantham

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and Ciric in [12] defined mixed g-monotone property and studied coupled coincidence point in partially ordered metric spaces. Samet [24] investigated coupled fixed point results in the setting of partial ordered metric spaces for a generalized Meir- Keeler type contractive condition. Very recently, Sarwar et al. [15] studied the common coupled fixed point results satisfying rational type contractive conditions in b-metric spaces. Many researchers studied the coupled fixed point and discussed it's application. (see [3, 9, 25, 26]).

Berinde and Borcut [4] introduced the notion of tripled fixed point and established some results in the setting of partially ordered metric spaces. Karapinar [11] studied some quadruple fixed point results for non-linear contraction partially ordered metric spaces.

Imdad*et al.*[13] introduced the concept of n-tupled coincidence as well as n-tupled fixed point (for even n) and obtained n-tupled coincidence as well as n-tupled common fixed point theorems for nonlinear ϕ-contraction. Paknazar *et al.* [14] introduced the concept of a new g-monotone mapping and defined the notions of n-fixed point and n-coincidence point and proved some related theorems for nonlinear contractive mappings in partially ordered complete metric spaces. Soliman *et al.* [1] proved some n-tupled coincidence point theorems for nonlinear ϕ-contraction mappings in partially ordered complete asymptotically regular metric spaces. In [22] the authors introduced the notion of compatibility for n-tupled coincidence points and proved n-tupled fixed point for compatible mappings satisfying contractive type conditions in partially ordered metric spaces. Murthy *et al.* [17] introduced n-tupled fixed points (for all positive integers) and proved n-tupled fixed points theorems for contractive type mappings in fuzzy metric spaces. Husain *et al.* [23] present some n-tupled coincidence point results for a pair of mappings without mixed monotone property satisfying a rational type contractive condition in metric spaces equipped with a partial ordering as well as present results on the existence and uniqueness of n-tupled common fixed points.

The aim of this manuscript is to study n-tupled fixed point results via rational type contraction in complete b-metric spaces. The established result generalizes some recent results (particularly the result of Sarwar *et al.* [15] and Malhotra and Bansal [16]) from the existing literature in b-metric spaces.

Throughout this paper $\mathbb{R}$ is the set of real and $\mathbb{R}^+$ is a set of non-negative real numbers.

Definition 1. *[10] Let X be a non empty set and $s \geq 1$, $s \in \mathbb{R}$. A function $d : X \times X \to \mathbb{R}^+$ is called b-metric if for each $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then the pair (X, d) with parameter s is called b-metric space.

Example 1. *[6] The l_p space, $0 < p < 1$, $l_p = \{(x_n) \in \mathbb{R} : \sum |x_n|^p < \infty\}$, and function is defined as $d : l_p \times l_p \to \mathbb{R}$ by $d(x, y) = (\sum |x_n - y_n|^p)^{\frac{1}{p}}$, $x = (x_n)$, $y = (y_n) \in l_p$, then (X, d) is called b-metric space with parameter $s = 2^{\frac{1}{2}}$ provided that $d(x, z) \leq 2^{\frac{1}{2}}[d(x, y) + d(y, z)]$.*

Definition 2. *[6] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ is said be converge to $x \in X$ if for each $\epsilon > 0$ there exists $j(\epsilon) \in \mathbb{N}$, such that $d(x_n, x) < \epsilon \ \forall \ n \geq j(\epsilon)$.*

Definition 3. *[6] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ is said be a Cauchy sequence if for each $\epsilon > 0$ there exists $j(\epsilon) \in N$, such that $d(x_n, x_m) < \epsilon \ \forall \ n$, $m \geq j(\epsilon)$.*

Definition 4. *[13] Let X be a non empty set. An element $(x^1, x^2, \cdots, x^n) \in X^n$ is called an n-tupled fixed point of a given mapping $T : X^n \to X$ if*

$$\begin{aligned}
&x^1 = T(x^1, x^2, \cdots, x^n),\\
&x^2 = T(x^2, x^3, \cdots, x^n, x^1),\\
&x^3 = T(x^3, \cdots, x^n, x^1, x^2),\\
&\vdots\\
&x^n = T(x^n, x^1, x^2 \cdots, x^{n-1}).
\end{aligned}$$

Definition 5. *[13] Let X be a non empty set. An element $(x^1, x^2, \cdots, x^n) \in X^n$ is called an n-tupled coincidence point of the given mappings $S, T : X^n \to X$ if $S(x^1, x^2, \cdots, x^n) = T(x^1, x^2, \cdots, x^n)$,*

$$\begin{aligned}
&S(x^2, x^3, \cdots, x^n, x^1) = T(x^2, x^3, \cdots, x^n, x^1),\\
&S(x^3, \cdots, x^n, x^1, x^2) = T(x^3, \cdots, x^n, x^1, x^2),\\
&\vdots\\
&S(x^n, x^1, x^2, \cdots, x^{n-1}) = T(x^n, x^1, x^2, \cdots, x^{n-1}).
\end{aligned}$$

Example 2. *Suppose $X = \mathbb{R}$ and $S, T : X^n \to X$ be defined by*
$S(x^1, x^2, x^3, \cdots, x^n) = \frac{x^1+x^2+x^3+\cdots+x^n}{n}$ *and* $T(x^1, x^2, x^3, \cdots, x^n) = x^1x^2x^3 \cdots x^n$ *for each* x^1, x^2, x^3,$\cdots$, $x^n \in X^n$*. Then clearly* $(0, 0, 0, \cdots, 0)$ *and* $(1, 1, 1, \cdots, 1)$ *are n-tupled coincidence points of S and T.*

Definition 6. *[1] Let X be a non empty set. An element $(x^1, x^2, \cdots, x^n) \in X^n$ is called an n-tupled common fixed point of the mappings $S, T : X^n \to X$ if*

$$\begin{aligned}
&x^1 = S(x^1, x^2, \cdots, x^n) = T(x^1, x^2, \cdots, x^n),\\
&x^2 = S(x^2, x^3, \cdots, x^n, x^1) = T(x^2, x^3, \cdots, x^n, x^1),\\
&x^3 = S(x^3, \cdots, x^n, x^1, x^2) = T(x^3, \cdots, x^n, x^1, x^2),\\
&\vdots\\
&x^n = S(x^n, x^1, x^2, \cdots, x^{n-1}) = T(x^n, x^1, x^2, \cdots, x^{n-1}).
\end{aligned}$$

2. Main results

We begin with the following theorem.

