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Modifications of the Multistep Optimal Homotopy Asymptotic Method to some Nonlinear KdV-Equations

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Abstract. In this article, we have introduced the mathematical theory of Multistep Optimal Homotopy Asymptotic Method (MOHAM). The proposed method is implemented to different models having system of partial differential equations (PDEs). The results obtained by the proposed method are compared with Homotopy Analysis Method (HAM) and closed form solutions. The comparisons of these results show that (MOHAM) is simpler in applicability, effective, explicit, control the convergence through optimal constants, involve less computational work. The (MOHAM) is independent of the assumption of initial conditions and small parameters like Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), Variational Iteration Method (VIM), Adomian Decomposition Method (ADM) and Perturbation Method (PM).

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1. Introduction

The nonlinear physical phenomena can be understood by the effort of finding the exact solution. The exact solutions of some boundary value problems (BVPs) may be found by using transformation based techniques like invariance group analysis method [1], Lie infinitesimal criterion [2], the symbolic computation [3] and Backlund transformation [4]. These methods reduced the complex equations into simple equations by using the transformation. The exact solutions of all the nonlinear problems are either not available or difficult to find because (PDEs) have infinitely many solutions. So the attentions of the researchers are attracted to develop the approximation tools for the nonlinear (BVPs) like Variational Iteration Method (VIM) [5], Adomian Decomposition Method (ADM) [6], Differential Transform Method (DTM) [7], Homotopy Perturbation Method (HPM) [8] and Perturbation Method (PM) [9–11] have been used for the solution of nonlinear (PDEs). These methods contain a small parameter which cannot be found easily. The homotopy was combined with perturbation techniques such as Homotopy Analysis Method (HAM) [12] and Homotopy Perturbation Method (HPM) [8]. For these methods an initial solution is needed to assume.

Marinca et al. introduced (OHAM) [14–16] for the solution of nonlinear problems which made the perturbation methods independent of the assumption of small parameters and huge computational work.

The motivation of this paper is to formulate and implement the (MOHAM) for the solution of system of (PDEs). In [19–23], (OHAM) has been proved to be valuable for obtaining approximate solutions of various boundary value problems. Here we have proved that (MOHAM) is useful

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and reliable for the complex nonlinear (PDEs), showing its validity and great potential for the solution of transient physical phenomenon in science and engineering.
In the succeeding section, the basic idea of (MOHAM) is formulated. The effectiveness and efficiency of (MOHAM) is shown in Section 3.

2. Fundamental Mathematical Theory of (MOHAM) for a system of coupled PDEs

Consider a system of n partial differential equations of the form

$$\begin{cases} A_1\Big(f_1(\zeta,t), f_2(\zeta,t), ..., f_n(\zeta,t)\Big) + s_1(\zeta,t) = 0, \\ A_2\Big(f_1(\zeta,t), f_2(\zeta,t), ..., f_n(\zeta,t)\Big) + s_2(\zeta,t) = 0, \\ . \\ . \\ . \\ A_n\Big(f_1(\zeta,t), f_2(\zeta,t), ..., f_n(\zeta,t)\Big) + s_n(\zeta,t) = 0, \ \zeta \in \Omega \end{cases} \tag{1}$$

with boundary conditions

$$\begin{cases} B_1\Big(f_1(\zeta,t), \dfrac{df_1(\zeta,t)}{d\zeta}\Big) = 0, \\ B_2\Big(f_2(\zeta,t), \dfrac{df_2(\zeta,t)}{d\zeta}\Big) = 0, \\ . \\ . \\ . \\ B_n\Big(f_n(\zeta,t), \dfrac{df_n(\zeta,t)}{d\zeta}\Big) = 0, \ \zeta \in \Gamma \end{cases} \tag{2}$$

where $A_1, A_2, ..., A_n$ are differential operators, $f_1(\zeta,t), f_2(\zeta,t), ..., f_n(\zeta,t)$ are unknown functions, ζ and t denote spatial variables, respectively, Γ is the boundaries of $\Omega$and $s_1(\zeta,t), s_2(\zeta,t), ..., s_n(\zeta,t)$ are known analytic functions.
$A_1, A_2, ..., A_n$ can be divided into two parts:

$$\begin{cases} A_1 = L_1 + N_1, \\ A_2 = L_2 + N_2, \\ . \\ . \\ . \\ A_n = L_n + N_n \end{cases} \tag{3}$$

$L_1, L_2, ..., L_n$ contain the linear parts while $N_1, N_2, ..., N_n$ contains the nonlinear parts of the system of partial differential equations.

According to (OHAM), one can construct a family of equations

$$\begin{cases} (1-r)\Big[L_1\Big(\varphi(\zeta,t,r)\Big)+s_1(\zeta,t)\Big]=H_1(r)\Big[L_1\Big(\varphi(\zeta,t,r)\Big)+N_1\Big(\varphi(\zeta,t,r)\Big)+s_1(\zeta,t)\Big]=0,\\ (1-r)\Big[L_2\Big(\varphi(\zeta,t,r)\Big)+s_2(\zeta,t)\Big]=H_2(r)\Big[L_2\Big(\varphi(\zeta,t,r)\Big)+N_2\Big(\varphi(\zeta,t,r)\Big)+s_2(\zeta,t)\Big]=0,\\ .\\ .\\ .\\ (1-r)\Big[L_n\Big(\varphi(\zeta,t,r)\Big)+s_n(\zeta,t)\Big]=H_n(r)\Big[L_n\Big(\varphi(\zeta,t,r)\Big)+N_n\Big(\varphi(\zeta,t,r)\Big)+s_n(\zeta,t)\Big]=0 \end{cases} \tag{4}$$

where the auxiliary functions $H_1((\zeta,t,r)), H_2((\zeta,t,r)), ..., H_n((\zeta,t,r))$ are nonzero for $r \neq 0$. (4) is called optimal homotopy equations. Clearly, we have

$$\begin{cases} r=0 \Rightarrow L_1\Big(\varphi(\zeta,t,0)\Big)+s_1(\zeta,t)=0,\\ r=0 \Rightarrow L_2\Big(\varphi(\zeta,t,0)\Big)+s_2(\zeta,t)=0,\\ .\\ .\\ .\\ r=0 \Rightarrow L_n\Big(\varphi(\zeta,t,0)\Big)+s_n(\zeta,t)=0 \end{cases} \tag{5}$$

$$\begin{cases} r=1 \Rightarrow L_1\Big(\varphi(\zeta,t,1)\Big)+s_1(\zeta,t)=0,\\ r=1 \Rightarrow L_2\Big(\varphi(\zeta,t,1)\Big)+s_2(\zeta,t)=0,\\ .\\ .\\ .\\ r=1 \Rightarrow L_n\Big(\varphi(\zeta,t,1)\Big)+s_n(\zeta,t)=0 \end{cases} \tag{6}$$

