



Rank- k perturbation of Hamiltonian systems with periodic coefficients and applications

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Abstract. Jordan canonical forms of a rank- k perturbation of symplectic matrices and the fundamental solutions of Hamiltonian systems are presented on the basis of work done by C. Mehl et, al.. Small rank- k perturbations of Mathieu systems are analyzed. More precisely, it is shown that the rank- k perturbations of coupled or non-coupled double pendulums and the motion of an ion through a quadrupole analyzer slightly perturb the behavior of their spectra and their stabilities.

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1. Introduction

Let $J, W \in \mathbb{R}^{2N \times 2N}$ such that J is a skew-symmetric matrix. We say that the matrix W is J -symplectic or J -orthogonal if and only if $W^T J W = J$ [4, 18]. These types of matrices generally appear in control theory [3, 11, 15, 18], especially in optimal control [11] and in parametric resonance theory [15]. The spectra of the symplectic matrices is generally composed of three groups with respect to the unit circle (see e.g. [7, 8, 18]) : N_0 eigenvalues outside the unit circle, $N_0 = N_\infty$ eigenvalues inside the unit circle and $2N_1 = 2(N - N_0)$ eigenvalues on the unit circle. A symplectic matrix W is stable if all its powers are bounded. In other words, if the eigenvalues of W lie on the unit circle and are semi-simple. Some classifications of eigenvalues of W are given by the following definitions [4, 7, 10, 18]

Definition 1. *Let λ be a semi-simple eigenvalue of W lying on the unit circle.*

- (i) *Then λ is called an eigenvalue of the first (second) kind if the quadratic form (iJx, x) is positive (negative) on the eigenspace associated with λ . When $(Jx, x) = 0$, then λ is of mixed kind.*

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(ii) Then λ is an eigenvalue of a red (green) color or in short r -eigenvalue (g -eigenvalue) if $(S^{(0)}x, x)$ is positive (negative) on the eigenspace associated with λ where $S^{(0)} = \frac{1}{2}(JW + (JW)^T)$.

This leads us to the characterization of the strong stability by the following theorem (see [6, 7])

Theorem 1. *A symplectic matrix W is strongly stable if all the eigenvalues are on the unit circle and verifies one of the following assertion :*

(i) *the eigenvalues are either of the first or second kind and there is a sufficient gap between the eigenvalues of first and second kind. In other words, the quantity*

$$\delta_{KGL}(W) = \min \left\{ |e^{i\theta_k} - e^{i\theta_l}| \text{ such that } e^{i\theta_k}, e^{i\theta_l} \text{ are eigenvalues of } W \text{ of different kinds} \right\}$$

should not be close to zero.

(ii) *the eigenvalues are either of red color or green color and there is a sufficient gap between the eigenvalues of red and green color. In other words, the quantity*

$$\delta_S(W) = \min \left\{ |e^{i\theta_k} - e^{i\theta_l}| \text{ such that } e^{i\theta_k}, e^{i\theta_l} \text{ are } r\text{- and } g\text{-eigenvalues of } W \right\}$$

should not be close to zero.

These symplectic matrices are often obtained as solutions of Hamiltonian systems with periodic coefficients i.e. the differential systems of the form

$$J \frac{dx(t)}{dt} = H(t)x(t), \quad t \in \mathbb{R} \tag{1}$$

where $H(t) \in \mathbb{R}^{2N \times 2N}$ is symmetric and P -periodic (i.e. $H(t + P) = H(t) = (H(t))^T$). We know that the fundamental solution $X(t)$ of (1), in other words, the solution of the system

$$\begin{cases} J \frac{dX(t)}{dt} = H(t)X(t), & t \in \mathbb{R} \\ X(0) = I \end{cases}, \tag{2}$$

satisfies the relationship $X(t + nP) = X(t)X^n(P) (\neq X^n(P)X(t))$, $\forall (t, n) \in \mathbb{R} \times \mathbb{N}$ and is J -symplectic [18, Vol. 1, chap. 2]. Regarding stability (strong stability), we have the following definitions [6, 18]

Definition 2. (i) *System (1) is stable if each of its solutions $x(t)$ remains bounded for $t \in \mathbb{R}$.*

(ii) *System (1) is strongly stable if any Hamiltonian system with P -periodic coefficients to sufficiently close to (1), is stable.*

Specifically, system (1) (or system (2)) is strongly stable if there exists $\varepsilon > 0$ such that any Hamiltonian system with P -periodic coefficients of the form $J \frac{dx(t)}{dt} = \tilde{H}(t)x(t)$ and satisfying $\|H - \tilde{H}\| = \int_0^T \|H(t) - \tilde{H}(t)\| dt < \varepsilon$, is stable. We also have the following theorem [18, p. 196]

Theorem 2. *System 1 is strongly stable if and only if the J -symplectic matrix $X(P)$ is strongly stable.*

Since the strong stability analysis of the Hamiltonian systems with P -periodic coefficients is related to the study of their perturbation, we will dwell on a type of the perturbation which we call rank- k perturbation. Thus, we present in section 2, preliminaries necessary to the study of the isotropic subspaces, on the rank- k perturbation of a symplectic matrix and rank- k perturbation of a Hamiltonian system with P -periodic coefficients. In sections 3 and 4, we give respectively the Jordan canonical forms of a rank- k perturbation of a symplectic matrix and of a rank- k perturbation of the fundamental solution of (1). Finally in section 5, we present some applications for some systems of Mathieu: Namely systems that describe the movement of a double pendulum with oscillating support and those that describe the motion of an ion through a quadrupole analyzer.

2. Preliminaries

2.1. Isotropic subspaces

Definition 3. *A subspace $\mathcal{X} \subseteq \mathbb{R}^{2N}$ is called isotropic if $\mathcal{X} \perp J\mathcal{X}$. A maximal isotropic subspace is called Lagrangian.*

The maximum isotropic subspaces containing \mathcal{X} are of dimension N . Hence the following definition (see [9])

Definition 4. *A subspace \mathcal{L} of \mathbb{R}^N is called a Lagrangian subspace if it has the dimension N and*

$$x^T J y = 0, \quad \forall x, y \in \mathcal{L}.$$

In other words, we say that a subspace \mathcal{L} is Lagrangian if and only if every matrix L whose columns span \mathcal{L} satisfies $\text{rank} L = N$ and $L^T J L = 0$.

We list a set of properties on the isotropic subspaces in the following proposition

Proposition 1. *(i) Let \mathcal{X} be an isotropic subspace. Then the dimension of \mathcal{X} is less than or equal to N .*

(ii) Every isotropic subspace is contained in a Lagrangian subspaces.

(iii) Let $S = [S_1 \ S_2] \in \mathbb{R}^{2N \times 2N}$ be a symplectic matrix with $S_i \in \mathbb{R}^{2N \times N}$, $i = 1, 2$; then the columns of S_1 and S_2 span isotropic subspaces.