Theorem 1. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and let the mapping S, $T : X^n \to X$ satisfy:*

$$d(S(x^1, x^2, \cdots, x^n), T(y^1, y^2, \cdots, y^n)) \leq \alpha_1 \frac{d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)}{n}$$

$$
\begin{aligned}
&+\alpha_2 \frac{d(x^1, S(x^1, x^2, \cdots, x^n))d(y^1, T(y^1, y^2, \cdots, y^n))}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_3 \frac{d(y^1, S(x^1, x^2, \cdots, x^n))d(x^1, T(y^1, y^2, \cdots, y^n))}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_4 \frac{d(S(x^1, x^2, \cdots, x^n), T(y^1, y^2, \cdots, y^n))d(x^1, y^1)}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_5 \frac{d(S(x^1, x^2, \cdots, x^n), T(y^1, y^2, \cdots, y^n))d(x^2, y^2)}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_6 \frac{d(y^1, T(y^1, y^2, \cdots, y^n))d(x^2, y^2)}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_7 \frac{d(y^1, S(x^1, x^2, \cdots, x^n))d(x^1, y^1)}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_8 \frac{d(y^1, S(x^1, x^2, \cdots, x^n))d(x^2, y^2)}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_9 \frac{d(y^1, T(y^1, y^2, \cdots, y^n))d(x^n, y^n)}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)} \\
&+\alpha_{10} \frac{d(y^1, S(x^1, x^2, \cdots, x^n))d(x^n, y^n)}{1 + d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)}.
\end{aligned} \tag{1}
$$

For all x^1, x^2, x^3,$\cdots$, x^n and y^1, y^2,y^3,$\cdots$, $y^n \in X$ and $\alpha_i \geq 0$, $i = 1, 2, \cdots, 10$ with the conditions $s\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_9 < 1$ and $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_{10} < 1$. Then S and T have unique common n-fixed point in X.

Proof. Taking "n" arbitrary points x_0^1, x_0^2, x_0^3,$\cdots$, x_0^n, in X, define the sequence by the following rules

$x_{2k+1}^1 = S(x_{2k}^1, x_{2k}^2, x_{2k}^3, \cdots, x_{2k}^n)$,
$x_{2k+1}^2 = S(x_{2k}^2, x_{2k}^1, x_{2k}^3, \cdots, x_{2k}^n)$,
$x_{2k+1}^3 = S(x_{2k}^3, x_{2k}^2, x_{2k}^1, \cdots, x_{2k}^n)$
$\vdots$
$x_{2k+1}^n = S(x_{2k}^n, x_{2k}^{n-1}, x_{2k}^{n-2}, \cdots, x_{2k}^2, x_{2k}^1)$,
and
$x_{2k+2}^1 = T(x_{2k+1}^1, x_{2k+1}^2, x_{2k+1}^3, \cdots, x_{2k+1}^n)$,
$x_{2k+2}^2 = T(x_{2k+1}^2, x_{2k+1}^1, x_{2k+1}^3, \cdots, x_{2k+1}^n)$,
$x_{2k+2}^3 = T(x_{2k+1}^3, x_{2k+1}^2, x_{2k+1}^1, \cdots, x_{2k+1}^n)$,
$\vdots$
$x_{2k+2}^n = T(x_{2k+1}^n, x_{2k+1}^{n-1}, x_{2k+1}^{n-2}, \cdots, x_{2k+1}^2, x_{2k+1}^1)$ for k=0,1,2,$\cdots$.
Consider

$$d(x_{2k+1}^1, x_{2k+2}^1) = d(S(x_{2k}^1, x_{2k}^2, x_{2k}^3, \cdots, x_{2k}^n), T(x_{2k+1}^1, x_{2k+1}^2, x_{2k+1}^3, \cdots, x_{2k+1}^n)).$$

Then by using contractive condition (1) of Theorem 1, we have

$$d(x^1_{2k+1}, x^1_{2k+2}) \leq \alpha_1 \frac{d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}{n}$$

$$+\alpha_2 \frac{d(x^1_{2k}, S(x^1_{2k}, x^2_{2k}, \cdots, x^n_{2k}))d(x^1_{2k+1}, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_3 \frac{d(x^1_{2k+1}, S(x^1_{2k}, x^2_{2k}, \cdots, x^n_{2k}))d(x^1_{2k}, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_4 \frac{d(S(x^1_{2k}, x^2_{2k}, \cdots, x^n_{2k}), T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))d(x^1_{2k}, x^1_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_5 \frac{d(S(x^1_{2k}, x^2_{2k}, \cdots, x^n_{2k}), T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))d(x^2_{2k}, x^2_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$\alpha_6 \frac{d(x^1_{2k+1}, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))d(x^2_{2k}, x^2_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_7 \frac{d(x^1_{2k+1}, S(x^1_{2k}, x^2_{2k}, \cdots, x^n_{2k}))d(x^1_{2k}, x^1_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_8 \frac{d(x^1_{2k+1}, S(x^1_{2k}, x^2_{2k}, \cdots, x^n_{2k}))d(x^2_{2k}, x^2_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_9 \frac{d(x^1_{2k+1}, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))d(x^n_{2k}, x^n_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_{10} \frac{d(x^1_{2k+1}, S(x^1_{2k}, x^2_{2k}, \cdots, x^n_{2k}))d(x^n_{2k}, x^n_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$= \alpha_1 \frac{d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}{n}$$

$$+\alpha_2 \frac{d(x^1_{2k}, x^1_{2k+1})d(x^1_{2k+1}, x^1_{2k+2})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_3 \frac{d(x^1_{2k+1}, x^1_{2k+1})d(x^1_{2k}, x^1_{2k+2})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_4 \frac{d(x^1_{2k+1}, x^1_{2k+2})d(x^1_{2k}, x^1_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_5 \frac{d(x^1_{2k+1}, x^1_{2k+2})d(x^2_{2k}, x^2_{2k+1})}{1 + d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})}$$

$$+\alpha_6\frac{d(x^1_{2k+1},x^1_{2k+2})d(x^2_{2k},x^2_{2k+1})}{1+d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})}$$

$$+\alpha_7\frac{d(x^1_{2k+1},x^1_{2k+1})d(x^1_{2k},x^1_{2k+1})}{1+d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})}$$

$$+\alpha_8\frac{d(x^1_{2k+1},x^1_{2k+1})d(x^2_{2k},x^2_{2k+1})}{1+d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})}$$

$$+\alpha_9\frac{d(x^1_{2k+1},x^1_{2k+2})d(x^n_{2k},x^n_{2k+1})}{1+d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})}$$

$$+\alpha_{10}\frac{d(x^1_{2k+1},x^1_{2k+1})d(x^n_{2k},x^n_{2k+1})}{1+d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})}$$

$$\leq \frac{\alpha_1}{n}[d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})]+\alpha_2 d(x^1_{2k+1},x^1_{2k+2})$$

$$+\alpha_4 d(x^1_{2k+1},x^1_{2k+2})+\alpha_5 d(x^1_{2k+1},x^1_{2k+2})+\alpha_6 d(x^1_{2k+1},x^1_{2k+2})+\alpha_9 d(x^1_{2k+1},x^1_{2k+2}).$$

Which implies that

$$(1-\alpha_2-\alpha_4-\alpha_5-\alpha_6-\alpha_9)d(x^1_{2k+1},x^1_{2k+2})\leq$$

$$\frac{\alpha_1}{n}[d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})]$$

$$d(x^1_{2k+1},x^1_{2k+2})\leq\frac{\alpha_1[d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})]}{n(1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9))}. \qquad (A_1)$$

Similarly

$$d(x^2_{2k+1},x^2_{2k+2})\leq\frac{\alpha_1[d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})]}{n(1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9))}. \qquad (A_2)$$

Proceeding n-times, one can write

$$d(x^n_{2k+1},x^n_{2k+2})\leq\frac{\alpha_1[d(x^1_{2k},x^1_{2k+1})+d(x^1_{2k},x^1_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})]}{n(1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9))}. \qquad (A_n)$$