Obviously, when $r=0$ and $r=1$, we obtain

$$\begin{cases} \varphi_1(\zeta,t,0)=(f_1)_0(\zeta,t),\\ \varphi_2(\zeta,t,0)=(f_2)_0(\zeta,t),\\ .\\ .\\ .\\ \varphi_n(\zeta,t,0)=(f_n)_0(\zeta,t) \end{cases} \tag{7}$$

$$\begin{cases} \varphi_1(\zeta,t,1)=f_1(\zeta,t),\\ \varphi_2(\zeta,t,1)=f_2(\zeta,t),\\ .\\ .\\ .\\ \varphi_n(\zeta,t,1)=f_n(\zeta,t) \end{cases} \tag{8}$$

respectively.

When r varies from 0 to 1, the solution $\varphi_1(\zeta,t,r), \varphi_2(\zeta,t,r), ..., \varphi_n(\zeta,t,r)$ approaches from

$(f_1)_0(\zeta,t),(f_2)_0(\zeta,t),...,(f_n)_0(\zeta,t)$ to $f_1(\zeta,t),f_2(\zeta,t),...,f_1(\zeta,t)$

$$\begin{cases} L_1\Big(((f_1)_0)(\zeta,t)\Big)+s_1(\zeta,t)=0, \quad B_1\Big((f_1)_0(\zeta,t),\dfrac{d(f_1)_0(\zeta,t)}{d\zeta}\Big)=0, \\ L_2\Big(((f_2)_0)(\zeta,t)\Big)+s_2(\zeta,t)=0, \quad B_2\Big((f_2)_0(\zeta,t),\dfrac{d(f_2)_0(\zeta,t)}{d\zeta}\Big)=0, \\ . \\ . \\ . \\ L_n\Big(((f_n)_0)(\zeta,t)\Big)+s_n(\zeta,t)=0, \quad B_n\Big((f_n)_0(\zeta,t),\dfrac{d(f_n)_0(\zeta,t)}{d\zeta}\Big)=0 \end{cases} \tag{9}$$

We choose auxiliary functions $H_1(r),H_2(r),...,H_n(r)$ in the form

$$\begin{cases} H_1(r)=rC_{11}+r^2C_{12}+...+r^nC_{1n}, \\ H_2(r)=rC_{21}+r^2C_{22}+...+r^nC_{2n}, \\ . \\ . \\ . \\ H_n(r)=rC_{n1}+r^2C_{n2}+...+r^nC_{nn} \end{cases} \tag{10}$$

To get the approximate solutions, we expand $\varphi_1(\zeta,t,r,C_{1i}),\varphi_2(\zeta,t,r,C_{2i}),...,\varphi_n(\zeta,t,r,C_{ni})$ by Taylor's series about γ in the following manner,

$$\begin{cases} \varphi_1(\zeta,t,r,C_{1i})=(f_1)_0(\zeta,t)+\sum\limits_{k\geq 1}(f_1)_k(\zeta,t,C_{1i})r^k, \\ \varphi_2(\zeta,t,r,C_{2i})=(f_2)_0(\zeta,t)+\sum\limits_{k\geq 1}(f_2)_k(\zeta,t,C_{2i})r^k, \\ . \\ . \\ . \\ \varphi_n(\zeta,t,r,C_{ni})=(f_n)_0(\zeta,t)+\sum\limits_{k\geq 1}(f_n)_k(\zeta,t,C_{ni})r^k \end{cases} \tag{11}$$

where $k=1,2,....$ Now substituting Equations (10-11) into Equation (4) and equating the coefficient of like powers of r, we obtain Zeroth order system, given by Equation (9), the first and second order systems given by Equations (12-14) respectively and the general governing equations are given by Equation (15)

$$\begin{cases} L_1\Big((f_1)_1(\zeta,t)\Big)-L_1\Big((f_1)_0(\zeta,t)\Big)=C_{11}\Big(L_1\Big((f_1)_0(\zeta,t)\Big)+N_1\Big((f_1)_0(\zeta,t)\Big)\Big), \; B_1\Big((f_1)_1(\zeta,t),\dfrac{d(f_1)_1(\zeta,t)}{d\zeta}\Big), \\ L_2\Big((f_2)_1(\zeta,t)\Big)-L_2\Big((f_2)_0(\zeta,t)\Big)=C_{21}\Big(L_2\Big((f_2)_0(\zeta,t)\Big)+N_2\Big((f_2)_0(\zeta,t)\Big)\Big), \; B_2\Big((f_2)_1(\zeta,t),\dfrac{d(f_2)_1(\zeta,t)}{d\zeta}\Big), \\ . \\ . \\ . \\ L_n\Big((f_n)_1(\zeta,t)\Big)-L_n\Big((f_n)_0(\zeta,t)\Big)=C_{n1}\Big(L_n\Big((f_n)_0(\zeta,t)\Big)+N_n\Big((f_n)_0(\zeta,t)\Big)\Big), \; B_n\Big((f_n)_1(\zeta,t),\dfrac{d(f_n)_1(\zeta,t)}{d\zeta}\Big) \end{cases} \tag{12}$$