Recall the two lemmas below (see [13])

Lemma 1. *Let $\mathcal{X}_S \subseteq \mathbb{R}^{2N}$ be a subspace that is invariant under a Hamiltonian matrix S which has all its eigenvalues associated with \mathcal{X}_S having their real part negative. Then \mathcal{X}_S is isotropic.*

Lemma 2. *Let $S \in \mathbb{R}^{2N \times 2N}$ be a skew-Hamiltonian matrix and $X \in \mathbb{R}^{2N \times k} (k \leq N)$ with orthogonal columns. Then the columns of X span an isotropic invariant subspace of S if and only if there exists an orthogonal symplectic matrix $U = [X, Z, J^T X, J^T Z]$ with some $Z \in \mathbb{R}^{2N \times (N-k)}$ so that*

$$U^T S U = \begin{matrix} & k & N-k & k & N-k \\ \begin{matrix} k \\ N-k \\ k \\ N-k \end{matrix} & \begin{bmatrix} A_{11} & A_{12} & G_{11} & G_{12} \\ 0 & A_{22} & -G_{12}^T & G_{22} \\ 0 & 0 & A_{11}^T & 0 \\ 0 & H_{22} & A_{12}^T & A_{22}^T \end{bmatrix} \end{matrix}$$

On the other hand, if we consider the Krylov subspace defined below

$$\mathcal{K}_m \equiv \mathcal{K}_m(A, v) = \text{span} \{v, Av, A^2v, \dots, A^{m-1}v\},$$

where $A \in \mathbb{R}^{n \times m}$ (with $n > 1$) and $v \in \mathbb{R}^m$. Then we have the following proposition which shows that we can construct isotropic invariant subspaces from Krylov process (see [17, p. 399])

Proposition 2. *Let $S \in \mathbb{R}^{2N \times 2N}$ be a skew-Hamiltonian matrix and $u \in \mathbb{R}^{2N}$ be an arbitrary nonzero vector. Then the Krylov subspace $\mathcal{K}_j(S, u)$ is isotropic for all j .*

2.2. Rank- k perturbation of symplectic matrices

Let $W \in \mathbb{R}^{2N \times 2N}$ and \mathcal{L} be respectively a symplectic matrix and a J -Lagrangian subspace. Consider k vectors u_1, \dots, u_k of \mathcal{L} , where $k \leq N$. Setting

$$U = [u_1; \dots; u_k], \quad \text{and} \quad \widetilde{W} = (I + UU^T J)W,$$

we have the following proposition

Proposition 3. *The matrix \widetilde{W} is J -symplectic.*

Proof. For the proof, see [2].

Definition 5. *We call rank- k perturbation of W , any matrix of the form*

$$\widetilde{W} = (I + UU^T J)W, \tag{3}$$

where U is a matrix of rank k whose columns belong to a J -Lagrangian subspace.

The matrix \widetilde{W} can be put in the form

$$\widetilde{W} = \left(I + \sum_{j=1}^k u_j u_j^T J \right) W.$$

More specially, this shows that any rank- k perturbation of W is k rank-one perturbations of the symplectic matrix W . We have

$$\left(\prod_{j=1}^k (I + u_j u_j^T J) \right) W = \left(I + \sum_{j=1}^k u_j u_j^T J \right) W.$$

Consider a symplectic matrix of function $(X(t))_{t \in \mathbb{R}}$; we can consider for example the solution of system (2) which is J -symplectic. We have the following definition

Definition 6. We call rank- k perturbation of $X(t)$ any matrix function of the form

$$\widetilde{X}(t) = (I + UU^T J)X(t), \tag{4}$$

where $\text{rank}(U) = k$ and the columns of U belong in a J -Lagrangian subspace.

Remark 1. Since the matrix function $(X(t))_{t \in \mathbb{R}}$ is J -symplectic, its rank- k perturbation will be J -symplectic.

2.3. Rank- k perturbation of Hamiltonian system with periodic coefficients

Let $U \in \mathbb{R}^{2N \times k}$ (with $k \leq N$) be a constant matrix of rank k such that its columns belong to a J -Lagrangian subspace and $(X(t))_{t \geq 0}$ be the fundamental solution of (2). We have the following proposition

Proposition 4. a Consider the following perturbed Hamiltonian system

$$J \frac{d\widetilde{X}(t)}{dt} = [H(t) + E(t)] \widetilde{X}(t), \tag{5}$$

where

$$E(t) = (JU U^T H(t))^T + JU U^T H(t) + (UU^T J)^T H(t)(UU^T J).$$

(i) Then $\widetilde{X}(t) = (I + UU^T J)X(t)$ is a solution of system (5).

(ii) Equation (5) can be put in the form

$$\begin{cases} J \frac{d\widetilde{X}(t)}{dt} = (I - UU^T J)^T H(t) (I - UU^T J) \widetilde{X}(t), & t \in \mathbb{R}_+, \\ \widetilde{X}(0) = I + UU^T J \end{cases} \tag{6}$$

(iii) Any solution $(\tilde{X}(t))_{t \geq 0}$ of perturbed system (5) of system (2), is of the form

$$\tilde{X}(t) = (I + UU^T J)X(t),$$

where $(X(t))_{t \geq 0}$ is the fundamental solution of system (2).

Proof. For the proof, see [2].

System (6) can be written as below

$$\begin{cases} J \frac{d\tilde{X}(t)}{dt} = \left(I - \sum_{j=1}^k u_j u_j^T J \right)^T H(t) \left(I - \sum_{j=1}^k u_j u_j^T J \right) \tilde{X}(t) \\ \tilde{X}(0) = \left(I + \sum_{j=1}^k u_j u_j^T J \right) \end{cases}, \quad (7)$$

where each vector $(u_j)_{1 \leq j \leq k} \subset \mathbb{R}^{2N}$ belongs to a same J -Lagrangian subspace. We can immediately see that the rank- k perturbation of (2) can be interpreted as k rank-one perturbations of (2). In fact, since

$$I - UU^T J = I - \sum_{j=1}^k u_j u_j^T J = \prod_{j=1}^k (I - u_j u_j^T J),$$

we easily see that system (7) can be put in the following form

$$\begin{cases} J \frac{d\tilde{X}(t)}{dt} = \left(\prod_{j=1}^k (I - u_j u_j^T J) \right)^T H(t) \left(\prod_{j=1}^k (I - u_j u_j^T J) \right) \tilde{X}(t) \\ \tilde{X}(0) = \prod_{j=1}^k (I + u_j u_j^T J) \end{cases} \quad (8)$$

which is the same as the bellow system, for all $p \in \{1, 2, \dots, k - 1\}$:

$$\begin{cases} J \frac{d\tilde{X}(t)}{dt} = \left(\prod_{j=p+1}^k (I - u_j u_j^T J) \right)^T H^{(p)}(t) \left(\prod_{j=p+1}^k (I - u_j u_j^T J) \right) \tilde{X}(t) \\ \tilde{X}(0) = \left(\prod_{j=p+1}^k (I + u_{(k+p-j+1)} u_{(k+p-j+1)}^T J) \right) \bar{X}^{(p)}(0) \end{cases}, \quad (9)$$

where

$$H^{(p)}(t) = \left(\prod_{j=1}^p (I - u_j u_j^T J) \right)^T H(t) \left(\prod_{j=1}^p (I - u_j u_j^T J) \right) \quad \text{and}$$

$$\bar{X}^{(p)}(0) = \prod_{j=1}^p (I + u_{(p-j+1)} u_{(p-j+1)}^T J).$$

3. Jordan canonical form of rank- k perturbation of a symplectic matrix

Let $W, J \in \mathbb{R}^{2N \times 2N}$ be two matrices and $\lambda \in \mathbb{C}$ such that J is skew-symmetric, W is J -symplectic and λ an eigenvalue of W . We have the following theorem

Theorem 3. *Suppose that W has the following Jordan canonical form :*

$$\left(\bigoplus_{j=1}^{l_1} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \mathcal{J},$$

where $n_1 > \cdots > n_m$, $m \in \mathbb{N}^*$ such that the algebraic multiplicity a of λ is of the form $a = \sum_{j=1}^m l_j n_j$ and \mathcal{J} contains all the forms in Jordan blocks associated with eigenvalues of

W that are different from λ . Moreover let $B = UU^T JW$ where $U \in \mathbb{R}^{2N \times k}$ is such that its columns generate an isotropic subspace.

- (1) *If $\lambda \notin \{-1, 1\}$, then generally with respect to the components of U , the matrix $W + B$ has the Jordan canonical form*

$$\left\{ \begin{array}{l} \left(\bigoplus_{j=1}^{l_1-k} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \tilde{\mathcal{J}}, \quad \text{if } k < l_1 \\ \left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \tilde{\mathcal{J}}, \quad \text{if } \begin{cases} k = \sum_{s=1}^{i-1} l_s + k_i, \\ \text{with } k_i < l_i \\ \text{and } i > 1 \end{cases} \end{array} \right.$$

where $\tilde{\mathcal{J}}$ contains all the forms in Jordan blocks of $W + B$ associated with eigenvalues different from λ .