Adding (A_1), (A_2), $\cdots$, and (A_n), we get

$$d(x^1_{2k+1},x^1_{2k+2})+d(x^2_{2k+1},x^2_{2k+2})+\cdots+d(x^n_{2k+1},x^n_{2k+2})\leq$$

$$\frac{\alpha_1[d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})]}{1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9)}$$

$$=h[d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})].$$

Where

$$h=\frac{\alpha_1}{1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9)}<1.$$

Also,

$$d(x^1_{2k+2}, x^1_{2k+3}) \leq \frac{\alpha_1[d(x^1_{2k+1}, x^1_{2k+2}) + d(x^2_{2k+1}, x^2_{2k+2}) + \cdots + d(x^n_{2k+1}, x^n_{2k+2})]}{n(1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9))}. \qquad (B_1)$$

$$d(x^2_{2k+2}, x^2_{2k+3}) \leq \frac{\alpha_1[d(x^1_{2k+1}, x^1_{2k+2}) + d(x^2_{2k+1}, x^2_{2k+2}) + \cdots + d(x^n_{2k+1}, x^n_{2k+2})]}{n(1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9))}. \qquad (B_2)$$

$\vdots$

$$d(x^n_{2k+2}, x^n_{2k+3}) \leq \frac{\alpha_1[d(x^1_{2k+1}, x^1_{2k+2}) + d(x^2_{2k+1}, x^2_{2k+2}) + \cdots + d(x^n_{2k+1}, x^n_{2k+2})]}{n(1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9))}. \qquad (B_n)$$

Adding equations, (B_1), (B_2), $\cdots$, and (B_n), we get

$$\begin{aligned}
&d(x^1_{2k+2}, x^1_{2k+3}) + d(x^2_{2k+2}, x^2_{2k+3}) + \cdots + d(x^n_{2k+2}, x^n_{2k+3}) \leq \\
&\qquad \frac{\alpha_1[d(x^1_{2k+1}, x^1_{2k+2}) + d(x^2_{2k+1}, x^2_{2k+2}) + \cdots + d(x^n_{2k+1}, x^n_{2k+2})]}{1-(\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9)} \\
&\qquad = h[d(x^1_{2k+1}, x^1_{2k+2}) + d(x^2_{2k+1}, x^2_{2k+2}) + \cdots + d(x^n_{2k+1}, x^n_{2k+2})] \\
&\qquad \leq h^2[d(x^1_{2k}, x^1_{2k+1}) + d(x^2_{2k}, x^2_{2k+1}) + \cdots + d(x^n_{2k}, x^n_{2k+1})].
\end{aligned}$$

Therefore one can write,

$$\begin{aligned}
&d(x^1_n, x^1_{n+1}) + d(x^2_n, x^2_{n+1}) + \cdots + d(x^n_n, x^n_{n+1}) \\
&\qquad \leq h[d(x^1_{n-1}, x^1_n) + d(x^2_{n-1}, x^2_n) + \cdots + d(x^n_{n-1}, x^n_n)] \\
&\qquad \leq h^2[d(x^1_{n-2}, x^1_{n-1}) + d(x^2_{n-2}, x^2_{n-1}) + \cdots + d(x^n_{n-2}, x^n_{n-1})] \\
&\qquad \leq \cdots \leq h^n[d(x^1_0, x^1_1) + d(x^2_0, x^2_1) + \cdots + d(x^n_0, x^n_1)].
\end{aligned}$$

If we set $d(x^1_n, x^1_{n+1}) + d(x^2_n, x^2_{n+1}) + \cdots + d(x^n_n, x^n_{n+1}) = \psi_n$.

Then $\psi_n \leq h\psi_{n-1} \leq h^2\psi_{n-2} \leq \cdots \leq h^n\psi_0$.

For m > n,

$$\begin{aligned}
&[d(x^1_n, x^1_m) + d(x^2_n, x^2_m) + \cdots + d(x^n_n, x^n_m)] \leq \\
&s[d(x^1_n, x^1_{n+1}) + d(x^2_n, x^2_{n+1}) + \cdots + d(x^n_n, x^n_{n+1})] \\
&+s^2[d(x^1_{n+1}, x^1_{n+2}) + d(x^2_{n+1}, x^2_{n+2}) + \cdots + d(x^n_{n+1}, x^n_{n+2})] \\
&+\cdots + s^{m-n}[d(x^1_{m-1}, x^1_m) + d(x^2_{m-1}, x^2_m) + \cdots + d(x^n_{m-1}, x^n_m)] \\
&\leq sh^n\psi_0 + s^2h^{n+1}\psi_0 + \cdots s^{m-n}h^{m-1}\psi_0 \\
&< sh^n[1 + sh + (sh)^2 + \cdots]\psi_0 \\
&= \tfrac{sh^n}{1-sh} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

This shows that $\{x^1_n\}$, $\{x^2_n\}$, $\cdots$, $\{x^n_n\}$ are Cauchy sequences in X. As X is complete

b-metric space, so there exists $x^1, x^2, x^3, \cdots, x^n \in X$ such that $x_n^1 \longrightarrow x^1$, $x_n^2 \longrightarrow x^2$,$\cdots$, $x_n^n \longrightarrow x^n$ as n$\longrightarrow\infty$.

Now we will prove that $x^1 = S(x^1, x^2, \cdots, x^n)$, $x^2 = S(x^2, x^3, x^4, \cdots, x^n, x^1)$,$\cdots$, $x^n = S(x^n, x^1, x^2 \cdots, x^{n-1})$.

Suppose on contrary that

$x^1 \neq S(x^1, x^2, \cdots, x^n), x^2 \neq S(x^2, x^3, x^4, \cdots, x^n, x^1), \cdots, x^n \neq S(x^n, x^1, x^2 \cdots, x^{n-1})$.

Then

$d(x^1, S(x^1, x^2, \cdots, x^n)) = l_1 > 0, d(x^2, S(x^2, x^3, x^4, \cdots, x^n, x^1)) = l_2 > 0, \cdots, d(x^n, S(x^n, x^1, x^2 \cdots, x^{n-1}))$ $l_3 > 0$.