$$\begin{cases} L_1\Big((f_1)_2(\zeta,t)\Big) - L_1\Big((f_1)_1(\zeta,t)\Big) = C_{11}\Big(L_1\Big((f_1)_1(\zeta,t)\Big)\Big) + N_1\Big((f_1)_1(\zeta,t),(f_1)_0(\zeta,t)\Big) \\ + C_{12}\Big(L_1\Big((f_1)_0(\zeta,t)\Big) + N_1\Big((f_1)_0(\zeta,t)\Big)\Big),\ B_1\Big((f_1)_2(\zeta,t), \dfrac{d(f_1)_2(\zeta,t)}{d\zeta}\Big) = 0, \\ L_2\Big((f_2)_2(\zeta,t)\Big) - L_2\Big((f_2)_1(\zeta,t)\Big) = C_{21}\Big(L_2\Big((f_2)_1(\zeta,t)\Big)\Big) + N_2\Big((f_2)_1(\zeta,t),(f_2)_0(\zeta,t)\Big) \\ + C_{22}\Big(L_2\Big((f_2)_0(\zeta,t)\Big) + N_2\Big((f_2)_0(\zeta,t)\Big)\Big),\ B_2\Big((f_2)_2(\zeta,t), \dfrac{d(f_2)_2(\zeta,t)}{d\zeta}\Big) = 0, \\ . \\ . \\ . \\ L_n\Big((f_n)_n(\zeta,t)\Big) - L_n\Big((f_n)_1(\zeta,t)\Big) = C_{n1}\Big(L_n\Big((f_n)_1(\zeta,t)\Big)\Big) + N_n\Big((f_n)_1(\zeta,t),(f_n)_0(\zeta,t)\Big) \\ + C_{n2}\Big(L_n\Big((f_n)_0(\zeta,t)\Big) + N_n\Big((f_n)_0(\zeta,t)\Big)\Big),\ B_n\Big((f_n)_2(\zeta,t), \dfrac{d(f_n)_2(\zeta,t)}{d\zeta}\Big) = 0 \end{cases} \tag{13}$$

$$\begin{cases} L_1\Big((f_1)_3(\zeta,t)\Big) - L_1\Big((f_1)_2(\zeta,t)\Big) = C_{11}\Big(L_1\Big((f_1)_2(\zeta,t)\Big) + N_1\Big((f_1)_2(\zeta,t),(f_1)_1(\zeta,t),(f_1)_0(\zeta,t)\Big)\Big) + C_{12}\Big(L_1\Big((\zeta \\ + N_1\Big((f_1)_1(\zeta,t),(f_1)_0(\zeta,t)\Big)\Big) + C_{13}\Big(L_1\Big((f_1)_0(\zeta,t)\Big) + N_1\Big((f_1)_0(\zeta,t)\Big)\Big),\ B_1\Big((f_1)_3(\zeta,t), \dfrac{d(f_1)_3(\zeta,t)}{d\zeta}\Big) = 0, \\ L_2\Big((f_2)_3(\zeta,t)\Big) - L_2\Big((f_2)_2(\zeta,t)\Big) = C_{21}\Big(L_2\Big((f_2)_2(\zeta,t)\Big) + N_2\Big((f_2)_2(\zeta,t),(f_2)_1(\zeta,t),(f_2)_0(\zeta,t)\Big)\Big) + C_{22}\Big(L_2\Big((\zeta \\ + N_2\Big((f_2)_1(\zeta,t),(f_2)_0(\zeta,t)\Big)\Big) + C_{23}\Big(L_2\Big((f_2)_0(\zeta,t)\Big) + N_2\Big((f_2)_0(\zeta,t)\Big)\Big),\ B_2\Big((f_2)_3(\zeta,t), \dfrac{d(f_2)_3(\zeta,t)}{d\zeta}\Big) = 0, \\ . \\ . \\ . \\ L_n\Big((f_n)_3(\zeta,t)\Big) - L_n\Big((f_n)_2(\zeta,t)\Big) = C_{n1}\Big(L_n\Big((f_n)_2(\zeta,t)\Big) + N_n\Big((f_n)_2(\zeta,t),(f_n)_1(\zeta,t),(f_n)_0(\zeta,t)\Big)\Big) \\ + C_{n2}\Big(L_n\Big((f_n)_1(\zeta,t)\Big) + N_n\Big((f_n)_1(\zeta,t),(f_n)_0(\zeta,t)\Big)\Big) + C_{n3}\Big(L_n\Big((f_n)_0(\zeta,t)\Big) \\ + N_n\Big((f_n)_0(\zeta,t)\Big)\Big),\ B_n\Big((f_n)_3(\zeta,t), \dfrac{d(f_n)_3(\zeta,t)}{d\zeta}\Big) = 0 \end{cases} \tag{14}$$

$$\begin{cases} L_1\Big((f_1)_k(\zeta,t)\Big) - L_1\Big((f_1)_{k-1}(\zeta,t)\Big) = \sum_{i=1}^{k} C_{1i}\Big[L_1\Big((f_1)_{k-1}(\zeta,t)\Big) + N_1\Big((f_1)_{k-1}(\zeta,t), (f_1)_{k-2}(\zeta,t), ..., (f_1)_0(\zeta,t) \\ k=2,3,..., \; B_1\Big((f_1)_k(\zeta,t), \dfrac{d(f_1)_k(\zeta,t)}{d\zeta}\Big) = 0, \\ L_2\Big((f_2)_k(\zeta,t)\Big) - L_2\Big((f_2)_{k-1}(\zeta,t)\Big) = \sum_{i=1}^{k} C_{2i}\Big[L_2\Big((f_2)_{k-1}(\zeta,t)\Big) + N_2\Big((f_2)_{k-1}(\zeta,t), (f_2)_{k-2}(\zeta,t), ..., (f_2)_0(\zeta,t) \\ k=2,3,..., \; B_2\Big((f_2)_k(\zeta,t), \dfrac{d(f_2)_k(\zeta,t)}{d\zeta}\Big) = 0, \\ . \\ . \\ . \\ L_n\Big((f_n)_k(\zeta,t)\Big) - L_n\Big((f_n)_{k-1}(\zeta,t)\Big) = \sum_{i=1}^{k} C_{ni}\Big[L_n\Big((f_n)_{k-1}(\zeta,t)\Big) + N_n\Big((f_n)_{k-1}(\zeta,t), (f_n)_{k-2}(\zeta,t), ..., (f_n)_0(\zeta, \\ k=2,3,..., \; B_n\Big((f_n)_k(\zeta,t), \dfrac{d(f_n)_k(\zeta,t)}{d\zeta}\Big) = 0 \end{cases} \tag{15}$$

It has been observed that the convergence of the series Equations (11) depends upon the auxiliary constants $C_{11}, C_{12}, ..., C_{21}, C_{22}, ..., C_{n1}, C_{n2}, ...$
If it is convergent at , one has

$$\begin{cases} \varphi_1(\zeta,t,C_{1i}) = (f_1)_0)(\zeta,t) + \sum_{k\geq 1}(f_1)_k(\zeta,t,C_{1i}), i=1,2,...,m, \\ \varphi_2(\zeta,t,C_{2i}) = (f_2)_0)(\zeta,t) + \sum_{k\geq 1}(f_2)_k(\zeta,t,C_{1i}), i=1,2,...,m, \\ . \\ . \\ . \\ \varphi_n(\zeta,t,C_{ni}) = (f_n)_0)(\zeta,t) + \sum_{k\geq 1}(f_n)_k(\zeta,t,C_{ni}), i=1,2,...,m \end{cases} \tag{16}$$

