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Approximation of a function in Hölder class using double Karamata ($K^{\lambda,\mu}$) method

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Abstract. In this paper, we establish a new theorem on the best approximation of a function of two variables belonging to Hölder class by double Karamata ($K^{\lambda,\mu}$) means of its double Fourier series.

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1. Introduction

K^{λ}-method was first introduced by Karamata [7]. Lotosky [10] re-introduced the special case $\lambda = 1$. Only after the study of Agnew [1], an intensive study of these and similar cases took place. Vuĉkoviĉ [19] applied this method for summability of Fourier series. Kathal[8] extended the result of Vuĉkoviĉ [19]. The approximation of a 2π-periodic function $f(x)$ in different Lipschitz classes using Cesàro, Nörlund and K^{λ} summability methods of Fourier series and conjugate Fourier series has been studied by the researchers [2, 5, 6, 12, 14–16].

The approximation of a 2π-periodic function $f(x)$ in Hölder metric by different summability transforms of Fourier series has been studied by the researchers like [4, 11, 13]. The approximation of a function $f(x, y)$ (2π-periodic with respect to the variables x, y) of their Fourier series has been studied by [17, 18]. Lal [9] studied the approximation of a function in Lipschitz class by matrix means of its double Fourier series. But nothing seems to have done to obtain the best approximation of the function $f(x, y)$, a 2π-periodic with respect to the variable x, y, of its double Fourier series. Thus, in this paper, we obtain the best approximation of the function $h(\zeta, \Theta)$ in Hölder class by $K^{\lambda,\mu}$ method of its double Fourier series.

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2. Preliminaries

Under usual assumptions, Fourier series of $h(t)$ is defined as

$$h(t) \sim \frac{1}{2}a_0 + \sum_{\rho=1}^{\infty}(a_\rho \cos \rho t + b_\rho \sin \rho t). \tag{1}$$

Under usual assumptions, double Fourier series of $h(\zeta, \Theta)$ is given by

$$h(\zeta,\Theta) \sim \sum_{\nu=0}^{\infty}\sum_{\rho=0}^{\infty}\alpha_{\nu,\rho}[a_{\nu,\rho}\cos\nu\zeta\cos\rho\Theta + b_{\nu,\rho}\sin\nu\zeta\cos\rho\Theta + c_{\nu,\rho}\cos\nu\zeta\sin\rho\Theta + d_{\nu,\rho}\sin\nu\zeta\sin\rho\Theta], \tag{2}$$

where

$$\alpha_{\nu,\rho} = \begin{cases} \frac{1}{4} & \text{for} \quad \nu = \rho = 0 \\ \frac{1}{2} & \text{for} \quad \nu > 0, \rho = 0 \text{ and } \nu = 0, \ \rho > 0 \\ 1 & \text{for} \quad \nu > 0, \rho > 0. \end{cases} \tag{3}$$

and

$$a_{\nu,\rho} = \frac{1}{\pi^2}\iint_{S^2} h(\zeta,\Theta)\cos\nu\zeta\cos\rho\Theta \, d\zeta \, d\Theta \tag{4}$$

with similar expressions for $b_{\nu,\rho}, c_{\nu,\rho}$ and $d_{\nu,\rho}$ for $\nu = 0, 1, 2, \ldots$ and $\rho = 0, 1, 2, \ldots$, where S^2 denotes the fundamental square $(-\pi, \pi; -\pi, \pi)$.
The partial sums of (2) can be denoted by

$$s_{\nu,\rho}(h;\zeta,\Theta) = \sum_{i=0}^{\nu}\sum_{j=0}^{\rho}[a_{ij}\cos i\zeta\cos j\Theta + b_{ij}\sin i\zeta\cos j\Theta + c_{ij}\cos i\zeta\sin j\Theta + d_{ij}\sin i\zeta\sin j\Theta].$$

We can also write

$$s_{\nu,\rho}(\zeta,\Theta) = \frac{1}{\pi^2}\iint_{S^2} h(\zeta+\sigma, \Theta+\tau)\frac{\sin\left(\nu+\frac{1}{2}\right)\sigma\sin\left(\rho+\frac{1}{2}\right)\tau}{4\sin\left(\frac{\sigma}{2}\right).\sin\left(\frac{\tau}{2}\right)} \, d\sigma \, d\tau.$$

The number $\begin{bmatrix}\nu\\p\end{bmatrix}$ for $0 \le p \le \nu$ and $\nu \in \mathbb{N} \cup \{0\}$ is defined by

$$\prod_{l=0}^{\nu-1}(\zeta+l) = \zeta(\zeta+1)\cdots(\zeta+\nu-1) = \sum_{p=0}^{\nu}\begin{bmatrix}\nu\\p\end{bmatrix}\zeta^p = \frac{\Gamma(\zeta+\nu)}{\Gamma\zeta}.$$