Consider the following and using condition (1) of Theorem 1, we get

$$
\begin{aligned}
l_1 = d(x^1, S(x^1, x^2, \cdots, x^n)) &\leq s[d(x^1, x^1_{2k+2}) + d(x^1_{2k+2}, S(x^1, x^2, \cdots, x^n))] \\
&= sd(x^1, x^1_{2k+2}) + sd(S(x^1, x^2, \cdots, x^n), x^1_{2k+2}) \\
&= sd(x^1, x^1_{2k+2}) + sd(S(x^1, x^2, \cdots, x^n), T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1})) \\
&\leq sd(x^1, x^1_{2k+2}) + s\alpha_1 \frac{d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}{n} \\
&+ s\alpha_2 \frac{d(x^1, S(x^1, x^2, \cdots, x^n)) d(x^1_{2k+1}, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_3 \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n)) d(x^1, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1}))}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_4 \frac{d(S(x^1, x^2, \cdots, x^n), T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1})) d(x^1, x^1_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_5 \frac{d(S(x^1, x^2, \cdots, x^n), T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1})) d(x^2, x^2_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_6 \frac{d(x^1_{2k+1}, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1})) d(x^2, x^2_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_7 \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n)) d(x^1, x^1_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_8 \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n)) d(x^2, x^2_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_9 \frac{d(x^1_{2k+1}, T(x^1_{2k+1}, x^2_{2k+1}, \cdots, x^n_{2k+1})) d(x^n, x^n_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})} \\
&+ s\alpha_{10} \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n)) d(x^n, x^n_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}
\end{aligned}
$$

$$= sd(x^1, x^1_{2k+2}) + s\alpha_1 \frac{d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}{n}$$
$$+ s\alpha_2 \frac{d(x^1, S(x^1, x^2, \cdots, x^n))d(x^1_{2k+1}, x^1_{2k+2})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_3 \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n))d(x^1, x^1_{2k+2})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_4 \frac{d(S(x^1, x^2, \cdots, x^n), x^1_{2k+2})d(x^1, x^1_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_5 \frac{d(S(x^1, x^2, \cdots, x^n), x^1_{2k+2})d(x^2, x^2_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_6 \frac{d(x^1_{2k+1}, x^1_{2k+2})d(x^2, x^2_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_7 \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n))d(x^1, x^1_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_8 \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n))d(x^2, x^2_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_9 \frac{d(x^1_{2k+1}, x^1_{2k+2})d(x^n, x^n_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}$$
$$+ s\alpha_{10} \frac{d(x^1_{2k+1}, S(x^1, x^2, \cdots, x^n))d(x^n, x^n_{2k+1})}{1 + d(x^1, x^1_{2k+1}) + d(x^2, x^2_{2k+1}) + \cdots + d(x^n, x^n_{2k+1})}.$$

Using the concept that $\{x^1_n\}$, $\{x^2_n\}$,$\cdots$, $\{x^n_n\}$ are convergent sequences, so it's subsequences. Taking limit as $k \to \infty$ we get $l_1 \leq 0$.

Which implies that
$d(x^1, S(x^1, x^2, \cdots, x^n)) = 0$, so $x^1 = S(x^1, x^2, \cdots, x^n)$.
Similarly we can prove that
$x^2 = S(x^2, x^3, x^4, \cdots, x^n, x^1)$,$\cdots$, $x^n = S(x^n, x^1, x^2 \cdots, x^{n-1})$.
Analogously we have
$x^1 = T(x^1, x^2, \cdots, x^n)$, $x^2 = T(x^2, x^3, \cdots, x^n, x^1)$,$\cdots$, $x^n = T(x^n, x^1, x^2 \cdots, x^{n-1})$.
Thus we have proved that $(x^1, x^2, \cdots, x^n)$ is a common n-tupled fixed point of S and T.

Uniqueness

Let $(x^1_*, x^2_*, \cdots, x^n_*) \in X^n$ be second common n-tupled fixed point of S and T. From condition (1) of Theorem 1, we can write

$$d(x^1, x^1_*) = d(S(x^1, x^2, \cdots, x^n), T(x^1_*, x^2_*, \cdots, x^n_*))$$
$$\leq \alpha_1 \frac{d(x^1, x^1_*) + d(x^2, x^2_*) + \cdots + d(x^n, x^n_*)}{n}$$

$$
\begin{aligned}
&+\alpha_2\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1_*,T(x^1_*,x^2_*,\cdots,x^n_*))}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_3\frac{d(x^1_*,S(x^1,x^2,\cdots,x^n))d(x^1,T(x^1_*,x^2_*,\cdots,x^n_*))}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_4\frac{d(S(x^1,x^2,\cdots,x^n),T(x^1_*,x^2_*,\cdots,x^n_*))d(x^1,x^1_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_5\frac{d(S(x^1,x^2,\cdots,x^n),T(x^1_*,x^2_*,\cdots,x^n_*))d(x^2,x^2_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_6\frac{d(x^1_*,T(x^1_*,x^2_*,\cdots,x^n_*))d(x^2,x^2_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_7\frac{d(x^1_*,S(x^1,x^2,\cdots,x^n))d(x^1,x^1_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_8\frac{d(x^1_*,S(x^1,x^2,\cdots,x^n))d(x^2,x^2_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_9\frac{d(x^1_*,T(x^1_*,x^2_*,\cdots,x^n_*))d(x^n,x^n_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_{10}\frac{d(x^1_*,S(x^1,x^2,\cdots,x^n))d(x^n,x^n_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&=\alpha_1\frac{d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}{n}\\
&+\alpha_2\frac{d(x^1,x^1)d(x^1_*,x^1_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_3\frac{d(x^1_*,x^1)d(x^1,x^1_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_4\frac{d(x^1,x^1_*)d(x^1,x^1_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_5\frac{d(x^1,x^1_*)d(x^2,x^2_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_6\frac{d(x^1_*,x^1_*)d(x^2,x^2_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_7\frac{d(x^1_*,x^1)d(x^1,x^1_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}\\
&+\alpha_8\frac{d(x^1_*,x^1)d(x^2,x^2_*)}{1+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}
\end{aligned}
$$

$$+\alpha_9\frac{d(x_*^1,x_*^1)d(x^n,x_*^n)}{1+d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)}$$
$$+\alpha_{10}\frac{d(x_*^1,x^1)d(x^n,x_*^n)}{1+d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)}$$
$$\leq \alpha_1\frac{d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)}{n}+\alpha_3 d(x^1,x_*^1)$$
$$+\alpha_4 d(x^1,x_*^1)+\alpha_5 d(x^1,x_*^1)+\alpha_7 d(x^1,x_*^1)+\alpha_8 d(x^1,x_*^1)+\alpha_{10} d(x^1,x_*^1)$$

which implies that

$$(1-\frac{\alpha_1}{n}-\alpha_3-\alpha_4-\alpha_5-\alpha_7-\alpha_8-\alpha_{10})d(x^1,x_*^1)\leq$$
$$\alpha_1\frac{d(x^2,x_*^2)+d(x^3,x_*^3)+\cdots+d(x^n,x_*^n)}{n}$$
$$d(x^1,x_*^1)\leq\frac{\alpha_1[d(x^2,x_*^2)+d(x^3,x_*^3)+\cdots+d(x^n,x_*^n)]}{(n-\alpha_1-n\alpha_3-n\alpha_4-n\alpha_5-n\alpha_7-n\alpha_8-n\alpha_{10})}\tag{C_1}$$

similarly,

$$d(x^2,x_*^2)\leq\frac{\alpha_1[d(x^1,x_*^1)+d(x^3,x_*^3)+\cdots+d(x^n,x_*^n)]}{(n-\alpha_1-n\alpha_3-n\alpha_4-n\alpha_5-n\alpha_7-n\alpha_8-n\alpha_{10})}.\tag{C_2}$$

Proceeding n-times, one can write

$$d(x^n,x_*^n)\leq\frac{\alpha_1[d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^{n-1},x_*^{n-1})]}{(n-\alpha_1-n\alpha_3-n\alpha_4-n\alpha_5-n\alpha_7-n\alpha_8-n\alpha_{10})}.\tag{C_n}$$

Adding, C_1, C_2, $\cdots$, and C_n, we get

$$d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)\leq\frac{(n-1)\alpha_1[d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)]}{(n-\alpha_1-n\alpha_3-n\alpha_4-n\alpha_5-n\alpha_7-n\alpha_8-n\alpha_{10})}$$
$$\left[1-\frac{(n-1)\alpha_1[d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)]}{(n-\alpha_1-n\alpha_3-n\alpha_4-n\alpha_5-n\alpha_7-n\alpha_8-n\alpha_{10})}\right]\leq 0$$
$$\frac{n(1-\alpha_1-\alpha_3-\alpha_4-\alpha_5-\alpha_7-\alpha_8-\alpha_{10})}{(n-\alpha_1-n\alpha_3-n\alpha_4-n\alpha_5-n\alpha_7-n\alpha_8-n\alpha_{10})}[d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)]\leq 0$$

since $\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_7+\alpha_8+\alpha_{10}<1$.