- (2) *If $\lambda \in \{-1, 1\}$, then*

- (2a) *if $k = \sum_{s=1}^{i-1} l_s + k_i$ where the n_1, n_2, \dots, n_i are even and $k_i < l_i$, then generally with respect to the components of U , then matrix $W + B$ has the Jordan canonical form*

$$\left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \tilde{\mathcal{J}},$$

where $\tilde{\mathcal{J}}$ contains all the forms in Jordan blocks of $W + B$ associated with eigenvalues different from λ .

(2b) if $k = \sum_{s=1}^{i-1} l_s + 2k_i - 1$ with $2k_i \leq l_i$ and n_i is odd, then l_i is even and generally with respect to the components of U , then matrix $W + B$ has the Jordan canonical form

$$\mathcal{J}_{n_{i+1}}(\lambda) \oplus \left(\bigoplus_{j=1}^{l_i-2k_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \tilde{\mathcal{J}},$$

where $\tilde{\mathcal{J}}$ contains all the forms in Jordan blocks of $W + B$ associated with eigenvalues, different from λ .

Proof. We know that the rang- k perturbation $\tilde{W} = W + B$ of W can be written in the form

$$\tilde{W} = \left[\prod_{j=1}^k (I + u_{k-j+1} u_{k-j+1}^T J) \right] W,$$

where each vector u_j is a column of U . Therefore

1) If $\lambda \notin \{-1, 1\}$, then

- For $k < l_1$;

- Set $\tilde{W}_1 = (I + u_1 u_1^T J) W$. According to 1) of Theorem 7.1 of [16], \tilde{W}_1 has the Jordan canonical

$$\left(\bigoplus_{j=1}^{l_1-1} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \tilde{\mathcal{J}}_1,$$

where $\tilde{\mathcal{J}}_1$ contains all the forms in blocks of Jordan of \tilde{W}_1 associated with eigenvalues different from λ .

- Set $\tilde{W}_2 = (I + u_2 u_2^T J) \tilde{W}_1$. According to 1) of Theorem 7.1 of [16], \tilde{W}_2 has the Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_1-2} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \tilde{\mathcal{J}}_2,$$

with $\tilde{\mathcal{J}}_2$ containing all the forms in Jordan blocks of \tilde{W}_2 associated with eigenvalues different from λ .

- On the other hand,

$$\tilde{W}_k = W + B = (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_3 u_3^T J) \tilde{W}_2,$$

by applying $(k - 2)$ -times 1) of Theorem 7.1 of [16] to the matrix \widetilde{W}_2 , we get that \widetilde{W}_k has the following Jordan canonical form :

$$\left(\bigoplus_{j=1}^{l_1-k} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of $W + B$ associated with eigenvalues different from λ .

- For $k = \sum_{s=1}^{i-1} l_s + k_i$, with $k_i < l_i$:
 - $i = 2$, we have $k = l_1 + k_2$ with $k_2 < l_2$.
We know that

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{l_1+1} u_{l_1+1}^T J) \times \\ &\quad \underbrace{(I + u_{l_1} u_{l_1}^T J) (I + u_{l_1-1} u_{l_1-1}^T J) \times \cdots \times (I + u_1 u_1^T J)}_{\widetilde{W}_{l_1}} W \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{l_1+1} u_{l_1+1}^T J) \widetilde{W}_{l_1} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{k-k_2+1} u_{k-k_2+1}^T J) \widetilde{W}_{l_1}, \end{aligned}$$

because $l_1 = k - k_2$. As \widetilde{W}_{l_1} has the following Jordan canonical form :

$$\left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{l_1},$$

where $\widetilde{\mathcal{J}}_{l_1}$ contains all the forms in Jordan blocks of \widetilde{W}_{l_1} associated with eigenvalues different from λ because \widetilde{W}_k is a rang- k_2 perturbation of \widetilde{W}_{l_1} , then \widetilde{W}_k has the following Jordan canonical form :

$$\left(\bigoplus_{j=1}^{l_2-k_2} \mathcal{J}_{n_2}(\lambda) \right) \left(\bigoplus_{j=1}^{l_3} \mathcal{J}_{n_3}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{k_2},$$

where $\widetilde{\mathcal{J}}_{k_2}$ contains all the forms in Jordan blocks of $\widetilde{W}_k = W + B$ associated with eigenvalues different from λ .

- $i > 2$, $k = \sum_{s=1}^{i-1} l_s + k_i$, with $k_i < l_i$.

Set

$$\gamma(i) = \sum_{s=1}^i l_s, \forall i \geq 1.$$

We have

$$\begin{aligned}
 \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\
 &\quad \times (I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \cdots \times (I + u_{\gamma(3)+1} u_{\gamma(3)+1}^T J) \\
 &\quad (I + u_{\gamma(3)} u_{\gamma(3)}^T J) \times \cdots \times (I + u_{\gamma(2)+1} u_{\gamma(2)+1}^T J) \\
 &\quad \underbrace{(I + u_{\gamma(2)} u_{\gamma(2)}^T J) \times \cdots \times (I + u_1 u_1^T J)}_{\widetilde{W}_{\gamma(2)}} W \\
 &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\
 &\quad \times (I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \cdots \times (I + u_{\gamma(3)+1} u_{\gamma(3)+1}^T J) \\
 &\quad \times \underbrace{(I + u_{\gamma(3)} u_{\gamma(3)}^T J) \times \cdots \times (I + u_{\gamma(2)+1} u_{\gamma(2)+1}^T J)}_{\widetilde{W}_{\gamma(3)}} \widetilde{W}_{\gamma(2)} \\
 &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\
 &\quad \times \underbrace{(I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \cdots \times (I + u_{\gamma(3)+1} u_{\gamma(3)+1}^T J)}_{\widetilde{W}_{\gamma(i-1)}} \widetilde{W}_{\gamma(3)} \\
 &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \widetilde{W}_{\gamma(i-1)} \\
 &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \cdots \times (I + u_{k-k_i+1} u_{k-k_i+1}^T J) \widetilde{W}_{\gamma(i-1)},
 \end{aligned}$$

where $\widetilde{W}_{\gamma(i-1)}$ is $\gamma(i-1)$ rank-one perturbations of the symplectic matrix W . Then $\widetilde{W}_{\gamma(i-1)}$ has the following Jordan canonical form :

$$\left(\bigoplus_{j=1}^{l_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{\gamma(i-1)},$$

where $\widetilde{\mathcal{J}}_{\gamma(i-1)}$ contains all the forms in Jordan blocks of $\widetilde{W}_{\gamma(i-1)}$ associated with eigenvalues different from λ . Finally, using the fact that \widetilde{W}_k is k_i rank one perturbation of $\widetilde{W}_{\gamma(i-1)}$, according to 1) of Theorem 7.1 of [16] and it follows that the Jordan canonical form of $\widetilde{W} = \widetilde{W}_k$ is given by

$$\left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda) \right) \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the form in Jordan blocks of \widetilde{W}_k associated with eigenvalues different from λ .

2) If $\lambda \in \{-1, 1\}$, then :

2a) if $k = \sum_{s=1}^{i-1} l_s + k_i$, where the n_1, n_2, \dots, n_i are even and $k_i < l_i$, then

– $i = 1$, we have $k = k_1$ and n_1 is even.

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_2 u_2^T J) \underbrace{(I + u_1 u_1^T J) W}_{=\widetilde{W}_1} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times \underbrace{(I + u_2 u_2^T J) \widetilde{W}_1}_{=\widetilde{W}_2} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_3 u_3^T J) \widetilde{W}_2. \end{aligned}$$

As \widetilde{W}_1 is a rank-one perturbation of W , according to 2a) of Theorem 7.1 of [16], it has the following Jordan canonical form :

$$\left(\bigoplus_{j=1}^{l_1-1} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_1,$$

where $\widetilde{\mathcal{J}}_1$ contains all the forms in Jordan blocks of \widetilde{W}_1 associated with eigenvalues different from λ . Using the fact that \widetilde{W}_2 is a rank-one perturbation of \widetilde{W}_1 , 2a) of Theorem 7.1 of [16] implies that the Jordan canonical form of \widetilde{W}_2 is given by

$$\left(\bigoplus_{j=1}^{l_1-2} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_2,$$

where $\widetilde{\mathcal{J}}_2$ contains all the forms in Jordan blocks of \widetilde{W}_2 associated with eigenvalues different from λ . Hence, applying $(k_1 - 2)$ -times 2a) of Theorem 7.1 of [16] to the matrix \widetilde{W}_2 , we have the following Jordan canonical form of \widetilde{W}_{k_1}

$$\left(\bigoplus_{j=1}^{l_1-k_1} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{k_1},$$

where $\widetilde{\mathcal{J}}_{k_1}$ contains all the forms in Jordan blocks of \widetilde{W}_{k_1} associated with eigenvalues different from λ .

