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Direct estimates for certain integral type Operators

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Abstract. In this note, we study approximation properties of a family of linear positive operators and establish asymptotic formula, rate of convergence, local approximation theorem, global approximation theorem, weighted approximation theorem and better approximation for this family of linear positive operators.

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1. Introduction

In the year 2008, Miheşan [38] constructed an important generalization of the well-known Szász operators depending on $\alpha \in R$ as

$$\mathcal{G}_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \ x \in [0,\infty) \tag{1}$$

where

$$m_{n,k}^{(\alpha)}(x) = \frac{(\alpha)_k}{k!}.\frac{(\frac{nx}{\alpha})^k}{(1+\frac{nx}{\alpha})^{\alpha+k}},$$

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and $(\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1), (\alpha)_0 = 1$, is the rising factorial and $\alpha + nx > 0$. The operator $\mathcal{G}_n^{(\alpha)}$ preserve the linear polynomials, and for special values of α, one can obtain some well-known operators.

Recently, Kajla [15] introduced a new sequence of summation-integral type operators and established some approximation properties e.g. weighted approximation, asymptotic formula and error estimation in terms of modulus of smoothness. Very recently, Gupta and Agrawal [11] proposed the integral modification of the operators (1) by taking weights of Beta basis functions as follows:

$$M_n^{(\alpha)}(f;x) = \sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt + \left(\frac{\alpha}{\alpha+nx}\right)^{\alpha} f(0), \tag{2}$$

where

$$b_{n,k}(t) = \frac{1}{B(n+1,k)} . \frac{t^{k-1}}{(1+t)^{k+n+1}},$$

and $B(m,n)$ being the Beta function defined as

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \ m,n > 0.$$

They obtain different approximation properties for these operators. For the different values of α, we get different special cases. Some of the special cases are discussed in [11].

In [42], Stancu introduced and investigated a new parameter-dependent linear positive operators of Bernstein type associated to a function $f \in C[0,1]$. The new construction of his operators shows that the new sequence of Bernstein polynomials present a better approach with the suitable selection of the parameters.

In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [1], [16], [23], [24], [25], [33] etc.

Inspired by the above work, We introduce the Stancu type generalization of the operators (2):

$$M_{n,\alpha}^{(\beta,\gamma)}(f;x) = \sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x) \int_0^{\infty} b_{n,k}(t) f\left(\frac{nt+\beta}{n+\gamma}\right) dt + \left(\frac{\alpha}{\alpha+nx}\right)^{\alpha} f\left(\frac{\beta}{n+\gamma}\right). \tag{3}$$

In this present work, our focus is to study the approximation properties of the operators (3) in terms of first and second order modulus of continuity. We estimate the rate of convergence of these operators in terms of modulus of continuity. Furthermore, we investigate weighted approximation theorems. Lastly we study King type modification of the operators (3).

2. Moment estimates

In the sequel, we shall need the following auxiliary results which will be necessary to prove our main results.

Lemma 1. *[11] For the operators $M_n^{(\alpha)}(f;x)$, we have*

(i) $M_n^{(\alpha)}(1;x) = 1$,

(ii) $M_n^{(\alpha)}(t;x) = x$,

(iii) $M_n^{(\alpha)}(t^2;x) = \dfrac{x[nx(\alpha+1)+2\alpha]}{\alpha(n-1)}$.

Lemma 2. *For the operators $M_{n,\alpha}^{(\beta,\gamma)}(f;x)$, we have*

(i) $M_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1$,

(ii) $M_{n,\alpha}^{(\beta,\gamma)}(t;x) = \dfrac{nx+\beta}{n+\gamma}$,

(iii) $M_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = \left\{\dfrac{n^3(\alpha+1)}{\alpha(n-1)(n+\gamma)^2}\right\}x^2 + \left\{\dfrac{2n^2+2n\beta(n-1)}{(n-1)(n+\gamma)^2}\right\}x + \dfrac{\beta^2}{(n+\gamma)^2}$.