Therefore

$$\frac{n(1-\alpha_1-\alpha_3-\alpha_4-\alpha_5-\alpha_7-\alpha_8-\alpha_{10})}{(n-\alpha_1-n\alpha_3-n\alpha_4-n\alpha_5-n\alpha_7-n\alpha_8-n\alpha_{10})}>0.$$

Hence

$$[d(x^1,x_*^1)+d(x^2,x_*^2)+\cdots+d(x^n,x_*^n)]\leq 0$$

which implies that $x^1=x_*^1$, $x^2=x_*^2,\cdots$, $x^n=x_*^n$.

So $(x^1,x^2,\cdots,x^n)=(x_*^1,x_*^2,\cdots,x_*^n)$.

Thus, S and T have unique common n-fixed point.

Theorem 1 yields the following corollary.

Corollary 1. *Let (X,d) be a complete b-metric space with parameter $s \geq 1$ and let the mapping $T : X^n \to X$ satisfy:*

$$d(T(x^1,x^2,\cdots,x^n),T(y^1,y^2,\cdots,y^n)) \leq \alpha_1 \frac{d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}{n}$$
$$+\alpha_2 \frac{d(x^1,T(x^1,x^2,\cdots,x^n))d(y^1,T(y^1,y^2,\cdots,y^n))}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_3 \frac{d(y^1,T(x^1,x^2,\cdots,x^n))d(x^1,T(y^1,y^2,\cdots,y^n))}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_4 \frac{d(T(x^1,x^2,\cdots,x^n),T(y^1,y^2,\cdots,y^n))d(x^1,y^1)}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_5 \frac{d(T(x^1,x^2,\cdots,x^n),T(y^1,y^2,\cdots,y^n))d(x^2,y^2)}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_6 \frac{d(y^1,T(y^1,y^2,\cdots,y^n))d(x^2,y^2)}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_7 \frac{d(y^1,T(x^1,x^2,\cdots,x^n))d(x^1,y^1)}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_8 \frac{d(y^1,T(x^1,x^2,\cdots,x^n))d(x^2,y^2)}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_9 \frac{d(y^1,T(y^1,y^2,\cdots,y^n))d(x^n,y^n)}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$
$$+\alpha_{10} \frac{d(y^1,T(x^1,x^2,\cdots,x^n))d(x^n,y^n)}{1+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}$$

for all x^1, x^2, x^3,$\cdots$, x^n and y^1, y^2, y^3, $\cdots$, $y^n \in X$ and $\alpha_i \geq 0$, $i = 1,2,\cdots,10$ with the conditions $s\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_9 < 1$ and $\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_7+\alpha_8+\alpha_{10} < 1$. Then T has unique common n-tupled fixed point in X.

Proof. Proof is very easy if we take $S = T$ in Theorem 1.

Theorem 2. *Let (X,d) be a complete b metric space with parameter $s \geq 1$ and let the mappings $S,T : X^n \longrightarrow X$ satisfy:*

$$d(S(x^1,x^2,\cdots,x^n),T(y^1,y^2,\cdots,y^n)) \leq \alpha_1 \frac{d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}{n}$$
$$+\beta \frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(y^1,T(y^1,y^2,\cdots,y^n))}{1+s[d(x^1,T(y^1,\cdots,y^n))+d(y^1,S(x^1,\cdots,x^n))+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)]}$$

$$+\gamma\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1,T(y^1,y^2,\cdots,y^n))}{1+s[d(x^1,T(y^1,\cdots,y^n))+d(y^1,S(x^1,\cdots,x^n))+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)]}. \tag{2}$$

For all x^1, x^2, x^3,$\cdots$, x^n and y^1, y^2, y^3, $\cdots$, $y^n \in X$ and α, β, γ are non-negative real numbers with $s(\alpha+\beta+\gamma)<1$. Then S and T have unique common n-tupled fixed point.

Proof. Taking n arbitrary points x_0^1, x_0^2, x_0^3,$\cdots$, x_0^n, in X, define
$x_{2k+1}^1=S(x_{2k}^1,x_{2k}^2,x_{2k}^3,\cdots,x_{2k}^n)$,
$x_{2k+1}^2=S(x_{2k}^2,x_{2k}^1,x_{2k}^3,\cdots,x_{2k}^n)$,
$x_{2k+1}^3=S(x_{2k}^3,x_{2k}^2,x_{2k}^1,\cdots,x_{2k}^n)$,
$\vdots$
$x_{2k+1}^n=S(x_{2k}^n,x_{2k}^{n-1},x_{2k}^{n-2},\cdots,x_{2k}^2,x_{2k}^1)$,
and
$x_{2k+2}^1=T(x_{2k+1}^1,x_{2k+1}^2,x_{2k+1}^3,\cdots,x_{2k+1}^n)$,
$x_{2k+2}^2=T(x_{2k+1}^2,x_{2k+1}^1,x_{2k+1}^3,\cdots,x_{2k+1}^n)$,
$x_{2k+2}^3=T(x_{2k+1}^3,x_{2k+1}^2,x_{2k+1}^1,\cdots,x_{2k+1}^n)$,
$\vdots$
$x_{2k+2}^n=T(x_{2k+1}^n,x_{2k+1}^{n-1},x_{2k+1}^{n-2},\cdots,x_{2k+1}^2,x_{2k+1}^1)$ for k=0,1,2,$\cdots$.