Generally speaking the solution of Equations (1) can be approximately written as

$$\begin{cases} f_1^m(\zeta,t,C_{1i}) = (f_1)_0)(\zeta,t) + \sum_{k\geq 1}^{m}(f_1)_k(\zeta,t,C_{1i}), i=1,2,..., \\ f_2M(\zeta,t,C_{2i}) = (f_2)_0)(\zeta,t) + \sum_{k\geq 1}^{m}(f_2)_k(\zeta,t,C_{1i}), i=1,2,..., \\ . \\ . \\ . \\ f_n^m(\zeta,t,C_{ni}) = (f_n)_0)(\zeta,t) + \sum_{k\geq 1}^{m}(f_n)_k(\zeta,t,C_{ni}), i=1,2,... \end{cases} \tag{17}$$

where m is the order of approximation. Substituting Equations (17) into Equations (1), it results the following expression for residuals

$$\begin{cases} R_1(\zeta,t,C_{1i}) = L_1\Big(f_1^m(\zeta,t,C_{1i})\Big) + s_1(\zeta,t) + N_1\Big(f_1^m(\zeta,t,C_{1i})\Big), \\ R_2(\zeta,t,C_{2i}) = L_2\Big(f_2^m(\zeta,t,C_{2i})\Big) + s_2(\zeta,t) + N_2\Big(f_2^m(\zeta,t,C_{2i})\Big) \\ . \\ . \\ . \\ R_n(\zeta,t,C_{ni}) = L_n\Big(f_n^m(\zeta,t,C_{ni})\Big) + s_n(\zeta,t) + N_n\Big(f_n^m(\zeta,t,C_{ni})\Big) \end{cases} \tag{18}$$

If $R_1(\zeta,t,C_{1i}) = 0,\ R_2(\zeta,t,C_{2i}) = 0, ...,\ R_n(\zeta,t,C_{ni}) = 0$, then $\varphi_1(\zeta,t,C_{1i}),\ \varphi_2(\zeta,t,C_{2i}), ...,\ \varphi_n(\zeta,t,C_{ni})$ will be the exact solutions of the problem. Generally it doesnt happen, especially in nonlinear problems.

For the computation of auxiliary constants, $C_{1i}, C_{2i}, ..., C_{ni}, i = 1, 2, ..., m$, there are different methods like Galerkins Method, Ritz Method, Least Squares Method and Collocation Method. One can apply the Method of Least Squares as under

$$\begin{cases} J_1(C_{1i}) = \int_\Gamma \int_\Omega R_1^2(\zeta,t,C_{1i})\ d\zeta\ dt, \\ J_2(C_{2i}) = \int_\Gamma \int_\Omega R_2^2(\zeta,t,C_{2i})\ d\zeta\ dt, \\ . \\ . \\ . \\ J_n(C_{ni}) = \int_\Gamma \int_\Omega R_n^2(\zeta,t,C_{ni})\ d\zeta\ dt \end{cases} \tag{19}$$

and

$$\begin{cases} \frac{\partial J_1}{\partial C_{11}} = \frac{\partial J_1}{\partial C_{12}} = ... = \frac{\partial J_1}{\partial C_{1m}}, \\ \frac{\partial J_2}{\partial C_{21}} = \frac{\partial J_2}{\partial C_{22}} = ... = \frac{\partial J_2}{\partial C_{2m}}, \\ . \\ . \\ . \\ \frac{\partial J_n}{\partial C_{n1}} = \frac{\partial J_n}{\partial C_{n2}} = ... = \frac{\partial J_n}{\partial C_{nm}} \end{cases} \tag{20}$$

The m th order approximate solution can be obtained by these constants. The more general auxiliary functions $H_1(r), H_2(r), ..., H_n(r)$ are useful for convergence, which depends upon constants $C_{11}, C_{12}, ..., C_{n1}, C_{n2}, ...,$ can be optimally identified by Equations (19) and is useful in error minimization.

The same procedure is repeating for the next iteration and so on.

The (MOHAM) solution is given by

$$\tilde{f}_1(\zeta) = \begin{cases} f_1(\zeta) & ,\ z_0 \le t \le z_1 \\ .... & \\ f_N(\zeta) & ,\ z_{N-1} \le t \le T \end{cases}$$

$$\tilde{f}_2(\zeta) = \begin{cases} f_2(\zeta) & ,\ z_0 \le t \le z_1 \\ .... & \\ f_N(\zeta) & ,\ z_{N-1} \le t \le T \end{cases}$$

$$\vdots$$

$$\tilde{f}_n(\zeta) = \begin{cases} f_n(\zeta) & , \ z_0 \le t \le z_1 \\ \dots\dots\dots & \\ f_N(\zeta) & , \ z_{N-1} \le t \le T \end{cases}$$

3. Implementation of the MOHAM formulation of a system of PDEs

Model(1): Application of (MOHAM) to a system of three (PDEs) KdV equations of the form

$$\begin{cases} \dfrac{\partial u(\zeta,t)}{\partial t} = 0.5\dfrac{\partial^3 u(\zeta,t)}{\partial \zeta^3} - 3u(\zeta,t)\dfrac{\partial u(\zeta,t)}{\partial \zeta} + 3\dfrac{\partial}{\partial \zeta}\Big(v(\zeta,t)w(\zeta,t)\Big), \\ \dfrac{\partial v(\zeta,t)}{\partial t} = -\dfrac{\partial^3 v(\zeta,t)}{\partial^3 \zeta^3} + 3u(\zeta,t)\dfrac{\partial v(\zeta,t)}{\partial \zeta}, \\ \dfrac{\partial w(\zeta,t)}{\partial t} = -\dfrac{\partial^3 w(\zeta,t)}{\partial \zeta^3} + 3u(\zeta,t)\dfrac{\partial w(\zeta,t)}{\partial t} \end{cases} \tag{21}$$

with

$$\begin{cases} u(\zeta,0) = \dfrac{1}{3}(\beta - 8k^2) + 4k^2\tanh^2(k\zeta), \\ v(\zeta,0) = \dfrac{-4k^2\Big(3k^2c_0 - 2\beta c_2 + 4k^2c_2\Big)}{3c_2^2} + \dfrac{4k^2}{c_2}\tanh^2(k\zeta), \\ w(\zeta,0) = c_0 + c_2\tanh^2(k\zeta) \end{cases} \tag{22}$$