Proof. For $x \in [0,\infty)$, in view of Lemma 1, we have

$$M_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1.$$

The first order moment is given by

$$M_{n,\alpha}^{(\beta,\gamma)}(t;x) = \frac{n}{n+\gamma}M_n^{(\alpha)}(t;x) + \frac{\beta}{n+\gamma} = \frac{nx+\beta}{n+\gamma}.$$

The second order moment is given by

$$\begin{aligned} M_{n,\alpha}^{(\beta,\gamma)}(t^2;x) &= \left(\frac{n}{n+\gamma}\right)^2 M_n^{(\alpha)}(t^2;x) + \frac{2n\beta}{(n+\gamma)^2}M_n^{(\alpha)}(t;x) + \left(\frac{\beta}{n+\gamma}\right)^2 \\ &= \left\{\frac{n^3(\alpha+1)}{\alpha(n-1)(n+\gamma)^2}\right\}x^2 + \left\{\frac{2n^2+2n\beta(n-1)}{(n-1)(n+\gamma)^2}\right\}x + \frac{\beta^2}{(n+\gamma)^2}. \end{aligned}$$

Lemma 3. *For $f \in C_B[0,\infty)$ (space of all real valued bounded functions on $[0,\infty)$ endowed with norm $\| f \|_{C_B[0,\infty)} = \sup\limits_{x\in[0,\infty)} |f(x)|$),*

$$\| M_{n,\alpha}^{(\beta,\gamma)}(f) \| \le \| f \| .$$

Proof. In view of (3) and Lemma 2, we get

$$\|M_{n,\alpha}^{(\beta,\gamma)}(f)\| \leq \|f\|M_{n,\alpha}^{(\beta,\gamma)}(1;x) = \|f\|.$$

Remark 1. *For every $x \in [0,\infty)$, we have*

$$M_{n,\alpha}^{(\beta,\gamma)}\left((t-x);x\right) = \frac{\beta-\gamma x}{n+\gamma}$$

and

$$\begin{aligned} M_{n,\alpha}^{(\beta,\gamma)}\left((t-x)^2;x\right) &= \left\{\frac{n^2(n+\alpha)+\alpha\gamma^2(n-1)}{\alpha(n-1)(n+\gamma)^2}\right\}x^2 + \left\{\frac{2n^2+2\beta\gamma(1-n)}{(n-1)(n+\gamma)^2}\right\}x + \frac{\beta^2}{(n+\gamma)^2} \\ &= \xi_{n,\alpha}^{(\beta,\gamma)}(x). \end{aligned}$$

3. Main results

Throughout this paper, we assume that $\alpha = \alpha(n) \to \infty$, as $n \to \infty$ and $\lim\limits_{n\to\infty} \dfrac{n}{\alpha(n)} = l(\in R)$. Let $e_i(t) = t^i,\ i = 0,1,2.$

Theorem 1. *Let $f \in C[0,\infty)$. Then $\lim\limits_{n\to\infty} M_{n,\alpha}^{(\beta,\gamma)}(f;x) = f(x)$, uniformly in each compact subset of $[0,\infty)$.*

Proof. In view of Lemma 2, we get

$$\lim_{n\to\infty} M_{n,\alpha}^{(\beta,\gamma)}(e_i;x) = x^i,\ i = 0,1,2,$$

uniformly in each compact subset of $[0,\infty)$. Applying Bohman-Korovkin theorem, it follows that $\lim\limits_{n\to\infty} M_{n,\alpha}^{(\beta,\gamma)}(f;x) = f(x)$, uniformly in each compact subset of $[0,\infty)$.

3.1. Voronovskaja type theorem

In this section we prove Voronvoskaja type asymptotic theorem for the operators $M_{n,\alpha}^{(\beta,\gamma)}$.

Theorem 2. *Let f be a bounded and integrable function on $[0,\infty)$, second derivative of f exists at a fixed point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n\left(M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)\right) = (\beta-\gamma x)f'(x) + \frac{1}{2}\left(2x+(l+1)x^2\right)f''(x).$$

Proof. Let $x \in [0,\infty)$ be fixed. Using Taylor's expansion formula of function f, it follows

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{4}$$

where $r(t,x)$ is a bounded function and $\lim_{t\to x} r(t,x)=0$.
Applying $M_{n,\alpha}^{(\beta,\gamma)}(f;x)$ on both sides of (4), we get

$$\begin{aligned} n\left(M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)\right) &= nf'(x)M_{n,\alpha}^{(\beta,\gamma)}((t-x);x)+\frac{1}{2}nf''(x)M_{n,\alpha}^{(\beta,\gamma)}\left((t-x)^2;x\right)\\ &\quad +nM_{n,\alpha}^{(\beta,\gamma)}\left((t-x)^2r(t,x);x\right). \end{aligned}$$

In view of Remark 1, we have

$$\lim_{n\to\infty} nM_{n,\alpha}^{(\beta,\gamma)}((t-x);x)=\beta-\gamma x \tag{5}$$

and

$$\lim_{n\to\infty} nM_{n,\alpha}^{(\beta,\gamma)}\left((t-x)^2;x\right)=2x+(l+1)x^2. \tag{6}$$