For the sake of simplicity, we take
$\lambda=d(x_{2k}^1,x_{2k+1}^1)+d(x_{2k}^2,x_{2k+1}^2)+\cdots+d(x_{2k}^n,x_{2k+1}^n)$.
Consider

$$d(x_{2k+1}^1,x_{2k+2}^1)=d(S(x_{2k}^1,x_{2k}^2,\cdots,x_{2k}^n),T(x_{2k+1}^1,x_{2k+1}^2,\cdots,x_{2k+1}^n)).$$

Then by using condition (2) of Theorem 2, we have

$$d(x_{2k+1}^1,x_{2k+2}^1)\le\alpha_1\frac{d(x_{2k}^1,x_{2k+1}^1)+d(x_{2k}^2,x_{2k+1}^2)+\cdots+d(x_{2k}^n,x_{2k+1}^n)}{n}$$
$$+\beta\frac{d(x_{2k}^1,S(x_{2k}^1,x_{2k}^2,\cdots,x_{2k}^n))d(x_{2k+1}^1,T(x_{2k+1}^1,x_{2k+1}^2,\cdots,x_{2k+1}^n))}{1+s[d(x_{2k}^1,T(x_{2k+1}^1,\cdots,x_{2k+1}^n))+d(x_{2k+1}^1,S(x_{2k}^1,\cdots,x_{2k}^n)+\lambda]}$$
$$+\gamma\frac{d(x_{2k}^1,S(x_{2k}^1,x_{2k}^2,\cdots,x_{2k}^n))d(x_{2k}^1,T(x_{2k+1}^1,x_{2k+1}^2,\cdots,x_{2k+1}^n))}{1+s[d(x_{2k}^1,T(x_{2k+1}^1,\cdots,x_{2k+1}^n))+d(x_{2k+1}^1,S(x_{2k}^1,\cdots,x_{2k}^n))+\lambda]}$$
$$=\alpha_1\frac{d(x_{2k}^1,x_{2k+1}^1)+d(x_{2k}^2,x_{2k+1}^2)+\cdots+d(x_{2k}^n,x_{2k+1}^n)}{n}$$
$$+\beta\frac{d(x_{2k}^1,x_{2k+1}^1)d(x_{2k+1}^1,x_{2k+2}^1)}{1+s[d(x_{2k}^1,x_{2k+2}^1)+d(x_{2k+1}^1,x_{2k+1}^1)+d(x_{2k}^1,x_{2k+1}^1)+d(x_{2k}^2,x_{2k+1}^2)+\cdots+d(x_{2k}^n,x_{2k+1}^n)]}$$
$$+\gamma\frac{d(x_{2k}^1,x_{2k+1}^1)d(x_{2k}^1,x_{2k+2}^1)}{1+s[d(x_{2k}^1,x_{2k+2}^1)+d(x_{2k+1}^1,x_{2k+1}^1)+d(x_{2k}^1,x_{2k+1}^1)+d(x_{2k}^2,x_{2k+1}^2)+\cdots+d(x_{2k}^n,x_{2k+1}^n)]}$$

$$
\begin{aligned}
&= \alpha_1 \frac{d(x_{2k}^1, x_{2k+1}^1) + d(x_{2k}^2, x_{2k+1}^2) + \cdots + d(x_{2k}^n, x_{2k+1}^n)}{n} \\
&+\beta \frac{d(x_{2k}^1, x_{2k+1}^1) d(x_{2k+1}^1, x_{2k+2}^1)}{1 + s[d(x_{2k+1}^1, x_{2k+2}^1) + d(x_{2k}^2, x_{2k+1}^2) + \cdots + d(x_{2k}^n, x_{2k+1}^n)]} \\
&+\gamma \frac{d(x_{2k}^1, x_{2k+1}^1) d(x_{2k}^1, x_{2k+2}^1)}{1 + s[d(x_{2k}^1, x_{2k+2}^1) + d(x_{2k}^1, x_{2k+1}^1) + d(x_{2k}^2, x_{2k+1}^2) + \cdots + d(x_{2k}^n, x_{2k+1}^n)]} \\
&\leq \alpha \frac{d(x_{2k}^1, x_{2k+1}^1) + d(x_{2k}^2, x_{2k+1}^2) + \cdots + d(x_{2k}^n, x_{2k+1}^n)}{n} + \beta d(x_{2k}^1, x_{2k+1}^1) + \gamma d(x_{2k}^1, x_{2k+1}^1)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
d(x_{2k+1}^1, x_{2k+2}^1) &\leq \frac{\alpha + n\beta + n\gamma}{n} d(x_{2k}^1, x_{2k+1}^1) \\
&+ \frac{\alpha}{n}[d(x_{2k}^2, x_{2k+1}^2) + d(x_{2k}^3, x_{2k+1}^3) + \cdots + d(x_{2k}^n, x_{2k+1}^n)]. \qquad (D_1)
\end{aligned}
$$

Similarly, we can prove

$$
\begin{aligned}
d(x_{2k+1}^2, x_{2k+2}^2) &\leq \frac{\alpha + n\beta + n\gamma}{n} d(x_{2k}^2, x_{2k+1}^2) \\
&+ \frac{\alpha}{n}[d(x_{2k}^1, x_{2k+1}^1) + d(x_{2k}^3, x_{2k+1}^3) + \cdots + d(x_{2k}^n, x_{2k+1}^n)]. \qquad (D_2)
\end{aligned}
$$

Proceeding similarly, one can write

$$
\begin{aligned}
d(x_{2k+1}^n, x_{2k+2}^n) &\leq \frac{\alpha + n\beta + n\gamma}{n} d(x_{2k}^n, x_{2k+1}^n) \\
&+ \frac{\alpha}{n}[d(x_{2k}^1, x_{2k+1}^1) + d(x_{2k}^2, x_{2k+1}^2) + \cdots + d(x_{2k}^{n-1}, x_{2k+1}^{n-1})]. \qquad (D_n)
\end{aligned}
$$

Adding equations (D_1), (D_2), $\cdots$, and (D_n), we get

$$
[d(x_{2k+1}^1, x_{2k+2}^1) + d(x_{2k+1}^2, x_{2k+2}^2) + \cdots + d(x_{2k+1}^n, x_{2k+2}^n)] \leq
$$

$$
(\alpha + \beta + \gamma)[d(x_{2k}^1, x_{2k+1}^1) + d(x_{2k}^2, x_{2k+1}^2) + \cdots + d(x_{2k}^n, x_{2k+1}^n)].
$$

Also

$$
\begin{aligned}
d(x_{2k+2}^1, x_{2k+3}^1) &\leq \frac{\alpha + n\beta + n\gamma}{n} d(x_{2k+1}^1, x_{2k+2}^1) \\
&+ \frac{\alpha}{n}[d(x_{2k+1}^2, x_{2k+2}^2) + d(x_{2k+1}^3, x_{2k+2}^3) + \cdots + d(x_{2k+1}^n, x_{2k+2}^n)]. \qquad (E_1) \\
d(x_{2k+2}^2, x_{2k+3}^2) &\leq \frac{\alpha + n\beta + n\gamma}{n} d(x_{2k+1}^2, x_{2k+2}^2) \\
&+ \frac{\alpha}{n}[d(x_{2k+1}^1, x_{2k+2}^1) + d(x_{2k+1}^3, x_{2k+2}^3) + \cdots + d(x_{2k+1}^n, x_{2k+2}^n)]. \qquad (E_2)
\end{aligned}
$$

$\vdots$

$$
d(x_{2k+2}^n, x_{2k+3}^n) \leq \frac{\alpha + n\beta + n\gamma}{n} d(x_{2k+1}^n, x_{2k+2}^n)
$$

$$+\frac{\alpha}{n}[d(x^1_{2k+1},x^1_{2k+2})+d(x^2_{2k+1},x^2_{2k+2})+\cdots+d(x^{n-1}_{2k+1},x^{n-1}_{2k+2})]. \qquad (E_n)$$