– For $i > 1$, we have $k = \sum_{s=1}^{i-1} l_s + k_i$, with $k_i < l_i$ and n_1, n_2, \dots, n_i are even.

With $\gamma(i) = \sum_{s=1}^i l_s, \forall i > 1$, we have :

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\ &\quad \times (I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \dots \times (I + u_{\gamma(3)+1} u_{\gamma(3)+1}^T J) (I + u_{\gamma(3)} u_{\gamma(3)}^T J) \\ &\quad \times \dots \times (I + u_{\gamma(2)+1} u_{\gamma(2)+1}^T J) \underbrace{(I + u_{\gamma(2)} u_{\gamma(2)}^T J) \times \dots \times (I + u_1 u_1^T J) W}_{\widetilde{W}_{\gamma(2)}} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\ &\quad \times (I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \dots \times (I + u_{\gamma(3)+1} u_{\gamma(3)+1}^T J) \\ &\quad \times \underbrace{(I + u_{\gamma(3)} u_{\gamma(3)}^T J) \times \dots \times (I + u_{\gamma(2)+1} u_{\gamma(2)+1}^T J)}_{\widetilde{W}_{\gamma(3)}} \widetilde{W}_{\gamma(2)} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\ &\quad \times \underbrace{(I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \dots \times (I + u_{\gamma(3)+1} u_{\gamma(3)+1}^T J)}_{\widetilde{W}_{\gamma(i-1)}} \widetilde{W}_{\gamma(3)} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \widetilde{W}_{\gamma(i-1)} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{k-k_i+1} u_{k-k_i+1}^T J) \widetilde{W}_{\gamma(i-1)}. \end{aligned}$$

Knowing that $\widetilde{W}_{\gamma(i-1)}$ is $\gamma(i-1)$ rank-one perturbations of the symplectic matrix W , according to 2a) of Theorem 7.1 of [16], $\widetilde{W}_{\gamma(i-1)}$ has the following Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{\gamma(i-1)},$$

where $\widetilde{\mathcal{J}}_{\gamma(i-1)}$ contains all the forms in Jordan blocks of $\widetilde{W}_{\gamma(i-1)}$ associated with eigenvalues different from λ . Finally, as \widetilde{W}_k is k_i rank-one perturbations of $\widetilde{W}_{\gamma(i-1)}$, according to 2a) of Theorem 7.1 of [16] the Jordan canonical form of \widetilde{W}_k is given by :

$$\left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda) \right) \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the form in Jordan blocks from \widetilde{W}_k associated with eigenvalues different from λ .

2b) if $k = \sum_{s=1}^{i-1} l_s + 2k_i - 1$, with $2k_i \leq l_i$ and n_i is even.

– For $i = 1$, we have $k = 2k_1 - 1$ with $2k_1 \leq l_1$ and n_1 is odd.

According to the property 2b) of Theorem 10 of [5], l_1 is even and we have

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_3 u_3^T J) (I + u_2 u_2^T J) \\ &\quad \times \underbrace{(I + u_1 u_1^T J) W}_{\widetilde{W}_1} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_3 u_3^T J) \underbrace{(I + u_2 u_2^T J) \widetilde{W}_1}_{\widetilde{W}_2} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_4 u_4^T J) \underbrace{(I + u_3 u_3^T J) \widetilde{W}_2}_{\widetilde{W}_3} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_4 u_4^T J) \widetilde{W}_3. \end{aligned}$$

We know that \widetilde{W}_1 is a rank-one perturbation of W and its Jordan canonical form in block of Jordan is given by (see [16, Theorem 7.1,2b])

$$\mathcal{J}_{n_1+1}(\lambda) \oplus \left(\bigoplus_{j=1}^{l_1-2} \mathcal{J}_{n_1}(\lambda) \right) \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_1,$$

where $\widetilde{\mathcal{J}}_1$ contains all the form in Jordan blocks of \widetilde{W}_1 associated with eigenvalues different from λ . So \widetilde{W}_2 has the following Jordan canonical form (see [16, 2b) Theorem 7.1]) :

$$\left(\bigoplus_{j=1}^{l_1-2} \mathcal{J}_{n_1}(\lambda) \right) \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_2,$$

where $\widetilde{\mathcal{J}}_2$ contains all the forms in Jordan blocks of \widetilde{W}_2 associated with eigenvalues different from λ , because it is a rank-one perturbation of \widetilde{W}_1 . Similarly the Jordan canonical form of \widetilde{W}_3 [16, part 2b) du theorem 7.1] is given by

$$\mathcal{J}_{n_1+1}(\lambda) \oplus \left(\bigoplus_{j=1}^{l_1-2 \times 2} \mathcal{J}_{n_1}(\lambda) \right) \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_3,$$

where $\widetilde{\mathcal{J}}_3$ contains all the forms in Jordan blocks of \widetilde{W}_3 associated with eigenvalues different from λ . Hence, applying $(2k_1 - 4)$ -times this process

to matrix \widetilde{W}_3 , we obtain the canonical form of Jordan below

$$\mathcal{J}_{n_1+1}(\lambda) \oplus \left(\bigoplus_{j=1}^{l_1-2k_1} \mathcal{J}_{n_1}(\lambda) \right) \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of $W + B$ associated with eigenvalues different from λ .

– For $i = 2$, we have $k = l_1 + 2k_2 - 1$ with $2k_2 \leq l_2$ and n_2 is odd.

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{l_1+1} u_{l_1+1}^T J) \\ &\times \underbrace{(I + u_{l_1} u_{l_1}^T J) \times \dots \times (I + u_1 u_1^T J)}_{\widetilde{W}_{l_1}} W \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{l_1+1} u_{l_1+1}^T J) \widetilde{W}_{l_1} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{k-2k_2+2} u_{k-2k_2+2}^T J) \widetilde{W}_{l_1}, \end{aligned}$$

knowing that $l_1 = k - 2k_2 + 1$.

i) If n_1 is even, according to the property 2a) of Theorem 7.1 of [16], \widetilde{W}_{l_1} has the Jordan canonical form bellow

$$\left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{l_1},$$

where $\widetilde{\mathcal{J}}_{l_1}$ contains all the forms in Jordan blocks of \widetilde{W}_{l_1} associated with the eigenvalues different from λ . Finally, as \widetilde{W}_k is $(2k_2 - 1)$ rank-one perturbations of \widetilde{W}_{l_1} and n_2 is odd, according to 2b) of Theorem 7.1 of [16], l_2 is even and the Jordan canonical form of \widetilde{W}_k is given by :

$$\mathcal{J}_{n_2+1}(\lambda) \oplus \left(\bigoplus_{j=1}^{l_2-2k_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_3} \mathcal{J}_{n_3}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_k,$$

where $\widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of \widetilde{W}_k associated to eigenvalues different from λ .

ii) If n_1 is odd, in this case l_1 is even [16, Theorem 7.1,2b]. Applying l_1 -times the 2b) of Theorem 7.1 of [16] to W , we have the Jordan canonical form of \widetilde{W}_{l_1}

$$\left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{l_1}, \tag{10}$$

where $\widetilde{\mathcal{J}}_{l_1}$ contains all the forms in Jordan blocks of \widetilde{W}_{l_1} associated with the eigenvalues different from λ . Since n_2 is odd and \widetilde{W}_k is $(2k_2 - 1)$ rank-one perturbations of the symplectic matrix \widetilde{W}_{l_1} , according the property 2b) of Theorem 7.1 of [16], l_2 is even and \widetilde{W}_k has the following Jordan canonical form :

$$\mathcal{J}_{n_2+1}(\lambda) \oplus \left(\bigoplus_{j=1}^{l_2-2k_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_k,$$

where $\widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of \widetilde{W}_k associated with eigenvalues different from λ .