Now, we shall show that

$$\lim_{n\to\infty} nM_{n,\alpha}^{(\beta,\gamma)}\Bigg(r(t,x)(t-x)^2;x\Bigg)=0.$$

By using Cauchy-Schwarz inequality, we have

$$M_{n,\alpha}^{(\beta,\gamma)}\Bigg(r(t,x)(t-x)^2;x\Bigg) \leq \left(M_{n,\alpha}^{(\beta,\gamma)}(r^2(t,x);x)\right)^{1/2}\left(M_{n,\alpha}^{(\beta,\gamma)}((t-x)^4;x)\right)^{1/2}. \tag{7}$$

We observe that $r^2(x,x)=0$ and $r^2(.,x)\in C_B[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} M_{n,\alpha}^{(\beta,\gamma)}(r^2(t,x);x)=r^2(x,x)=0. \tag{8}$$

Now, from (7) and (8) we obtain

$$\lim_{n\to\infty} nM_{n,\alpha}^{(\beta,\gamma)}\left(r(t,x)(t-x)^2;x\right)=0. \tag{9}$$

From (5), (6) and (9), we get the required result.

3.2. Local approximation

For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{\| f-g \| +\delta \| g'' \|\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. By, p. 177, Theorem 2.4 in [2], there exists an absolute constant $M>0$ such that

$$K_2(f,\delta)\leq M\omega_2(f,\sqrt{\delta}), \tag{10}$$

where

$$\omega_2(f,\sqrt{\delta}) \quad = \quad \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} \mid f(x+2h)-2f(x+h)+f(x) \mid$$

is the second order modulus of smoothness of f. By

$$\omega(f,\delta) \quad = \quad \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} \mid f(x+h)-f(x) \mid,$$

we denote the first order modulus of continuity of $f \in C_B[0,\infty)$.

Theorem 3. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, we have*

$$\mid M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x) \mid \quad \leq \quad M\omega_2\left(f,\zeta_{n,\alpha}^{(\beta,\gamma)}(x)\right)+\omega\left(f,\frac{|\beta-\gamma x|}{n+\gamma}\right),$$

where M is a positive constant and

$$\zeta_{n,\alpha}^{(\beta,\gamma)}(x) \quad = \quad \left(\xi_{n,\alpha}^{(\beta,\gamma)}(x)+\left(\frac{\beta-\gamma x}{n+\gamma}\right)^2\right)^{1/2}.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\overline{M}_{n,\alpha}^{(\beta,\gamma)}$ defined by

$$\overline{M}_{n,\alpha}^{(\beta,\gamma)}(f;x) \quad = \quad M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f\left(\frac{nx+\beta}{n+\gamma}\right)+f(x). \tag{11}$$

From Lemma 2, we observe that the operators $\overline{M}_{n,\alpha}^{(\beta,\gamma)}$ are linear and reproduce the linear functions.
Hence

$$\overline{M}_{n,\alpha}^{(\beta,\gamma)}((t-x);x) \quad = \quad 0. \tag{12}$$

Let $g \in W^2$ and $x,t \in [0,\infty)$. By Taylor's expansion we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-v)g''(v)dv.$$

Applying $\overline{M}_{n,\alpha}^{(\beta,\gamma)}$ on both sides of the above equation and using (12), we get

$$\overline{M}_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x) \quad = \quad \overline{M}_{n,\alpha}^{(\beta,\gamma)}\left(\int_x^t (t-v)g''(v)dv;x\right).$$

Thus, by (11) we get
$|\overline{M}_{n,\alpha}^{(\beta,\gamma)}(g;x)-g(x)|$

$$\leq \quad M_{n,\alpha}^{(\beta,\gamma)}\left(\left|\int_x^t (t-v)g''(v)dv\right|;x\right)+\left|\int_x^{\frac{nx+\beta}{n+\gamma}}\left(\frac{nx+\beta}{n+\gamma}-v\right)g''(v)dv\right|$$

$$
\begin{aligned}
&\leq \left(\xi_{n,\alpha}^{(\beta,\gamma)}(x) + \left(\frac{\beta-\gamma x}{n+\gamma}\right)^2\right) \parallel g'' \parallel \\
&\leq \left(\zeta_{n,\alpha}^{(\beta,\gamma)}(x)\right)^2 \parallel g'' \parallel . \qquad (13)
\end{aligned}
$$

On other hand, by (11) and Lemma 3, we have

$$
|\overline{M}_{n,\alpha}^{(\beta,\gamma)}(f;x)| \quad \leq \quad \parallel f \parallel . \qquad (14)
$$