The closed form solution of Equations (21) is given by [24]

$$\begin{cases} u(\zeta,t) = \dfrac{1}{3}(\beta - 8k^2) + 4k^2\tanh^2(k(\zeta+\beta t)), \\ v(\zeta,t) = \dfrac{-4k^2\Big(3k^2c_0 - 2\beta c_2 + 4k^2c_2\Big)}{3c_2^2} + \dfrac{4k^2}{c_2}\tanh^2(k(\zeta+\beta t)), \\ w(\zeta,0) = c_0 + c_2\tanh^2(k(\zeta+\beta t)) \end{cases} \tag{23}$$

Applying the formulation of Extended (OHAM) technique discussed in section 2.
We consider

$$\begin{cases} u = u_0 + \gamma u_1 + \gamma^2 u_2, \ v = v_0 + \gamma v_1 + \gamma^2 v_2, \\ w = w_0 + \gamma w_1 + \gamma^2 w_2, \ H_1(\gamma) = \gamma C_{11} + \gamma^2 C_{12}, \\ H_2(\gamma) = \gamma c_{21} + \gamma^2 C_{22}, \ H_3(\gamma) = \gamma C_{31} + \gamma^2 C_{32} \end{cases} \tag{24}$$

Zeroth Order System:

$$2\frac{\partial u_0}{\partial t} = 0, \ \frac{\partial v_0}{\partial t} = 0, \ \frac{\partial w_0}{\partial t} = 0, \tag{25}$$

with initial conditions

$$\begin{cases} u_0(\zeta,0) = \dfrac{1}{3}(\beta - 8k^2) + 4k^2\tanh^2(k\zeta), \\ v_0(\zeta,0) = \dfrac{-4k^2\Big(3k^2c_0 - 2\beta c_2 + 4k^2c_2\Big)}{3c_2^2} + \dfrac{4k^2}{c_2}\tanh^2(k\zeta), \\ w_0(\zeta,0) = c_0 + c_2\tanh^2(k\zeta) \end{cases} \tag{26}$$

Its solution is

$$\begin{cases} u_0(\zeta,t) = \dfrac{1}{3}\Big(\beta - 8k^2 + 12k^2\tanh^2(k\zeta)\Big), \\ v_0(\zeta,t) = \dfrac{-4\Big(3k^2c_0 + 4k^2c_2 - 2k^2\beta c_2 - 3k^2c_2\tanh^2(k\zeta)\Big)}{3c_2^2}, \\ w_0(\zeta,t) = c_0 + c_2\tanh^2(k\zeta) \end{cases} \tag{27}$$

First Order System:

$$\begin{cases} 2\dfrac{\partial u_1(\zeta,t)}{\partial t} = 2(1+C_{11})\dfrac{\partial u_0}{\partial t} + 6C_{11}\Big(u_0\dfrac{\partial u_0}{\partial \zeta} - w_0\dfrac{\partial v_0}{\partial \zeta} - v_0\dfrac{\partial w_0}{\partial \zeta}\Big) - C_{11}\dfrac{\partial^3 u_0}{\partial \zeta^3}, \\ \dfrac{\partial v_1(\zeta,t)}{\partial t} = (1+C_{21})\dfrac{\partial v_0}{\partial t} - 3C_{21}u_0\dfrac{\partial w_0}{\partial \zeta} - C_{21}\dfrac{\partial^3 w_0}{\partial \zeta^3}, \\ \dfrac{\partial w_1(\zeta,t)}{\partial t} = (1+C_{31})\dfrac{\partial w_0}{\partial t} - 3C_{31}w_0\dfrac{\partial w_0}{\partial \zeta} + C_{31}\dfrac{\partial^3 w_0}{\partial \zeta^3} \end{cases} \tag{28}$$

with

$$u_1(\zeta,0) = 0,\ v_1(\zeta,0) = 0,\ w_1(\zeta,0) = 0 \tag{29}$$

Its solution is

$$\begin{cases} u_1(\zeta,t,C_{11}) = \dfrac{8tC_{11}}{c_2}\Bigg[-3k^2 sech^2(k\zeta)c_0\tanh(k\zeta) + 3k^5 sech^2(k\zeta)c_0\tanh(k\zeta) - 4k^2 sech^2(k\zeta)c_2\tanh(k\zeta) \\ -k^3\beta sech^2(k\zeta)c_2\tanh(k\zeta) + 4k^5 sech^4(k\zeta)c_2\tanh(k\zeta) - 6k^3 sech^2(k\zeta)c_2\tanh^2(k\zeta) + 10k^3 sech^2(k\zeta)c_2\tanh^3(k \\ v_1(\zeta,t,C_{21}) = -\dfrac{8tC_{21}}{c_2}\Bigg[-8k^5 sech^2(k\zeta)\tanh(k\zeta) + k^3\beta sech^2(k\zeta)\tanh(k\zeta) + 8k^5 sech^4(k\zeta)\tanh(k\zeta) \\ +8k^5 sech^5(k\zeta)\tanh^3(k\zeta)\Bigg], \\ w_1(\zeta,t,C_{31}) = -2tC_{31}\Bigg[-8k^3 sech^2(k\zeta)c_2\tanh(k\zeta) + k\beta sech^2(k\zeta)c_2\tanh(k\zeta) + 8k^3 sech^4(k\zeta)c_2\tanh(k\zeta) \\ +8k^3 sech^2(k\zeta)c_2\tanh^3(k\zeta)\Bigg] \end{cases} \tag{30}$$

The approximate solution is obtained as

$$\begin{cases} u(\zeta,t,C_{11}) = u_0(\zeta,t) + u_1(\zeta,t,C_{11}), \\ v(\zeta,t,C_{21}) = v_0(\zeta,t) + v_1(\zeta,t,C_{21}), \\ w(\zeta,t,C_{31}) = w_0(\zeta,t) + w_1(\zeta,t,C_{31}) \end{cases} \tag{31}$$