Adding, (E_1), (E_2) $\cdots$, and (E_n), we get

$$\begin{aligned}
&[d(x^1_{2k+2},x^1_{2k+3})+d(x^2_{2k+2},x^2_{2k+3})+\cdots+d(x^n_{2k+2},x^n_{2k+3})]\\
&\quad\leq(\alpha+\beta+\gamma)[d(x^1_{2k+1},x^1_{2k+2})+d(x^2_{2k+1},x^2_{2k+2})+\cdots+d(x^n_{2k+1},x^n_{2k+2})]\\
&\quad\leq(\alpha+\beta+\gamma)^2[d(x^1_{2k},x^1_{2k+1})+d(x^2_{2k},x^2_{2k+1})+\cdots+d(x^n_{2k},x^n_{2k+1})].
\end{aligned}$$

Therefore we have the following

$$\begin{aligned}
&d(x^1_n,x^1_{n+1})+d(x^2_n,x^2_{n+1})+\cdots+d(x^n_n,x^n_{n+1})\\
&\quad\leq(\alpha+\beta+\gamma)[d(x^1_{n-1},x^1_n)+d(x^2_{n-1},x^2_n)+\cdots+d(x^n_{n-1},x^n_n)]\\
&\quad\leq(\alpha+\beta+\gamma)^2[d(x^1_{n-2},x^1_{n-1})+d(x^2_{n-2},x^2_{n-1})+\cdots+d(x^n_{n-2},x^n_{n-1})]\\
&\quad\leq\cdots\leq(\alpha+\beta+\gamma)^n[d(x^1_0,x^1_1)+d(x^2_0,x^2_1)+\cdots+d(x^n_0,x^n_1)]
\end{aligned}$$

where $h=\alpha+\beta+\gamma<1$.
Now if we set $d(x^1_n,x^1_{n+1})+d(x^2_n,x^2_{n+1})+\cdots+d(x^n_n,x^n_{n+1})=\psi_n$.
Then $\psi_n\leq h\psi_{n-1}\leq h^2\psi_{n-2}\leq\cdots\leq h^n\psi_0$.
So for $m>n$, we have

$$\begin{aligned}
&[d(x^1_n,x^1_m)+d(x^2_n,x^2_m)+\cdots+d(x^n_n,x^n_m)]\leq\\
&s[d(x^1_n,x^1_{n+1})+d(x^2_n,x^2_{n+1})+\cdots+d(x^n_n,x^n_{n+1})]\\
&+s^2[d(x^1_{n+1},x^1_{n+2})+d(x^2_{n+1},x^2_{n+2})+\cdots+d(x^n_{n+1},x^n_{n+2})]\\
&+\cdots+s^{m-n}[d(x^1_{m-1},x^1_m)+d(x^2_{m-1},x^2_m)+\cdots+d(x^n_{m-1},x^n_m)]\\
&\leq sh^n\psi_0+s^2h^{n+1}\psi_0+\cdots s^{m-n}h^{m-1}\psi_0\\
&<sh^n[1+sh+(sh)^2+\cdots]\psi_0\\
&=\tfrac{sh^n}{1-sh}\longrightarrow 0 \text{ as } n\longrightarrow\infty.
\end{aligned}$$

This shows that $\{x^1_n\}$, $\{x^1_n\}$, $\cdots$, $\{x^1_n\}$ are Cauchy sequences in X. As X is complete b-metric space, so there exists x^1, x^2, x^3,$\cdots$, $x^n\in X$ such that $x^1_n\longrightarrow x^1$, $x^2_n\longrightarrow x^2$,$\cdots$, $x^n_n\longrightarrow x^n$ as n$\longrightarrow\infty$.

Now we will prove that $x^1=S(x^1,x^2,\cdots,x^n)$, $x^2=S(x^2,x^3,x^4,\cdots,x^n,x^1)$,$\cdots$, $x^n=S(x^n,x^1,x^2\cdots,x^{n-1})$. On contrary suppose that $x^1\neq S(x^1,x^2,\cdots,x^n)$, $x^2\neq S(x^2,x^3,x^4,\cdots,x^n,x^1)$, $\cdots$, $x^n\neq S(x^n,x^1,x^2\cdots,x^{n-1})$.
Then

$d(x^1,S(x^1,x^2,\cdots,x^n))=l_1>0, d(x^2,S(x^2,x^3,x^4,\cdots,x^n,x^1))=l_2>0,\cdots, d(x^n,S(x^n,x^1,x^2\cdots,x^{n-1}))$
$l_3>0$.
For the sake of simplicity, we take again

$\lambda=d(x^1,x^1_{2k+1})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})$.
Now consider the following and using condition (2) of Theorem 2, we get

$$l_1=d(x^1,S(x^1,x^2,\cdots,x^n))\leq$$

$$
\begin{aligned}
& s[d(x^1,x^1_{2k+2})+d(x^1_{2k+2},S(x^1,x^2,\cdots,x^n))] \\
&= sd(x^1,x^1_{2k+2})+sd(S(x^1,x^2,\cdots,x^n),x^1_{2k+2})) \\
&= sd(x^1,x^1_{2k+2})+sd(S(x^1,x^2,\cdots,x^n),T(x^1_{2k+1},x^2_{2k+1},\cdots,x^n_{2k+1})) \\
&\le sd(x^1,x^1_{2k+2})+s\alpha\frac{d(x^1,x^1_{2k+1})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})}{n} \\
&+s\beta\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1_{2k+1},T(x^1_{2k+1},x^2_{2k+1},\cdots,x^n_{2k+1}))}{1+s[d(x^1,T(x^1_{2k+1},x^2_{2k+1},\cdots,x^n_{2k+1}))+d(x^1_{2k+1},S(x^1,x^2,\cdots,x^n))+\lambda]} \\
&+s\gamma\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1,T(x^1_{2k+1},x^2_{2k+1},\cdots,x^n_{2k+1}))}{1+s[d(x^1,T(x^1_{2k+1},x^2_{2k+1},\cdots,x^n_{2k+1}))+d(x^1_{2k+1},S(x^1,x^2,\cdots,x^n))+\lambda]} \\
&= s\alpha\frac{d(x^1,x^1_{2k+1})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})}{n} \\
&+s\beta\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1_{2k+1},x^1_{2k+2})}{1+s[d(x^1,x^1_{2k+2})+d(x^1_{2k+1},S(x^1,\cdots,x^n))+d(x^1,x^1_{2k+1})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})]} \\
&+s\gamma\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1,x^1_{2k+2})}{1+s[d(x^1,x^1_{2k+2})+d(x^1_{2k+1},S(x^1,\cdots,x^n))+d(x^1,x^1_{2k+1})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})]} \\
&= s\alpha\frac{d(x^1,x^1_{2k+1})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})}{n} \\
&+s\beta\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1_{2k+1},x^1_{2k+2})}{1+s[d(x^1_{2k+1},S(x^1,x^2,\cdots,x^n))+d(x^1_{2k+1},x^1_{2k+2})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})]} \\
&+s\gamma\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1,x^1_{2k+2})}{1+s[d(x^1_{2k+1},S(x^1,x^2,\cdots,x^n))+d(x^1_{2k+1},x^1_{2k+2})+d(x^2,x^2_{2k+1})+\cdots+d(x^n,x^n_{2k+1})]}.
\end{aligned}
$$

Taking limit $k\to\infty$ we get $l_1\le 0$

so $d(x^1,S(x^1,x^2,\cdots,x^n))=0\Rightarrow x^1=S(x^1,x^2,\cdots,x^n)$.