Using (13) and (14) in (11), we obtain
$|M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)|$

$$
\begin{aligned}
&\leq |\overline{M}_{n,\alpha}^{(\beta,\gamma)}(f-g;x)| + |(f-g)(x)| + |\overline{M}_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x)| + \left| f\left(\frac{nx+\beta}{n+\gamma}\right) - f(x) \right| \\
&\leq 2 \parallel f-g \parallel + \left(\zeta_{n,\alpha}^{(\beta,\gamma)}(x)\right)^2 \parallel g'' \parallel + \left| f\left(\frac{nx+\beta}{n+\gamma}\right) - f(x) \right|.
\end{aligned}
$$

Taking infimum over all $g \in W^2$, we get

$$
| \, M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x) \, | \quad \leq \quad K_2\left(f, (\zeta_{n,\alpha}^{(\beta,\gamma)}(x))^2\right) + \omega\left(f, \frac{|\beta-\gamma x|}{n+\gamma}\right).
$$

In view of (10), we get

$$
| \, M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x) \, | \quad \leq \quad M\omega_2\left(f, \zeta_{n,\alpha}^{(\beta,\gamma)}(x)\right) + \omega\left(f, \frac{|\beta-\gamma x|}{n+\gamma}\right),
$$

which proves the theorem.

Let $a_1, a_2 > 0$ be fixed. We define the following Lipschitz-type space (see [39]):

$$
Lip_M^{(a_1,a_2)}(r) = \left(f \in C[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^r}{(t + a_1x^2 + a_2x)^{r/2}};\ x, t \in [0,\infty)\right),
$$

where M is any positive constant and $0 < r \leq 1$.

Theorem 4. *Let $f \in Lip_M^{(a_1,a_2)}(r)$. Then, for all $x > 0$, we have*

$$
|M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \leq M\left(\frac{\xi_{n,\alpha}^{(\beta,\gamma)}(x)}{a_1x^2 + a_2x}\right)^{r/2}.
$$

Proof. First we prove the theorem for $r = 1$. Then, for $f \in Lip_M^{(a_1,a_2)}(1)$, and $x > 0$, we have

$$
\begin{aligned}
|M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| &\leq M_{n,\alpha}^{(\beta,\gamma)}(|f(t) - f(x)|;x) \\
&\leq MM_{n,\alpha}^{(\beta,\gamma)}\left(\frac{|t-x|}{(t + a_1x^2 + a_2x)^{1/2}};x\right)
\end{aligned}
$$

$$\leq \frac{M}{(a_1x^2+a_2x)^{1/2}} M_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x).$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| &\leq \frac{M}{(a_1x^2+a_2x)^{1/2}}\left(M_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x)\right)^{1/2}\\
&\leq M\left(\frac{\xi_{n,\alpha}^{(\beta,\gamma)}(x)}{a_1x^2+a_2x}\right)^{1/2}.
\end{aligned}$$

Thus the result holds for $r=1$.
Now, we prove that the result is true for $0<r<1$. Then, for $f \in Lip_M^{(a_1,a_2)}(r)$, and $x>0$, we get

$$|M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \leq \frac{M}{(a_1x^2+a_2x)^{r/2}} M_{n,\alpha}^{(\beta,\gamma)}(|t-x|^r;x).$$

Taking $p=\frac{1}{r}$ and $q=\frac{p}{p-1}$, applying the Hölders inequality, we have

$$|M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \leq \frac{M}{(a_1x^2+a_2x)^{r/2}}\left(M_{n,\alpha}^{(\beta,\gamma)}(|t-x|;x)\right)^{r}.$$

Finally by Cauchy-Schwarz inequality, we get

$$|M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \leq M\left(\frac{\xi_{n,\alpha}^{(\beta,\gamma)}(x)}{a_1x^2+a_2x}\right)^{r/2}.$$

Thus, the proof is completed.