For the computation of the constants $C_{11},\ C_{21}$, and C_{31} using (31) in (21) and applying the technique mentioned in (18 - 20) by taking $c_0 = 1.5,\ c_2 = 0.1,\ k = 0.1,\ \beta = 1.5$ and also repeating the same procedure for next iteration, We get the approximate solution as

$$u(\zeta,t) = \begin{cases} \frac{1}{3}[1.42 + 0.12\tanh^2(0.1\zeta)] + 80t[4.210289\times10^{-14} sech^2(0.1\zeta)\tanh(0.1\zeta)] & ,\ 0 \le t \le 0 \\ 80t[-3.65449\times10^{-17} sech^4(0.1\zeta)\tanh(0.1\zeta) + 5.39038\times10^{-15} sech^2(0.1\zeta)\tanh^3(0.1\zeta)] & ,\ 0.5 \le t \le \end{cases}$$

$$v(\zeta,t)=\begin{cases} -133.333(-0.00051-0.001\tanh^2(0.1\zeta)) & , \ 0\le t\le 0.5 \\ -133.333(-0.00051-0.002\tanh^2(0.1\zeta)) & , \ 0.5\le t\le 1 \end{cases}$$

$$w(\zeta,t)=\begin{cases} 1.5+0.1\tanh^2(0.1\zeta)-2t[-2.65863\times 10^{-21}sech^2(0.1\zeta)\tanh(0.1\zeta)] & , \ 0\le t\le \\ -2t[-1.49782\times 10^{-22}sech^4(0.1\zeta)\tanh(0.1\zeta)-1.49782\times 10^{-22}sech^2(0.1\zeta)\tanh^2(0.1\zeta)] & , \ 0.5\le t \end{cases}$$

for $C_{11}=-9.136232469244701\times 10^{-12}$, $C_{12}=C_{21}=C_{22}=0$, $C_{31}=-1.8722771193005665\times 10^{-19}$, $C_{32}=0$

Table 1: Comparison of (MOHAM), HAM and Closed form solutions for $u(\zeta,t)$ at $t=1$

ζ	(MOHAM) Solution	Closed Form Solution	(HAM) Solution
0	0.473333	0.47422	0.473333
20	0.510507	0.51122	0.51030
40	0.51328	0.513294	0.51324
60	0.513332	0.513333	0.513310
80	0.513333	0.513333	0.513333
100	0.513333	0.513333	0.513333

Table 2: Comparison of (MOHAM), HAM and Closed form solutions for $v(\zeta,t)$ at $t=1$

ζ	(MOHAM) Solution	Closed Form Solution	(HAM) Solution
0	0.334667	0.343533	0.334661
20	0.706406	0.713534	0.706402
40	0.73413	0.734269	0.73411
60	0.734657	0.734659	0.734654
80	0.734666	0.734667	0.734665
100	0.734667	0.734667	0.734667

Table 3: Comparison of (MOHAM), HAM and Closed form solutions for $w(\zeta,t)$ at $t=1$

ζ	(MOHAM) Solution	Closed Form Solution	(HAM) Solution
0	1.5	1.50222	1.5
20	1.59393	1.59472	1.59293
40	1.59987	1.5999	1.59985
60	1.6	1.6	1.6
80	1.6	1.6	1.6
100	1.6	1.6	1.6

Table 4: Absolute error of (MOHAM) solution $u(\zeta,t)$ corresponding to the Closed form Solution

ζ	$t=1$	$t=0.5$	$t=0.1$	$t=0.01$
0	3.27717×10^{-2}	1.61366×10^{-2}	8.8667×10^{-4}	8.99865×10^{-6}
20	2.6804×10^{-3}	2.17746×10^{-3}	7.128×10^{-4}	8.06039×10^{-5}
40	5.09658×10^{-5}	4.16635×10^{-5}	1.38951×10^{-5}	1.58421×10^{-6}
60	9.34118×10^{-7}	7.63709×10^{-7}	2.54789×10^{-7}	2.90535×10^{-8}
80	1.71092×10^{-8}	1.3988×10^{-8}	4.66673×10^{-9}	5.32147×10^{-10}
100	3.1336×10^{-10}	2.562×10^{-10}	8.54742×10^{-11}	9.74665×10^{-12}

Table 5: Absolute error of (MOHAM) solution $v(\zeta,t)$ corresponding to the Closed form Solution

ζ	$t=1$	$t=0.5$	$t=0.1$	$t=0.01$
0	0.327717	0.161366	8.8667×10^{-3}	8.99865×10^{-5}
20	2.6804×10^{-2}	2.17746×10^{-2}	7.128×10^{-3}	8.06039×10^{-4}
40	5.09658×10^{-4}	4.16635×10^{-4}	1.38951×10^{-4}	1.58421×10^{-5}
60	9.34118×10^{-6}	7.63709×10^{-6}	2.54789×10^{-6}	2.90535×10^{-7}
80	1.71092×10^{-7}	1.3988×10^{-7}	4.66673×10^{-8}	5.32146×10^{-9}
100	3.13366×10^{-9}	2.562×10^{-9}	8.54741×10^{-10}	9.7466×10^{-11}

Model(2): Application of the (MOHAM) to Kdv Equation of the form:

$$\begin{cases} \dfrac{\partial u(\zeta,t)}{\partial t} = 0.5u_{\zeta\zeta\zeta} - 3u^2u_\zeta + 1.5v_{\zeta\zeta} + 3uv_\zeta + 3u_\zeta v - 3\lambda u_\zeta, \\ \dfrac{\partial v(\zeta,t)}{\partial t} = -v_{\zeta\zeta\zeta} - 3vv_\zeta - 3u_\zeta v_\zeta + 3u^2v_\zeta + 3\lambda v_\zeta \end{cases} \tag{32}$$

with initial conditions

$$\begin{cases} u(\zeta,0) = \gamma\tanh(\gamma\zeta), \\ v(\zeta,0) = 0.5(4\gamma^2+\lambda) - 2\gamma^2\tanh^2(\gamma\zeta) \end{cases} \tag{33}$$

The Closed form solution of the problem is given as [25]

$$\begin{cases} u(\zeta,t) = \gamma\tanh\Big(\gamma\zeta t(\gamma^2+1.5\lambda)\Big), \\ v(\zeta,t) = 0.5(4\gamma^2+\lambda) - 2\gamma^2\tanh^2\Bigg(\gamma\Big(\zeta-(\gamma^2+1.5\lambda)t\Big)\Bigg) \end{cases} \tag{34}$$