Similarly we can prove that

$x^2=S(x^2,x^3,x^4,\cdots,x^n,x^1)$,$\cdots$, $x^n=S(x^n,x^1,x^2\cdots,x^{n-1})$.

Also we can prove that

$x^1=T(x^1,x^2,\cdots,x^n)$, $x^2=T(x^2,x^3,\cdots,x^n,x^1)$,$\cdots$, $x^n=T(x^n,x^1,x^2\cdots,x^{n-1})$.

Thus we have proved that $(x^1,x^2,\cdots,x^n)$ is a common n-fixed point of S and T.

Uniqueness:

Let $(x^1_*,x^2_*,\cdots,x^n_*)\in X^n$ be another common n-fixed point of S and T. Using condition (2) of Theorem 2 here, we get

$$
\begin{aligned}
d(x^1,x^1_*) &= d(S(x^1,x^2,\cdots,x^n),T(x^1_*,x^2_*,\cdots,x^n_*)) \\
&\le \alpha\frac{d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}{n}
\end{aligned}
$$

$$+\beta\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1_*,T(x^1_*,x^2_*,\cdots,x^n_*))}{1+s[d(x^1,T(x^1_*,\cdots,x^n_*))+d(x^1_*,S(x^1,\cdots,x^n))+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]}$$

$$+\gamma\frac{d(x^1,S(x^1,x^2,\cdots,x^n))d(x^1,T(x^1_*,x^2_*,\cdots,x^n_*))}{1+s[d(x^1,T(x^1_*,\cdots,x^n_*))+d(x^1_*,S(x^1,\cdots,x^n))+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]}$$

$$=\alpha\frac{d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}{n}$$

$$+\beta\frac{d(x^1,x^1)d(x^1_*,x^1_*)}{1+s[d(x^1,x^1_*)+d(x^1_*,x^1)+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]}$$

$$+\gamma\frac{d(x^1,x^1)d(x^1,x^1_*)}{1+s[d(x^1,x^1_*)+d(x^1_*,x^1)+d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]}$$

$$=\alpha\frac{d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}{n}$$

$$+\beta\frac{d(x^1,x^1)d(x^1_*,x^1_*)}{1+s[3d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]}$$

$$+\gamma\frac{d(x^1,x^1)d(x^1,x^1_*)}{1+s[3d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]}$$

$$d(x^1,x^1_*)\leq\alpha\frac{d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)}{n}.$$

Which implies that

$$d(x^1,x^1_*)[1-\frac{\alpha}{n}]\leq\alpha\frac{d(x^2,x^2_*)+d(x^3,x^3_*)+\cdots+d(x^n,x^n_*)}{n}$$

$$d(x^1,x^1_*)\leq\frac{\alpha}{n-\alpha}[d(x^2,x^2_*)+d(x^3,x^3_*)+\cdots+d(x^n,x^n_*)]. \qquad (F_1)$$

Similarly, we can prove that

$$d(x^2,x^2_*)\leq\frac{\alpha}{n-\alpha}[d(x^1,x^1_*)+d(x^3,x^3_*)+\cdots+d(x^n,x^n_*)] \qquad (F_2)$$

$$\vdots$$

$$d(x^n,x^n_*)\leq\frac{\alpha}{n-\alpha}[d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^{n-1},x^{n-1}_*)]. \qquad (F_n)$$

Adding, (F_1), (F_2), $\cdots$, and (F_n), we get

$$d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)\leq\frac{(n-1)\alpha}{n-\alpha}[d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]$$

which implies that

$$(1-\frac{(n-1)\alpha}{n-\alpha})[d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]\leq 0$$

so

$$(1-\alpha)[d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)]\leq 0.$$

Since $0<\alpha<1$.

Therefore

$d(x^1,x^1_*)+d(x^2,x^2_*)+\cdots+d(x^n,x^n_*)=0$.

Thus

$(x^1,x^2,\cdots,x^n)=(x^1_*,x^2_*,\cdots,x^n_*)$.

Hence S and T have unique common n-tupled fixed point.

Corollary 2. *Let (X,d) be a complete b metric space with parameter $s\geq 1$ and let the mapping $T:X^n\longrightarrow X$ satisfy:*

$$d(S(x^1,x^2,\cdots,x^n),T(y^1,y^2,\cdots,y^n))\leq \alpha_1\frac{d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)}{n}$$

$$+\beta\frac{d(x^1,T(x^1,x^2,\cdots,x^n))d(y^1,T(y^1,y^2,\cdots,y^n))}{1+s[d(x^1,T(y^1,\cdots,y^n))+d(y^1,T(x^1,\cdots,x^n))+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)]}$$

$$+\gamma\frac{d(x^1,T(x^1,x^2,\cdots,x^n))d(x^1,T(y^1,y^2,\cdots,y^n))}{1+s[d(x^1,T(y^1,\cdots,y^n))+d(y^1,T(x^1,\cdots,x^n))+d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)]}.$$

For all x^1, x^2, x^3,$\cdots$, x^n and y^1, y^2, y^3,$\cdots$, $y^n\in X$ and α, β, γ are non-negative real numbers with $s(\alpha+\beta+\gamma)<1$. Then T has unique common n-tupled fixed point.

Remarks:

- If we put $n=2$ and $\alpha_9=0$, $\alpha_{10}=0$ in Theorem 1, then we get coupled fixed point result of Sarwar *et al.* [15].
- If we put $n=2$ and $\alpha_i=0$, $i=4,5,\cdots,10$ in Theorem 1, then we get the result of Malhotra and Bansal [16].

Example 3. *Suppose S=$[0,1]$ and a b-metric $d:X\times X\to R$ defined by $d(x,y)=\frac{2}{3}(x-y)^2$ for each x, $y\in X$.*

Then (X,d) is b metric space having parameter $s=2$. If we define $S,T:X^n\to X$ by $S(x^1,x^2,x^3,\cdots,x^n)=\frac{x^1+x^2+x^3+\cdots+x^n}{n}$, $T(x^1,x^2,x^3,\cdots,x^n)=\frac{x^1+x^2+x^3+\cdots+x^n}{n+1}$ for each x^1, x^2, x^3,$\cdots$, $x^n\in X^n$. Then it can be proved simply that the maps S and T satisfy the contraction in Theorem 1 with $\alpha_1=\frac{1}{25},\alpha_2=\frac{2}{25},\alpha_3=\frac{3}{25},\alpha_4=\frac{4}{25},\alpha_5=\frac{1}{50},\alpha_6=\frac{3}{50},\alpha_7=\frac{7}{50},\alpha_8=\frac{9}{50},\alpha_9=\frac{1}{75},\alpha_{10}=\frac{2}{75}$.

Clearly $(0,0,0,\cdots,0)$ is a unique common n-tupled fixed point of S and T.

3. Conclusion

The derived results generalized the results of [16] and [15] in the setting of b metric spaces.

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