3.3. Global approximation

In this section, the first and the second order Ditzian-Totik moduli of smoothness are defined as

$$\bar{\omega}_\phi(f,\delta) = \sup_{0<|h|\leq\delta}\ \sup_{x+h\phi(x)\in[0,\infty)} |\ f(x+h\phi(x))-f(x)\ |$$

and

$$\omega_{2,\phi}(f,\sqrt{\delta}) = \sup_{0<|h|\leq\sqrt{\delta}}\ \sup_{x\pm h\phi(x)\in[0,\infty)} |\ f(x+h\phi(x))-2f(x)+f(x-h\phi(x))\ |,$$

respectively and the corresponding K-functional is

$$K_{2,\phi}(f,\delta) = \inf_{g\in W^2(\phi)}\{\|\ f-g\ \| + \delta \parallel \phi^2 g'' \parallel\},$$

where $\delta > 0$ and $W^2(\phi) = \{g \in C_B[0,\infty) : g' \in AC[0,\infty), \phi^2 g'' \in C_B[0,\infty)\}$ and $g' \in AC[0,\infty)$ means that g' is absolutely continuous on $[0,\infty)$. It is well known that (see [3]) $K_{2,\phi}(f,\delta) \sim \omega_{2,\phi}(f,\sqrt{\delta})$ which means that there exist an absolute constant $M > 0$ such that

$$M^{-1}\omega_{2,\phi}(f,\sqrt{\delta}) \leq K_{2,\phi}(f,\delta) \leq M\omega_{2,\phi}(f,\sqrt{\delta}). \tag{15}$$

In the following we will consider $\phi(x) = 1 + x^2$.

Theorem 5. *Let $f \in C_B[0,\infty)$ and $x \in [0,\infty)$. Then, there exist an absolute constant $M > 0$ such that*

$$\mid M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x) \mid \leq 4K_{2,\phi}\left(f, \frac{M}{2n}\right) + \bar{\omega}_{\phi}\left(f, \frac{\sqrt{M}}{n}\right),$$

for n sufficiently large.

Proof. Let $g \in W^2(\phi)$. Applying Taylor's expansion, we may write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying $\overline{M}_{n,\alpha}^{(\beta,\gamma)}$ on both sides of the above equation, we get
$|\overline{M}_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x)|$

$$\begin{aligned}
&\leq M_{n,\alpha}^{(\beta,\gamma)}\left(\left|\int_x^t |t-v||g''(v)|dv\right|;x\right) + \left|\int_x^{\frac{nx+\beta}{n+\gamma}} \left|\frac{nx+\beta}{n+\gamma} - v\right| |g''(v)|dv\right| \\
&\leq \frac{\parallel \phi^2 g'' \parallel}{\phi^2(x)}\left(\xi_{n,\alpha}^{(\beta,\gamma)}(x) + \left(\frac{\beta-\gamma x}{n+\gamma}\right)^2\right).
\end{aligned} \tag{16}$$

In view of Remark 1, it follows that there exist a positive constant $M > 0$ such that

$$\frac{\xi_{n,\alpha}^{(\beta,\gamma)}(x)}{\phi^2(x)} \leq \frac{M}{n}, \frac{1}{\phi^2(x)}\left(\frac{\beta-\gamma x}{n+\gamma}\right)^2 \leq \frac{M}{n^2}.$$

Thus,

$$|\overline{M}_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x)| \leq M \parallel \phi^2 g'' \parallel \left(\frac{1}{n} + \frac{1}{n^2}\right) \leq \frac{2M}{n} \parallel \phi^2 g'' \parallel.$$

Now,
$|M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)|$

$$\leq |\overline{M}_{n,\alpha}^{(\beta,\gamma)}(f-g;x)| + |(f-g)(x)| + |\overline{M}_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x)| + \left|f\left(\frac{nx+\beta}{n+\gamma}\right) - f(x)\right|$$

$$\leq \; 4 \parallel f-g \parallel + \frac{2M}{n} \parallel \phi^2 g'' \parallel + \left| f\left(\frac{nx+\beta}{n+\gamma}\right) - f(x) \right|.$$

Also, we obtain

$$\begin{aligned} \left| f\left(\frac{nx+\beta}{n+\gamma}\right) - f(x) \right| &= \left| f\left(x + \phi(x)\frac{\frac{nx+\beta}{n+\gamma} - x}{\phi(x)}\right) - f(x) \right| \\ &\leq \sup \left| f\left(x + \phi(x)\frac{\frac{\beta-\gamma x}{n+\gamma}}{\phi(x)}\right) - f(x) \right| \\ &\leq \bar{\omega}_\phi\left(f, \frac{\sqrt{M}}{n}\right). \end{aligned}$$

Using the above equations, we get

$$| \, M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x) \, | \;\; \leq \;\; 4\left(\parallel f-g \parallel + \frac{M}{2n} \parallel \phi^2 g'' \parallel\right) + \bar{\omega}_\phi\left(f, \frac{\sqrt{M}}{n}\right).$$

Now, applying (15), the theorem is completed.