Let us also define for $\rho \in \mathbb{N} \cup \{0\}$, the number $\begin{bmatrix}\rho\\q\end{bmatrix}$ for $0 \le q \le \rho$, by

$$\prod_{l=0}^{\rho-1}(\Theta+l) = \Theta(\Theta+1)\cdots(\Theta+\rho-1) = \sum_{q=0}^{\rho}\begin{bmatrix}\rho\\q\end{bmatrix}\Theta^q = \frac{\Gamma(\Theta+\rho)}{\Gamma\Theta}.$$

The numbers $\begin{bmatrix}\nu\\p\end{bmatrix}$ and $\begin{bmatrix}\rho\\q\end{bmatrix}$ are called the absolute values of Stirling numbers of first kind.

Let $\{s_{\nu,\rho}\}$ be the sequence of partial sums of double infinite series $\sum_{\nu=0}^{\infty}\sum_{\rho=0}^{\infty} a_{\nu,\rho}$ and write

$$s_{\nu,\rho}^{\lambda,\mu} = \frac{\Gamma\lambda}{\Gamma(\lambda+\nu)} \times \frac{\Gamma\mu}{\Gamma(\mu+\rho)} \sum_{p=0}^{\nu}\sum_{q=0}^{\rho} \begin{bmatrix}\nu\\p\end{bmatrix}\begin{bmatrix}\rho\\q\end{bmatrix} \lambda^p\mu^q s_{p,q} \tag{5}$$

to denote $(\nu,\rho)^{th}$ $K^{\lambda,\mu}$-means of order $(\lambda,\mu) > 0$. If

$$s_{\nu,\rho}^{\lambda,\mu} \to s \text{ as } (\nu,\rho) \to \infty, \tag{6}$$

then the sequence $s_{\nu,\rho}$ or the series $\sum_{\nu=0}^{\infty}\sum_{\rho=0}^{\infty} a_{\nu,\rho}$ summable to s by double Karamata ($K^{\lambda,\mu}$) method of order $(\lambda,\mu) > 0$.
Thus,

$$s_{\nu,\rho}^{\lambda,\mu} \to s(K^{\lambda,\mu}) \text{ as } (\nu,\rho) \to \infty.$$

The method $K^{\lambda,\mu}$ is regular for $(\lambda,\mu) > 0$. The regularity of the $K^{\lambda,\mu}$ method is supposed throughout the paper.
"Since $h(\zeta)$ is continuous and 2π-periodic function then the Hölder class for $h(\zeta)$ is defined as

$$H_\alpha = \{h \in C_{2\pi} : |h(\zeta) - h(\Theta)| \le K|\zeta - \Theta|\},$$

where K is a positive constant. It can be verified that H_α is a Banach space with the norm $\|.\|_\alpha$ defined by

$$\|h\|_\alpha = \|h\|_C + \sup_{\zeta\neq\Theta} \Delta^\alpha h(\zeta,\Theta), \tag{7}$$

where

$$\Delta^\alpha h(\zeta,\Theta) = \frac{|h(\zeta) - h(\Theta)|}{|\zeta - \Theta|^\alpha} \text{ for } \zeta \neq \Theta.$$

By convention $\Delta^0 h(\zeta,\Theta) = 0$ and $\|h\|_C = \sup_{\zeta\in[-\pi,\pi]} |h(\zeta)|$. The metric induced by the norm (7) on H_α is called the Hölder metric [13]."
Since $h(\zeta,\Theta)$ is continuous and 2π-periodic function then the Hölder class for $h(\zeta,\Theta)$ is defined as

$$H_{\alpha,\beta} = \left\{h : |h(\zeta,\Theta;z,w) := |h(\zeta,\Theta) - h(z,w)| \le C_1\left(|\zeta - z|^\alpha + |\Theta - w|^\beta\right)\right\}$$

for some $\alpha,\beta > 0$ and for all ζ,Θ,z,w. In above class of function, C_1 is some positive constant, which may depend on h, but not on ζ,Θ,t. $H_{\alpha,\beta}$ class of function is identical to Lip(α,β) class of function.
$H_{\alpha,\beta}$ is a Banach space, whose norm $\|.\|_{\alpha,\beta}$ is defined by

$$\|h\|_{\alpha,\beta} = \|h\|_C + \sup_{\substack{\zeta\neq z,\\ \Theta\neq w}} \Delta^{\alpha,\beta} h(\zeta,\Theta;z,w) \tag{8}$$

i.e.