Zeroth Order System:

$$2\frac{\partial u_0}{\partial t} = 0,\ \frac{\partial v_0}{\partial t} = 0, \tag{35}$$

with initial conditions

$$\begin{cases} u_0(\zeta,0) = \gamma\tanh(\gamma\zeta, \\ v_0(\zeta,0) = 0.5(4\gamma^2+\lambda) - 2\gamma^2\tanh^2(\gamma\zeta) \end{cases} \tag{36}$$

Its solution is

$$\begin{cases} u_0(\zeta,t) = \gamma\tanh(\gamma\zeta, \\ v_0(\zeta,t) = 0.5(4\gamma^2+\lambda) - 2\gamma^2\tanh^2(\gamma\zeta) \end{cases} \tag{37}$$

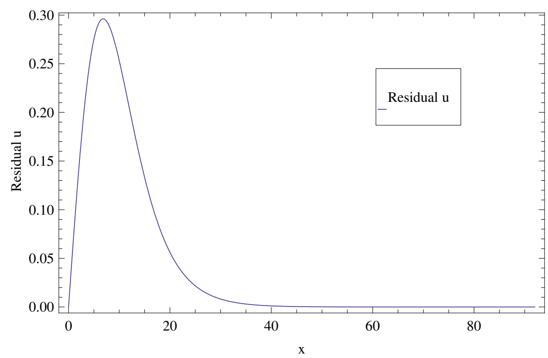
First Order System:

$$\begin{cases} 2\dfrac{\partial u_1(\zeta,t)}{\partial t} = 2(1+C_{11})\dfrac{\partial u_0}{\partial t} + 6C_{11}(\lambda-u_0^2)\dfrac{\partial u_0}{\partial\zeta} + 6C_{11}u_0\dfrac{\partial v_0}{\partial\zeta} + 3C_{11}\dfrac{\partial^2 v_0}{\partial\zeta^2} + C_{11}\dfrac{\partial^3 u_0}{\partial\zeta^3}, \\ \dfrac{\partial v_1}{\partial t} = (1+C_{21})\dfrac{\partial v_0}{\partial t} - 3C_{11}(\lambda+u_0^2)\dfrac{\partial v_0}{\partial\zeta} - 3(v_0+\dfrac{\partial u_0}{\partial\zeta})\dfrac{\partial v_0}{\partial\zeta} - C_{21}\dfrac{\partial^3 v_0}{\partial\zeta^3} \end{cases} \tag{38}$$

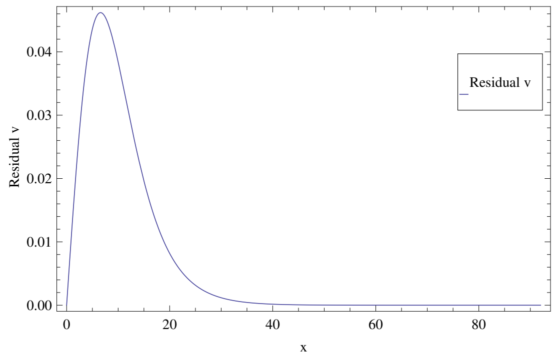
Table 6: Absolute error of (MOHAM) solution $w(\zeta,t)$ corresponding to the Closed form Solution

ζ	$t=1$	$t=0.5$	$t=0.1$	$t=0.01$
0	8.19293×10^{-2}	4.03414×10^{-2}	2.21668×10^{-3}	2.24966×10^{-5}
20	6.70099×10^{-3}	5.44365×10^{-3}	1.782×10^{-3}	2.0151×10^{-4}
40	1.27415×10^{-4}	1.04159×10^{-4}	3.47377×10^{-5}	3.96053×10^{-6}
60	2.33529×10^{-6}	1.90927×10^{-6}	6.36974×10^{-7}	7.26338×10^{-8}
80	4.27729×10^{-8}	3.49701×10^{-8}	1.16668×10^{-8}	1.33037×10^{-9}
100	7.83414×10^{-10}	6.40499×10^{-10}	2.13686×10^{-10}	2.43665×10^{-11}

(a) 2D - Residual of $u(\zeta,t)$ at $t=1$



(b) 2D - Residual of $v(\zeta,t)$ at $t=1$



(c) 2D - Residual of $w(\zeta,t)$ at $t=1$

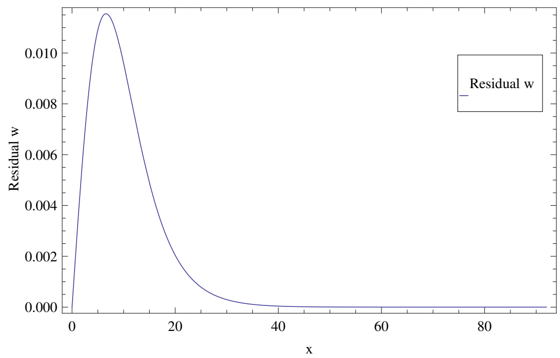


Figure 1: 2D - Residual

with initial conditions

$$u_1(\zeta,0)=0,\ \ v_1(\zeta,0)=0, \tag{39}$$

Its solution is

$$\begin{cases} u_1(\zeta,t,C_{11}) = \dfrac{t\gamma sech^2(\gamma\zeta)}{2}\Big[(2\gamma^2-3\lambda)C_{11})-6\lambda C_{11}\Big], \\ v_1(\zeta,t,C_{21}) = 2t\gamma^3 C_{21}\Big[-2\gamma^2+3\lambda\Big]sech^2(\gamma\zeta)\tanh(\gamma\zeta) \end{cases} \tag{40}$$

The approximate solution is obtained as

$$\begin{cases} u_1(\zeta,t,C_{11}) = u_0(\zeta,t)+u_1(\zeta,t,C_{11}), \\ v_1(\zeta,t,C_{21}) = v_0(\zeta,t)+v_1(\zeta,t,C_{21}) \end{cases} \tag{41}$$

Repeating the same step for next iteration we obtained the approximate solution as

$$u(\zeta,t)=\begin{cases} -0.0002304230sech^4(0.1\zeta)+0.1\tanh(0.1\zeta) & ,\ 0\le t\le 0.5 \\ tsech^2(0.1\zeta)(0.00150104-0.0000230423\tanh^2(0.1\zeta)) & ,\ 0.5\le t\le 1 \end{cases}$$

$$v(\zeta,t)=\begin{cases} 0.52+0.0000605366tsech^4(0.1\zeta)\tanh(0.1\zeta)-0.02\tanh^2(0.1\zeta) & , \ 0\le t\le 0.5 \\ tsech^2(0.1\zeta)(0.0174345+0.00005366\tanh^2(0.1\zeta)) & , \ 0.5\le t\le 1 \end{cases}$$

for $C_{11}=-0.3291761636279658$, $C_{12}=-0.00007616362$, $C_{21}=3.0068307906034297$, $C_{22}=0.020831245$, $\gamma=0.1$, $\lambda=1$.