3.4. Rate of convergence

Let $\omega_a(f,\delta)$ denote the usual modulus of continuity of f on the closed interval $[0,a], a > 0$, and defined as

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,a]} |f(t) - f(x)|.$$

We observe that for a function $f \in C_B[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.
Now, we give a rate of convergence theorem for the operators $M_{n,\alpha}^{(\beta,\gamma)}$.

Theorem 6. *Let $f \in C_B[0,\infty)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a > 0$. Then, we have*

$$|M_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \leq 6M_f(1+a^2)\xi_{n,\alpha}^{(\beta,\gamma)}(a) + 2\omega_{a+1}\left(f, \sqrt{\xi_{n,\alpha}^{(\beta,\gamma)}(a)}\right),$$

where $\xi_{n,\alpha}^{(\beta,\gamma)}(a)$ is defined in Remark 1 and M_f is a constant depending only on f.

Proof. For $x \in [0,a]$ and $t > a+1$. Since $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\ &\leq M_f(t-x)^2(2 + 3x^2 + 2(t-x)^2) \\ &\leq 6M_f(1+a^2)(t-x)^2. \end{aligned}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta)$$

with $\delta > 0$.
From the above, we have

$$|f(t)-f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),$$

for $x \in [0,a]$ and $t \geq 0$.
Thus

$$\begin{aligned} |M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| &\leq 6M_f(1+a^2)(M_{n,\alpha}^{(\beta,\gamma)}(t-x)^2;x) \\ &\quad +\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}(M_{n,\alpha}^{(\beta,\gamma)}(t-x)^2;x)^{\frac{1}{2}}\right). \end{aligned}$$

Applying Cauchy-Schwarz's inequality, we get

$$|M_{n,\alpha}^{(\beta,\gamma)}(f;x)-f(x)| \leq 6M_f(1+a^2)\xi_{n,\alpha}^{(\beta,\gamma)}(a) + 2\omega_{a+1}\left(f,\sqrt{\xi_{n,\alpha}^{(\beta,\gamma)}(a)}\right),$$

on choosing $\delta = \sqrt{\xi_{n,\alpha}^{(\beta,\gamma)}(a)}$. This completes the proof of theorem.

3.5. Weighted approximation

In this section we give some weighted approximation properties of the operators $M_{n,\alpha}^{(\beta,\gamma)}$. We do this for the following class of continuous functions defined on $[0,\infty)$.
Let $B_\nu[0,\infty)$ denote the weighted space of real-valued functions f defined on $[0,\infty)$ with the property $|f(x)| \leq M_f \nu(x)$ for all $x \in [0,\infty)$, where $\nu(x) = 1+x^2$ is a weight function and M_f is a constant depending on the function f. We also consider the weighted subspace $C_\nu[0,\infty)$ of $B_\nu[0,\infty)$ given by $C_\nu[0,\infty) = \{f \in B_\nu[0,\infty) : f \text{ is continuous on } [0,\infty)\}$ and $C_\nu^*[0,\infty)$ denotes the subspace of all functions $f \in C_\nu[0,\infty)$ for which $\lim\limits_{|x|\to\infty} \dfrac{f(x)}{\nu(x)}$ exists finitely.
It is obvious that $C_\nu^*[0,\infty) \subset C_\nu[0,\infty) \subset B_\nu[0,\infty)$. The space $B_\nu[0,\infty)$ is a normed linear space with the following norm:

$$\| f \|_\nu = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\nu(x)}.$$

Theorem 7. *For each $f \in C_\nu^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \| M_{n,\alpha}^{(\beta,\gamma)}(f) - f \|_\nu = 0.$$

Proof. From [6], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \parallel M_{n,\alpha}^{(\beta,\gamma)}(e_i) - e_i \parallel_\nu = 0, \; i = 0, 1, 2. \tag{17}$$

Since $M_{n,\alpha}^{(\beta,\gamma)}(1; x) = 1$, the condition in (17) holds true for $i = 0$.
By Lemma 2, we have

$$\begin{aligned} \parallel M_{n,\alpha}^{(\beta,\gamma)}(t) - x \parallel_\nu &= \sup_{x\in[0,\infty)} \frac{|M_{n,\alpha}^{(\beta,\gamma)}(t; x) - x|}{1 + x^2} \\ &\leq \frac{\gamma}{n+\gamma} \sup_{x\in[0,\infty)} \left(\frac{x}{1+x^2}\right) + \frac{\beta}{n+\gamma} \sup_{x\in[0,\infty)} \left(\frac{1}{1+x^2}\right) \\ &\leq \frac{\beta+\gamma}{n+\gamma} \end{aligned}$$

which implies that $\lim_{n\to\infty} \parallel M_{n,\alpha}^{(\beta,\gamma)}(t) - x \parallel_\nu = 0$.
Again by Lemma 2, we have