- For $i > 2$, we have $k = \sum_{s=1}^{i-1} l_s + 2k_i - 1$, with $2k_i \leq l_i$ and n_i is odd.

Set

$$\gamma(i) = \sum_{s=1}^i l_s, \forall i > 2.$$

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\ &\quad \times \underbrace{(I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \dots \times (I + u_1 u_1^T J)}_{\widetilde{W}_{\gamma(i-1)}} W \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\ &\quad \times \widetilde{W}_{\gamma(i-1)}. \end{aligned}$$

from 2a) and 2b) of Theorem 7.1 of [16], we deduce that $\widetilde{W}_{\gamma(i-1)}$ has the following Jordan canonical form :

$$\left(\bigoplus_{j=1}^{l_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{\gamma(i-1)},$$

where $\widetilde{\mathcal{J}}_{\gamma(i-1)}$ contains all the forms in Jordan blocks of $\widetilde{W}_{\gamma(i-1)}$ associated with the eigenvalues different from λ . Thus \widetilde{W}_k has the following Jordan canonical form :

$$\mathcal{J}_{n_{i+1}}(\lambda) \oplus \left(\bigoplus_{j=1}^{l_i-2k_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of \widetilde{W}_k associated with eigenvalues different from λ because \widetilde{W}_k is $(2k_i - 1)$ rank-one perturbation of $\widetilde{W}_{\gamma(i-1)}$.

Remark 2. In the property (2) of Theorem 3, if $k = \sum_{j=1}^{i-1} l_s + 2k_i$, with $2k_i < l_i$ and the n_i is odd, then the l_i are even and generally with respect to the components of U , the rank- k perturbation $\widetilde{W} = W + B$ of W has the following Jordan canonical form :

$$\left(\bigoplus_{j=1}^{l_i-2k_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_k,$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of \widetilde{W} associated with the eigenvalues different from λ .

Using (2a) and (2b) of Theorem 3, we have the following corollary :

Corollary 1. Suppose that $\lambda \in \{-1, 1\}$. If $k = \sum_{s=1}^{i-1} l_s + k_i$, with $k_i < l_i$ and only n_i is even, then generally with respect to the components of U , the $W + B$ has the Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}}$ contains all the forms in Jordan blocks of $W + B$ associated with the eigenvalues different from λ .

Proof.

- If $i = 1$, then $k = k_1$ and n_1 is even. According to (2a) of Theorem 3, \widetilde{W}_k has the following Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_1-k_1} \mathcal{J}_{n_1}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_{k_1}$ contains all the forms in Jordan blocks of \widetilde{W}_{k_1} associated with eigenvalues different from λ .

- If $i = 2$, then $k = l_1 + k_2$, with $(k_2 < l_2)$ and n_2 is even. We know that

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{l_1+1} u_{l_1+1}^T J) \\ &\quad \times \underbrace{(I + u_{l_1} u_{l_1}^T J) \times \dots \times (I + u_1 u_1^T J)}_{\widetilde{W}_{l_1}} W \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{l_1+1} u_{l_1+1}^T J) \widetilde{W}_{l_1}. \end{aligned}$$

So

- (a) if n_1 is even, according to the property 2a) of Theorem 3, \widetilde{W}_{l_1} has the following Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{l_1}, \tag{11}$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_{l_1}$ contains all the forms in Jordan blocks of \widetilde{W}_{l_1} associated with all eigenvalues different from λ . \widetilde{W}_k being k_2 rank-one perturbations of \widetilde{W}_{l_1} , according to 2a) of Theorem 3, the Jordan canonical form of \widetilde{W}_k is given by

$$\left(\bigoplus_{j=1}^{l_2-k_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_3} \mathcal{J}_{n_3}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of \widetilde{W}_k associated with eigenvalues different from λ .

- (b) If n_1 is odd, according to 2b) of Theorem 3, The Jordan canonical form of \widetilde{W}_{l_1} is given by (11). Knowing that n_2 is even and \widetilde{W}_k is k_2 rank-one perturbations of \widetilde{W}_{l_1} , we obtain from 2a) of Theorem 3 that \widetilde{W}_k has the following Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_2-k_2} \mathcal{J}_{n_2}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_3} \mathcal{J}_{n_3}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of \widetilde{W}_k associated with eigenvalues different from λ .

- For $i > 2$, we have $k = \sum_{s=1}^{i-1} l_s + k_i$, with $k_i < l_i$ and n_i is even.

Let's put $\gamma(i) = \sum_{s=1}^i l_s$. we know that

$$\begin{aligned} \widetilde{W}_k &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \\ &\quad \times \underbrace{(I + u_{\gamma(i-1)} u_{\gamma(i-1)}^T J) \times \dots \times (I + u_1 u_1^T J)}_{=\widetilde{W}_{\gamma(i-1)}} W \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{\gamma(i-1)+1} u_{\gamma(i-1)+1}^T J) \widetilde{W}_{\gamma(i-1)} \\ &= (I + u_k u_k^T J) (I + u_{k-1} u_{k-1}^T J) \times \dots \times (I + u_{k-k_i+1} u_{k-k_i+1}^T J) \widetilde{W}_{\gamma(i-1)}, \end{aligned}$$

because $\gamma(i-1) = k - k_i$. So $\widetilde{W}_{\gamma(i-1)}$ has the following Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}}_{\gamma(i-1)},$$

where $\widetilde{\mathcal{J}}_{\gamma(i-1)}$ contains all the forms in Jordan blocks of $\widetilde{W}_{\gamma(i-1)}$ associated with eigenvalues different from λ . Applying thereafter k_i times 2a) of Theorem 3 to the matrix $\widetilde{W}_{\gamma(i-1)}$, we obtain the Jordan canonical form of \widetilde{W}_k

$$\left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_m} \mathcal{J}_{n_m}(\lambda) \right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_k$ contains all the forms in Jordan blocks of $W + B$ associated with eigenvalues different from λ .

4. Jordan canonical form of $(\widetilde{X}(t))_{t>0}$

Theorem 4. Let $t > 0$, $J \in \mathbb{R}^{2N \times 2N}$ be a skew-symmetric and invertible matrix, $(X(t))_{t>0}$ be the fundamental solution of the system (2) and $\lambda(t) \in \mathbb{C}$ be an eigenvalue of $(X(t))_{t>0}$. Suppose that $X(t)$ has the following Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_1} \mathcal{J}_{n_1}(\lambda(t)) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda(t)) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_{m(t)}} \mathcal{J}_{n_{m(t)}}(\lambda(t)) \right) \oplus \mathcal{J}(t),$$

where $n_1 > \cdots > n_{m(t)}$ et $m : \mathbb{R} \rightarrow \mathbb{N}^*$ is a index function such that the algebraic multiplicity $a(t)$ of $\lambda(t)$ is of the form $a(t) = \sum_{j=1}^{m(t)} l_j n_j$ and $\mathcal{J}(t)$ contains all the forms in Jordan blocks associated with eigenvalues of $X(t)$ different from $\lambda(t)$. Moreover, Set $B(t) = U U^T J X(t)$ where $U \in \mathbb{R}^{2N \times k}$ is such that its columns generate an isotropic subspace.