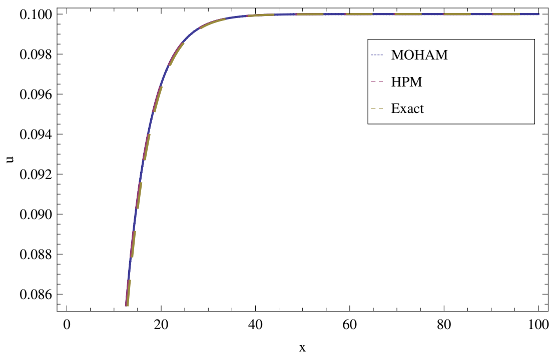
Table 7: Absolute error of (MOHAM) solution $u(\zeta,t)$ corresponding to the Exact Solution

ζ	$t=1$	$t=0.5$	$t=0.1$	$t=0.01$
0	1.64508×10^{-2}	8.144×10^{-3}	$4.49177\times10^{-}$	4.55951×10^{-9}
20	2.17387×10^{-3}	1.90041×10^{-3}	7.54522×10^{-4}	8.11496×10^{-5}
40	2.73879×10^{-4}	2.65752×10^{-5}	1.47276×10^{-5}	1.59572×10^{-6}
60	5.14463×10^{-6}	4.83366×10^{-7}	2.70061×10^{-7}	2.92649×10^{-8}
80	9.42707×10^{-8}	8.85201×10^{-9}	4.94645×10^{-9}	5.36017×10^{-10}
100	1.72664×10^{-9}	1.6213×10^{-10}	9.05975×10^{-11}	9.81748×10^{-12}

Table 8: Absolute error of (MOHAM) solution $v(\zeta,t)$ corresponding to the Exact Solution

ζ	$t=1$	$t=0.5$	$t=0.1$	$t=0.01$
0	1.64508×10^{-2}	8.144×10^{-3}	$4.49177\times10^{-}$	4.55951×10^{-9}
20	2.17387×10^{-3}	1.90041×10^{-3}	7.54522×10^{-4}	8.11496×10^{-5}
40	2.73879×10^{-4}	2.65752×10^{-5}	1.47276×10^{-5}	1.59572×10^{-6}
60	5.14463×10^{-6}	4.83366×10^{-7}	2.70061×10^{-7}	2.92649×10^{-8}
80	9.42707×10^{-8}	8.85201×10^{-9}	4.94645×10^{-9}	5.36017×10^{-10}
100	1.72664×10^{-9}	1.6213×10^{-10}	9.05975×10^{-11}	9.81748×10^{-12}

(a) Comparison of MOHAM and Exact solutions at $t=1$



(b) Comparisons of MOHAM and Exact solutions at $t=1$

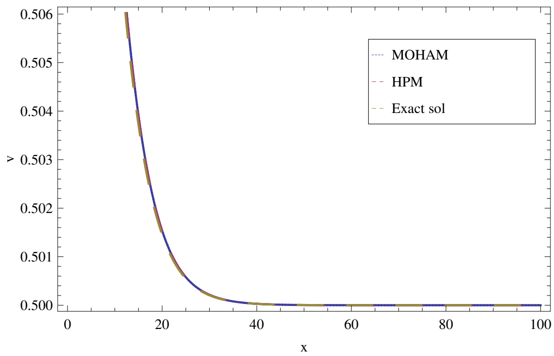
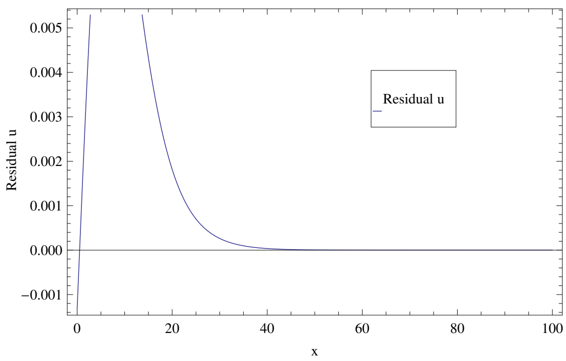


Figure 2:

(a) 2D - Residual of $u(\zeta, t)$ at $t = 1$



(b) 2D - Residual of $u(\zeta, t)$ at $t = 1$

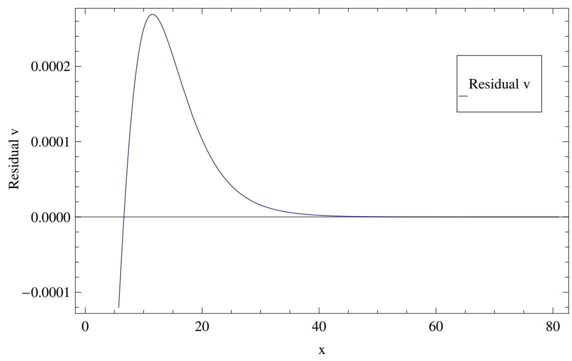


Figure 3: 2D - Residual

4. Results and Discussions

The mathematical theory of (MOHAM) provides highly accurate solutions for the system of BVP presented in section 3. We have used Mathematica 7 for our computational work. In Table 1-2, the (MOHAM) results are compared with closed form and (HAM) solutions at $t = 1$ for the generalized Hirota-Satsuma coupled KdV equation. The absolute errors at different values of t are given in Tables 1-6. The residuals are plotted in Figure 1. While for the (MKDV) equation, the Figure 2 shows the accuracy of (MOHAM) for $t = 1$. The Tables 7-8 presents the absolute errors at different values of t. The residuals of MKDV equations are plotted in Figure 3 at $t = 1$. From these Tables and Figure. it is evident that the (MOHAM) results are more accurate than (HAM) results and nearly identical to closed form solutions not only for small values of t but for large values of t. Here the results are very consistent with the decreasing time and by increasing displacement.

5. Conclusion

In this paper, we have seen the effectiveness of (MOHAM) different models of system of PDEs. The (MOHAM) is simpler in applicability, more convenient to control convergence and involved less computational overhead. Therefore, (MOHAM) shows its validity and great potential for the solution of nonlinear system of PDEs problems in science and engineering.

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