$$\begin{aligned} \parallel M_{n,\alpha}^{(\beta,\gamma)}(t^2) - x^2 \parallel_\nu &= \sup_{x\in[0,\infty)} \frac{|M_{n,\alpha}^{(\beta,\gamma)}(t^2; x) - x^2|}{1 + x^2} \\ &\leq \left|\frac{n^3(\alpha+1)}{\alpha(n-1)(n+\gamma)^2} - 1\right| + \left|\frac{2n^2 + 2n\beta(n-1)}{(n-1)(n+\gamma)^2}\right| + \frac{\beta^2}{(n+\gamma)^2}, \end{aligned}$$

which implies that $\lim_{n\to\infty} \parallel M_{n,\alpha}^{(\beta,\gamma)}(t^2) - x^2 \parallel_\nu = 0$.
This completes the proof of theorem.

3.6. Weighted L_p-approximation

Let w be positive continuous function on real axis $[0, \infty)$ satisfying the condition

$$\int_0^\infty x^{2p} w(x)dx < \infty.$$

We denote by $L_{p,w}[0, \infty)(1 \leq p < \infty)$ the linear space of p-absolutely integrable on $[0, \infty)$ with respect to the weight function w

$$L_{p,w}[0, \infty) = \left\{ f : [0, \infty) \to R, \|f\|_{p,w} = \left(\int_0^\infty |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty \right\}.$$

Theorem 8. *[8] Let $(L_n)_{n\geq 1}$ be a uniformly bounded sequence of positive linear operators from $L_{p,w}[0, \infty)$ into $L_{p,w}[0, \infty)$, satisfying the conditions*

$$\lim_{n\to\infty} \parallel L_n(t^k) - x^k \parallel_{p,w} = 0, \; k = 0, 1, 2. \tag{18}$$

Then for every $f \in L_{p,w}[0, \infty)$

$$\lim_{n\to\infty} \parallel L_n(f) - f \parallel_{p,w} = 0.$$

Now we choose $w(x) = \frac{1}{(1+x^{2r})^p}, 1 \leq p < \infty$ and consider analogue weighted L_p-space [5]:

$$L_{p,2r}[0,\infty) = \left\{ f : [0,\infty) \to R, \|f\|_{p,2r} = \left(\int_0^\infty \left| \frac{f(x)}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Theorem 9. *For every* $f \in L_{p,2r}[0,\infty), r > 1,$ *we have*

$$\lim_{n\to\infty} \| M_{n,\alpha}^{(\beta,\gamma)}(f) - f \|_{p,2r} = 0.$$

Proof. Using the Theorem 8, we see that it is sufficient to verify the three conditions (18). Since $M_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1$, the first condition is obvious for $k = 0$.
By Lemma 2, for $k = 1$, we have

$$\begin{aligned} \left(\int_0^\infty \left| \frac{M_{n,\alpha}^{(\beta,\gamma)}(t;x) - x}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} \leq & \frac{\gamma}{n+\gamma} \left(\int_0^\infty \left| \frac{x}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} \\ & + \frac{\beta}{n+\gamma} \left(\int_0^\infty \left| \frac{1}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

which implies that $\lim_{n\to\infty} \| M_{n,\alpha}^{(\beta,\gamma)}(t) - x \|_{p,2r} = 0$.
For $k = 2$, we can write

$$\begin{aligned} \left(\int_0^\infty \left| \frac{M_{n,\alpha}^{(\beta,\gamma)}(t^2;x) - x^2}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} \leq & \left(\frac{n^3(\alpha+1)}{\alpha(n-1)(n+\gamma)^2} - 1 \right) \left(\int_0^\infty \left| \frac{x^2}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} \\ & + \frac{2n^2 + 2n\beta(n-1)}{(n-1)(n+\gamma)^2} \left(\int_0^\infty \left| \frac{x}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} \\ & + \frac{\beta^2}{(n+\gamma)^2} \left(\int_0^\infty \left| \frac{1}{1+x^{2r}} \right|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

which implies that $\lim_{n\to\infty} \| M_{n,\alpha}^{(\beta,\gamma)}(t^2) - x^2 \|_{p,2r} = 0$.
This completes the proof of theorem.