(1) If $\lambda(t) \notin \{-1, 1\}$, then generally with respect to the components of U , $X(t) + B(t)$ has the following Jordan canonical form

$$\left\{ \begin{array}{l} \left(\bigoplus_{j=1}^{l_1-k} \mathcal{J}_{n_1}(\lambda(t)) \right) \oplus \left(\bigoplus_{j=1}^{l_2} \mathcal{J}_{n_2}(\lambda(t)) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_{m(t)}} \mathcal{J}_{n_{m(t)}}(\lambda(t)) \right) \oplus \widetilde{\mathcal{J}}(t), \text{ if } k < l_1 \\ \left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda(t)) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda(t)) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{l_{m(t)}} \mathcal{J}_{n_{m(t)}}(\lambda(t)) \right) \oplus \widetilde{\mathcal{J}}(t), \\ \text{if } \left\{ \begin{array}{l} k = \sum_{s=1}^{i-1} l_s + k_i, \\ \text{with } k_i < l_i \\ \text{et } i > 1 \end{array} \right. \end{array} \right.$$

where $\tilde{\mathcal{J}}(t)$ contains all the form in Jordan blocks of $X(t) + B(t)$ associated with eigenvalues different from $\lambda(t)$.

(2) if $\exists t_0 > 0$ such that $\lambda(t_0) \in \{-1, 1\}$, then

(2a) if $k = \sum_{s=1}^{i-1} l_s + k_i$ where the n_1, n_2, \dots, n_i are even and $k_i < l_i$, then generally with respect to the components of U , $X(t) + B(t)$ has the following Jordan canonical form

$$\left(\bigoplus_{j=1}^{l_i-k_i} \mathcal{J}_{n_i}(\lambda(t_0)) \right) \oplus \left(\bigoplus_{j=1}^{l_{i+1}} \mathcal{J}_{n_{i+1}}(\lambda(t_0)) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_{m(t_0)}} \mathcal{J}_{n_{m(t_0)}}(\lambda(t_0)) \right) \oplus \tilde{\mathcal{J}}(t_0),$$

where $\tilde{\mathcal{J}}(t_0)$ contains all the form in Jordan blocks of $X(t_0) + B(t_0)$ associated with eigenvalues different from $\lambda(t_0)$.

(2b) If $k = \sum_{s=1}^{i-1} l_s + 2k_i - 1$ with $2k_i \leq l_i$ and n_i is odd, then l_i is even and generally with respect to the components of U , $X(t_0) + B(t_0)$ has the following Jordan canonical form

$$\mathcal{J}_{n_{i+1}}(\lambda(t_0)) \oplus \left(\bigoplus_{j=1}^{l_i-2k_i} \mathcal{J}_{n_i}(\lambda(t_0)) \right) \oplus \dots \oplus \left(\bigoplus_{j=1}^{l_{m(t_0)}} \mathcal{J}_{n_{m(t_0)}}(\lambda(t_0)) \right) \oplus \tilde{\mathcal{J}}(t_0),$$

where $\tilde{\mathcal{J}}(t_0)$ contains all the form in Jordan blocks of $X(t_0) + B(t_0)$ associated with eigenvalues different from $\lambda(t_0)$.

Proof. It suffices to adapt the proof of Theorem 3 to the solution $(X(t))_{t \geq 0}$ of (2) and to its rank-k perturbation $(\tilde{X}(t))_{t \geq 0}$ (which is the fundamental solution of (6)).

5. Application to some Mathieu systems

5.1. Double pendulum with oscillating supports

Consider two identical simple pendulums attached to the same support. When the support of each pendulum is subjected to an oscillatory movement $f(t)$ of amplitude α and of a pulsation Ω , defined by $f(t) = \alpha \cos(\Omega t)$ [14], we present two cases:

Figure 1: Model of the uncoupled double pendulum with oscillating supports.

5.1.1. Uncoupled double pendulums with oscillating supports

According to [14], the differential equation of the movement of the two pendulums will be the same. Thus, the equation of motion is given by:

$$\frac{d^2x_i}{dt^2} + \frac{cg}{k_0^2} \left(1 - \frac{1}{g} \frac{d^2f}{dt^2} \right) x_i = 0, \quad i = 1, 2, \tag{12}$$

where k_0 is the radius of gyration of the pendulum around its point of suspension, and c is the distance between the point of suspension and the center of the pendulum.

Since $f(t) = \alpha \cos(\Omega t)$, then system (12) becomes:

$$\frac{d^2x_i}{dt^2} + \frac{cg}{k_0^2} \left(1 + \frac{\alpha\Omega^2}{g} \cos(\Omega t) \right) x_i = 0, \quad i = 1, 2, \tag{13}$$

and by the change of variable $\tau = \Omega t$ [14], equation (13) becomes:

$$\frac{d^2x_i}{d\tau^2} + (\delta + \varepsilon \cos(\tau)) x_i = 0, \quad i = 1, 2, \tag{14}$$

where

$$\varepsilon = \frac{c\alpha}{k_0^2} \text{ and } \delta = \frac{cg}{k_0^2\Omega^2}.$$

Finally, using the following change of variables

$$X(\tau) = \begin{bmatrix} x(\tau) \\ \frac{dx}{d\tau}(\tau) \end{bmatrix}, \quad J = \begin{bmatrix} 0_2 & -I_2 \\ I_2 & 0_2 \end{bmatrix} \text{ and } H(\tau, \delta, \varepsilon) = \begin{bmatrix} P(\tau, \delta, \varepsilon) & 0_2 \\ 0_2 & I_2 \end{bmatrix}, \tag{15}$$

with $x(\tau) = \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix}$ and $P(\tau, \delta, \varepsilon) = (\delta + \varepsilon \cos(\tau))I_2$, we obtain equation (1).

Now, consider the rank-2 perturbation of the fundamental solution $X(\tau, \delta, \varepsilon)$ of its corresponding Hamiltonian system by the following matrix of rank 2

$$E_a(\tau, \delta, \varepsilon) = U_a U_a^T J X(\tau, \delta, \varepsilon), \tag{16}$$

where

$$U_a = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } a \in [0, 1[. \tag{17}$$

According to [2, 5], its rank-2 perturbation

$$\tilde{X}_a(\tau, \delta, \varepsilon) = (I + U_a U_a^T J) X(\tau, \delta, \varepsilon). \tag{18}$$

is the solution of the following rank- k perturbation Hamiltonian system

$$\begin{cases} J \frac{d\tilde{X}_a(\tau, \delta, \varepsilon)}{d\tau} = \underbrace{(I - U_a U_a^T J)^T H(\tau, \delta, \varepsilon) (I - U_a U_a^T J)}_{\tilde{H}(\tau, \delta, \varepsilon, a)} \tilde{X}_a(\tau, \delta, \varepsilon), \\ \tilde{X}_a(0, \delta, \varepsilon) = I + U_a U_a^T J \end{cases} \quad (19)$$

Figure 2 represents the movement of eigenvalues of $\tilde{X}_a(\tau, \delta, \varepsilon)$ for $(\delta, \varepsilon) \in \{(1, 0.8), (1.93, 1.93)\}$ and $a \in \{0, 0.35\}$, with $\tau \in [0, 2\pi]$. These figures show that small rank- k

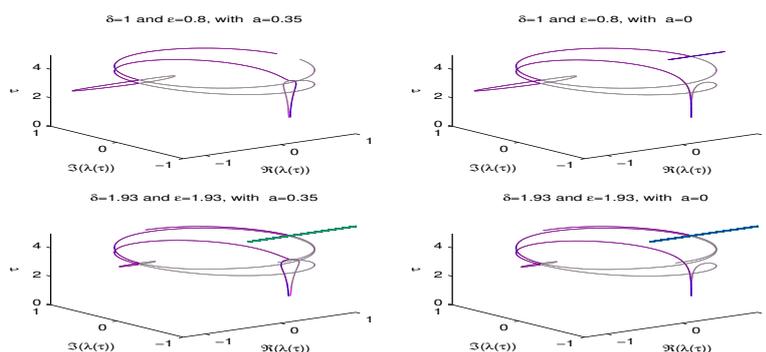


Figure 2: spectral portrait of $\tilde{X}_a(\tau, \delta, \varepsilon)$ for $\tau \in [0, 2\pi]$ and $(\delta, \varepsilon) \in \{(1, 0.8), (1.93, 1.93)\}$ with $a \in \{0, 0.35\}$.

perturbations on the movement of pendulums do not change the nature of the spectral portrait of $\tilde{X}_a(\tau, \delta, \varepsilon)$.