4. King type modification

In this section, we discuss better convergence rates by King type operators. To make the convergence faster, King [26] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2, where $e_i(t) = t^i, i = 0,1,2$. After this approach many researcher contributed in this direction.
As the operator $M_{n,\alpha}^{(\beta,\gamma)}(f;x)$ defined in (3) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators

preserve the constant as well as linear functions.
For this purpose the modification of (3) is defined as

$$\begin{aligned}\hat{M}_{n,\alpha}^{(\beta,\gamma)}(f;x) &= \sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(r_n(x)) \int_0^{\infty} b_{n,k}(t) f\left(\frac{nt+\beta}{n+\gamma}\right) dt \\ &+ \left(\frac{\alpha}{\alpha+nr_n(x)}\right)^{\alpha} f\left(\frac{\beta}{n+\gamma}\right) \end{aligned} \tag{19}$$

where $r_n(x) = \frac{(n+\gamma)x-\beta}{n}$ and $x \in I_n = [\frac{\beta}{n+\gamma}, \infty)$.

Lemma 4. *For every $x \in I_n$, we have*

(i) $\hat{M}_{n,\alpha}^{(\beta,\gamma)}(1;x) = 1$,

(ii) $\hat{M}_{n,\alpha}^{(\beta,\gamma)}(t;x) = x$,

(iii) $\hat{M}_{n,\alpha}^{(\beta,\gamma)}(t^2;x) = \dfrac{n(\alpha+1)x^2}{\alpha(n-1)} + \dfrac{(2n\alpha-2\alpha\beta-2n\beta)x}{\alpha(n-1)(n+\gamma)} + \dfrac{n\beta^2+\alpha\beta^2-2n\alpha\beta}{\alpha(n-1)(n+\gamma)^2}$.

Consequently, for each $x \in I_n$, we have the following equalities

$$\hat{M}_{n,\alpha}^{(\beta,\gamma)}((t-x);x) = 0$$

$$\begin{aligned}\hat{M}_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x) &= \frac{(n+\alpha)x^2}{\alpha(n-1)} + \frac{(2n\alpha-2\alpha\beta-2n\beta)x}{\alpha(n-1)(n+\gamma)} + \frac{n\beta^2+\alpha\beta^2-2n\alpha\beta}{\alpha(n-1)(n+\gamma)^2} \\ &= \lambda_{n,\alpha}^{(\beta,\gamma)}(x). \end{aligned} \tag{20}$$

Theorem 10. *For $f \in C_B(I_n)$, we have*

$$|\hat{M}_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)| \le M'\omega_2\left(f, \sqrt{\lambda_{n,\alpha}^{(\beta,\gamma)}(x)}\right),$$

where $\lambda_{n,\alpha}^{(\beta,\gamma)}(x)$ is given by (20) and M' is a positive constant.

Proof. Let $g \in W^2$ and $x,t \in I_n$. Using the Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying $\hat{M}_{n,\alpha}^{(\beta,\gamma)}$ on both sides and using Lemma **??**, we get

$$\hat{M}_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x) = \hat{M}_{n,\alpha}^{(\beta,\gamma)}\left(\int_x^t (t-v)g''(v)dv, x\right).$$

Obviously, we have $\left|\int_x^t (t-v)g''(v)dv\right| \leq (t-x)^2\|g''\|$.
Therefore

$$| \hat{M}_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x) | \leq \hat{M}_{n,\alpha}^{(\beta,\gamma)}((t-x)^2;x) \parallel g'' \parallel = \lambda_{n,\alpha}^{(\beta,\gamma)}(x) \parallel g'' \parallel .$$

Since $| \hat{M}_{n,\alpha}^{(\beta,\gamma)}(f;x) | \leq \|f\|$, we get

$$\begin{aligned} | \hat{M}_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x) | &\leq | \hat{M}_{n,\alpha}^{(\beta,\gamma)}(f-g;x) | + | (f-g)(x) | + | \hat{M}_{n,\alpha}^{(\beta,\gamma)}(g;x) - g(x) | \\ &\leq 2\|f-g\| + \lambda_{n,\alpha}^{(\beta,\gamma)}(x)\|g''\|. \end{aligned}$$

Finally, taking the infimum over all $g \in W^2$ and using (10) we obtain

$$| \hat{M}_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x) | \leq M'\omega_2\left(f, \sqrt{\lambda_{n,\alpha}^{(\beta,\gamma)}(x)}\right),$$

which proves the theorem.

Theorem 11. *Let $f \in C_B(I_n)$. If f', f'' exists at a fixed point $x \in I_n$, then we have*

$$\lim_{n\to\infty} n\left(\hat{M}_{n,\alpha}^{(\beta,\gamma)}(f;x) - f(x)\right) = \frac{x}{2}\left(2+(l+1)x\right)f''(x).$$

The proof follows along the lines of Theorem 2.

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