For all $(\delta, \varepsilon) \in [0, 1.98] \times [0, 2]$, Figure 3 shows the stability region of $\tilde{X}_a(2\pi, \delta, \varepsilon)$. The first figure (left) shows areas of stability in blue and of instability in red when our system is subject to a rank-2 perturbation (with $a = 0.35$). The second figure (right) also shows the zone of stability in blue and of instability in red of the unperturbed system (i.e. $a = 0$). Thus we notice a slight difference between the two figures due to the small rank- k perturbation of the system described by our two uncoupled pendulums.

5.1.2. Coupled double pendulums with oscillating supports

In this part, the two simple pendulums are coupled by a spring of constant stiffness k (see Figure 4). According to [14], the motion of the system is governed by the following differential system :

$$\frac{d^2x}{dt^2} + \left(B_0 - \frac{c}{k_0^2} \frac{d^2f}{dt^2} I_2 \right) x = 0, \quad (20)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad B_0 = \begin{bmatrix} \frac{cg}{k_0^2} + \frac{kb^2}{mk_0^2} & -\frac{kb^2}{mk_0^2} \\ -\frac{kb^2}{mk_0^2} & \frac{cg}{k_0^2} + \frac{kb^2}{mk_0^2} \end{bmatrix},$$

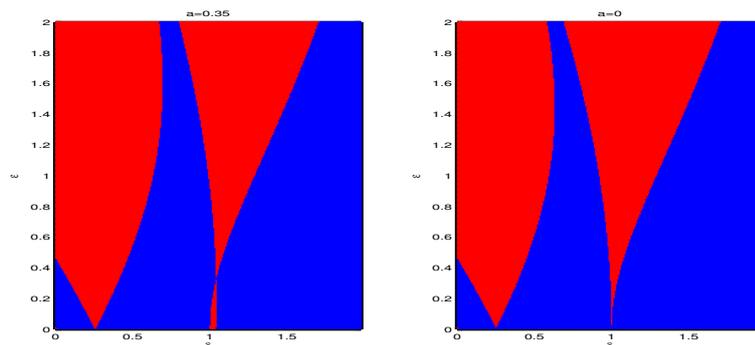


Figure 3: Stability zone of matrix $\tilde{X}_a(2\pi, \delta, \varepsilon), \forall (\delta, \varepsilon) \in [0, 1.98] \times [0, 2]$ and $a \in \{0, 0.35\}$.

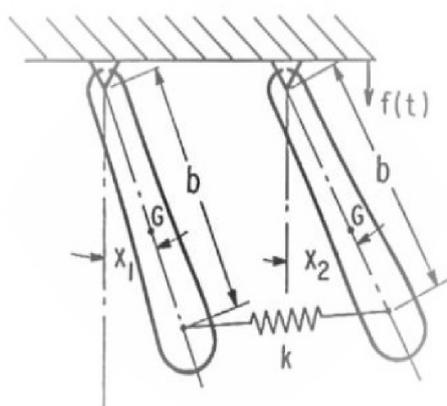


Figure 4: Model of the coupled double pendulum with oscillating supports.

with m is the mass of each pendulum and b is the distance between the point of suspension and the point of attachment of the coupling spring.

Replacing $f(t)$ by its expression in (20), equation (20) becomes:

$$\frac{d^2x}{dt^2} + \left(B_0 + \frac{c\alpha\Omega^2}{k_0^2} \cos(\Omega t) I_2 \right) x = 0. \tag{21}$$

Using successively the change of variables

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} \text{ and } \tau = \Omega t,$$

the equation of motion of the system can be reduced as (see [14])

$$\frac{d^2z_i}{d\tau^2} + (\delta_i + \varepsilon_i \cos(\tau)) z_i = 0, \quad i = 1, 2 \tag{22}$$

where

$$\delta_1 = \delta = \frac{cg}{k_0^2\Omega^2}, \quad \varepsilon_1 = \varepsilon = \frac{c\alpha}{k_0^2} \text{ and } \varepsilon_2 = \varepsilon + 2e, \text{ with } e = \frac{kb^2}{mk_0^2\Omega^2}.$$

Finally, using the change of variables given in (15) with $N = 2$, it is easy to see that the equation of motion of the coupled system can be reduce to form (1) with

$$H(\tau, \delta, \varepsilon, e) = \begin{pmatrix} P(\tau, \delta, \varepsilon, e) & 0_2 \\ 0_2 & I_2 \end{pmatrix} \quad \text{and}$$

$$P(\tau, \delta, \varepsilon, e) = \begin{pmatrix} \delta + \varepsilon \cos(\tau) & 0 \\ 0 & \delta + 2e + \varepsilon \cos(\tau) \end{pmatrix}.$$

Consider the rank-2 perturbation of the fundamental solution $\tilde{X}_a(\tau, \delta, \varepsilon, e)$ of its corresponding Hamiltonian system

$$E_a(\tau, \delta, \varepsilon, e) = U_a U_a^T J X(\tau, \delta, \varepsilon, e), \tag{23}$$

where U_a is defined in (17). In this case, it is easy to see that the equation of motion is of the form (19) [1, 2, 5].

Figure 5 represents the spectral portrait of the matrix $\tilde{X}_a(\tau, \delta, \varepsilon, e), \forall (\delta, \varepsilon, e) \in \{(1, 0.8, 0.5), (1.93, 1.93, 0.5)\}$ and $a \in \{0, 0.35\}, \forall \tau \in [0, 2\pi]$. The figures also show that the small

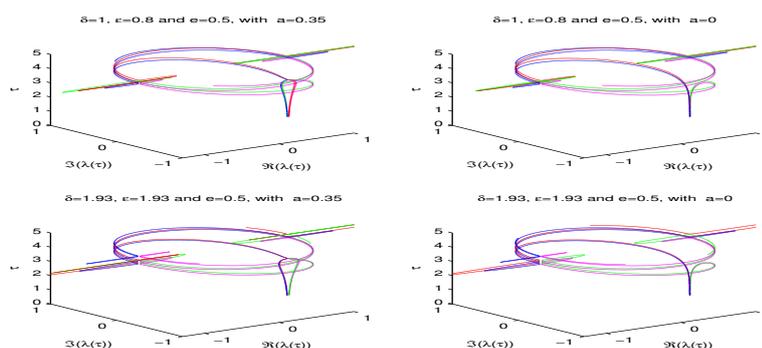


Figure 5: spectral portrait of $\tilde{X}_a(\tau, \delta, \varepsilon, e), \forall \tau \in [0, 2\pi]$ and $(\delta, \varepsilon, e) \in \{(1, 0.8, 0.5), (1.93, 1.93, 0.5)\}$, with $a \in \{0, 0.35\}$.

perturbation of rank-2 on the movement of pendulums do not change the nature of the spectral portrait of the fundamental solution $\tilde{X}(\tau, \delta, \varepsilon, e)$.

For all $(\delta, \varepsilon) \in [0, 1.98] \times [0, 2]$, Figure 6 shows the stability(strong) region of $\tilde{X}_a(2\pi, \delta, \varepsilon, e)$. The first figure (left) shows the zone of strong stability in white and instability in red when our system is subject to a rank-2 perturbation with $a = 0.35$. The second figure (right) also shows the zone of strong stability in white and instability in red of the unperturbed system ($a = 0$). However, we observe some points of stability in blue. Thus we notice a slight difference between the two figures due to the small rank-2 perturbations of the system described by our two coupled pendulums.

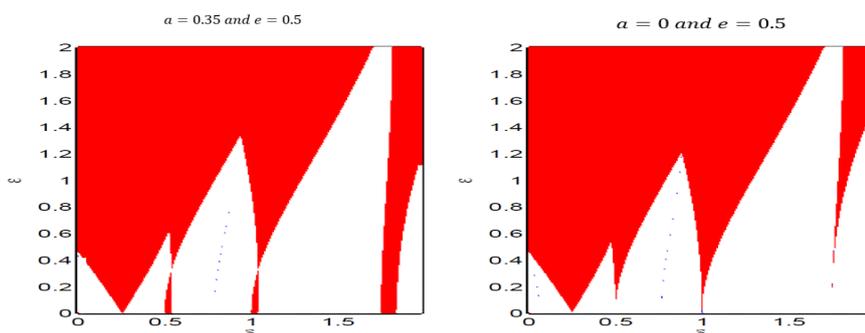


Figure 6: Stability(strong) of the matrix $\tilde{X}_\alpha(2\pi, \delta, \epsilon, e), \forall (\delta, \epsilon) \in [0, 1.98] \times [0, 2], \alpha \in \{0, 0.35\}$ and $e = 0.5$.

5.2. Motion of an ion through a quadrupole analyser

Consider an ion of mass m and of electric charge $|Ze|$ which moves with a velocity v through a quadrupole analyser of potential $\Phi_0 = U - V \cos(\omega t)$. Within the analyser, the ion experiences a force $f(x, y, t) = -Ze\nabla\mathcal{V}(x, y, t)$, where Z is the number of protons and e is the charge of a proton. We assume that the component of the electric field along the axis O_z is zero, and the component z of the velocity remains constant.

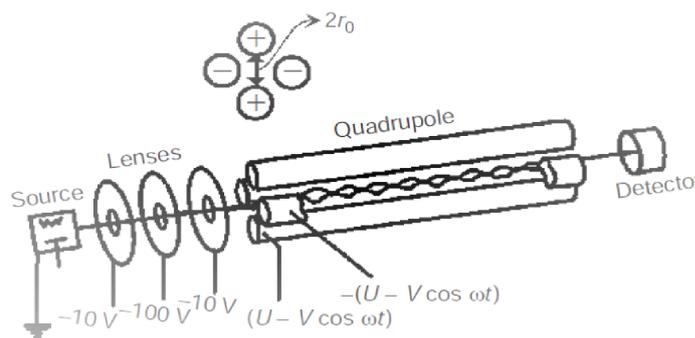


Figure 7: Model of a quadrupole analyzer.

According to [12], the motion of the ion through the analyser is governed by the following equation

$$\begin{cases} \frac{d^2x}{d\xi^2} + (\alpha - 2q \cos(2\xi)) x = 0 \\ \frac{d^2y}{d\xi^2} - (\alpha - 2q \cos(2\xi)) y = 0 \end{cases} \quad (24)$$

where

$$\alpha = \frac{8ZeU}{r_0^2 m \omega^2}, \quad q = \frac{4ZeV}{r_0^2 m \omega^2} \quad \text{and} \quad \xi = \frac{\omega t}{2}.$$

This equation was proposed in 1866 by physicist Mathieu to describe the propagation of waves in membranes. We apply the rank- k perturbation to this system in view of comparing the spectral portraits and the stability zones of the perturbed and unperturbed systems.

Using the change of variable given in (15) with $N = 2$, we obtain Hamiltonian system (1) with

$$H(\xi) = \begin{pmatrix} P(\xi) & 0_2 \\ 0_2 & I_2 \end{pmatrix} \text{ and } P(\xi) = \begin{pmatrix} \alpha - 2q \cos(2\xi) & 0 \\ 0 & -\alpha + 2q \cos(2\xi) \end{pmatrix}.$$

Considering that the motion of the ion is subjected to a perturbation of the type (16), the equation of the motion of the ion then becomes [2, 5]

$$\begin{cases} J \frac{d\tilde{X}_a(\xi, \alpha, q)}{d\xi} = \underbrace{(I - UU^T J)^T H(\xi, \alpha, q) (I - UU^T J)}_{\tilde{H}(\xi, \alpha, q)} \tilde{X}_a(\xi, \alpha, q), \\ \tilde{X}_a(0, \alpha, q) = I + UU^T J \end{cases} \quad (25)$$

where $U = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $a \in [0, 1[$.

Figure 8 represents the spectral portrait of the matrix $\tilde{X}_a(\alpha, q, \xi), \forall (\alpha, q) \in \{(0, 0.025), (0.1, 0.7)\}$, and $a = 0, 0.3003, \forall \xi \in [0, \pi]$. Once again, these figures show that the small

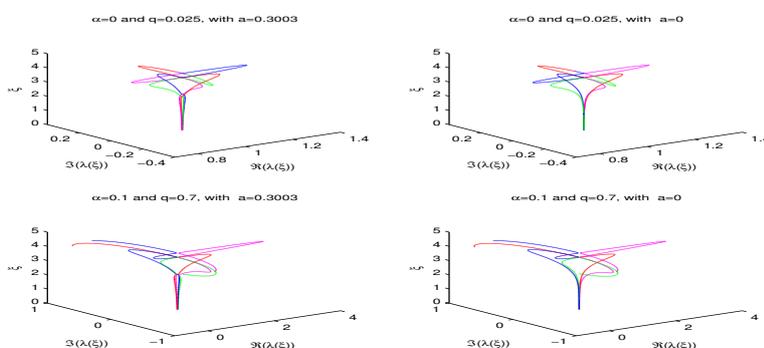


Figure 8: Spectral portrait of the matrix $\tilde{X}_a(\xi, \alpha, q), \forall \xi \in [0, \pi]$ and $(\alpha, q) \in \{(0, 0.025), (0.1, 0.7)\}$ with $a \in \{0, 0.3003\}$.

rang-2 perturbation on the movement of an ion through a quadrupole analyzer do not change the nature of the spectral portrait of $\tilde{X}_a(\xi, \alpha, q)$.

$\forall (\alpha, q) \in [0, 0.2] \times [0, 0.9]$, Figure 9 shows the stability(strong) zone of the matrix $\tilde{X}_a(\pi, \alpha, q)$. The first figure (to the left) shows the zone of strong stability in white color

and the zone of instability in red color when our system is subject to a rank-2 perturbation with $a = 0.35$. The second figure (to the right) also shows the zone of strong stability in white color and instability in red color of the unperturbed system ($a = 0$). However, we also see points of stability visible in blue on the first figure compared to the second. Thus we notice a slight difference between the two figures due to the small rank-2 perturbation of the system described by the movement of an ion through a quadrupole analyzer.

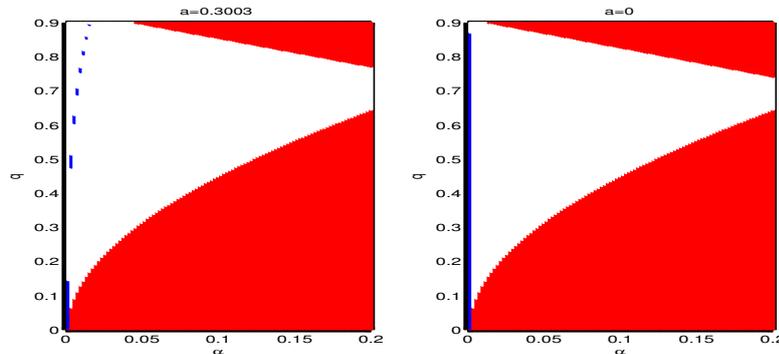


Figure 9: Stability(strong) zone of the matrix $\tilde{X}_a(\pi, \alpha, q)$, $\forall (\alpha, q) \in [0, 0.2] \times [0, 0.9]$ and $a \in \{0, 0.3003\}$.

6. Concluding remarks

From works by C. Mehl, et al.[16] on the perturbation theory of structured matrices, we presented Jordan canonical forms of rank- k perturbations of symplectic matrices and fundamental solutions of Hamiltonian system with periodic coefficients. These results show the effect of a k -rank perturbation on spectra of periodic Hamiltonian systems. Examples of applications on Mathieu systems have been proposed to check the small change of spectrum under small perturbations. Numerical simulations on the differential equations of the motion of two uncoupled or coupled pendulums and the movement of an ion through a quadrupole analyzer show a slight change in their spectra (thus in their stability zones).

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