$$\|h\|_{\alpha,\beta} = \|h\|_C + \sup_{\substack{\zeta\neq z,\\ \Theta\neq w}} \frac{|f(\zeta,\Theta) - f(z,w)|}{|\zeta - z|^\alpha + |\Theta - w|^\beta} \text{ for } \zeta \neq z, \Theta \neq w$$

where $\Delta^{\alpha,\beta}h(\zeta,\Theta;z,w)=\frac{|h(\zeta,\Theta)-h(z,w)|}{|\zeta-z|^{\alpha}+|\Theta-w|^{\beta}}$ $(\zeta\neq z,\Theta\neq w)$.
By convention $\Delta^{0,0}h(\zeta,\Theta;z,w)=0$ and

$$\|h\|_C=\sup_{(\zeta,\Theta)\in S^2}|h(\zeta,\Theta)|. \tag{9}$$

"The η-order error of approximation of a function $h\in C_{2\pi}$ is defined by

$$E_\eta(h)=\inf_{t_\eta}\|h-t_\eta\|,$$

where t_η is a trigonometric polynomial of degree η (Bernstein [3])".
"If $E_\eta(h)\to 0$ as $\eta\to\infty$, then $E_\eta(h)$ is said to be the best approximation of h ([20])".
We write

$$\begin{aligned}
\phi(t)&=h(\zeta+t)+h(\zeta-t)-2h(\zeta).\\
\Phi(t)&=\int_0^t|\phi(\sigma)|\,d\sigma.\\
\Phi(\sigma,\tau)&=\Phi(\zeta,\Theta;\sigma,\tau)\\
&=\frac{1}{4}[h(\zeta+\sigma,\Theta+\tau)+h(\zeta+\sigma,\Theta-\tau)+h(\zeta-\sigma,\Theta+\tau)\\
&+h(\zeta-\sigma,\Theta-\tau)-4h(\zeta,\Theta)]
\end{aligned}$$

where $\Psi(\sigma,\tau):=\Psi(z,w;\sigma,\tau)$.

Since $h(\zeta,\Theta)\in H_{\alpha,\beta}$, then $|F(\sigma,\tau)|=O(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta})$.

$$\begin{aligned}
K_\nu^\lambda(\sigma)&=\frac{\sum_{p=0}^{\nu}\begin{bmatrix}\nu\\p\end{bmatrix}\lambda^p\sin\left(p+\frac{1}{2}\right)\sigma}{\Gamma(\lambda+\nu)\sin\left(\frac{\sigma}{2}\right)}.\\
K_\rho^\mu(\tau)&=\frac{\sum_{q=0}^{\rho}\begin{bmatrix}\rho\\q\end{bmatrix}\mu^q\sin\left(q+\frac{1}{2}\right)\tau}{\Gamma(\mu+\rho)\sin\left(\frac{\tau}{2}\right)}.
\end{aligned}$$

3. Main Theorem

Theorem 1. *The best approximation of a 2π-periodic function $h(\zeta,\Theta)$ of two variables ζ and Θ and Lebesgue integrable over $S^2(-\pi,\pi;-\pi,\pi)$ in $H_{\alpha,\beta}$, $0<\alpha,\beta\leq 1$ class by double Karamata ($K^{\lambda,\mu}$) method of its double Fourier series is given by*

$$\begin{aligned}
&\|s_{\nu,\rho}^{\lambda,\mu}(\zeta,\Theta)-h(\zeta,\Theta)\|_{\alpha,\beta}\\
&=O\left[\frac{M\ N\ \Gamma\lambda\Gamma\mu}{(\nu+1)(\rho+1)}\left(\frac{1}{(\nu+1)^{\alpha}}+\frac{1}{(\rho+1)^{\beta}}+1\right)\right]\\
&+O\left[\frac{M\ \Gamma\lambda\Gamma\mu}{(\nu+1)\Gamma(\mu+\rho)}\left(1+\frac{\ln\pi(\rho+1)}{(\nu+1)^{\alpha}}+\ln\pi(\rho+1)\right)\right]
\end{aligned}$$

$$+ O\left[\frac{N\ \Gamma\lambda\Gamma\mu}{(\rho+1)\Gamma(\lambda+\nu)}\left(1+\frac{\ln\pi(\rho+1)}{(\rho+1)^{\beta}}+\ln\pi(\nu+1)\right)\right]$$
$$+ O\left[\frac{\Gamma\lambda\Gamma\mu}{\Gamma(\lambda+\nu)\Gamma(\mu+\rho)}\left(\ln\big((\nu+1)(\rho+1)\pi^2\big)+\{\ln\pi(\nu+1)\}\{\ln\pi(\rho+1)\}\right)\right]$$

where $M = \lambda\ln(\nu+1)+1$ *and* $N = \mu\ln(\rho+1)+1.$

4. Lemmas

Lemma 1. *"([19]). Let* $\lambda > 0$ *and* $0 < t < \frac{\pi}{2}$ *then*

$$\frac{Im\Gamma(\lambda e^{it}+\rho)}{\Gamma(\lambda\cos t+\rho)\sin\left(\frac{t}{2}\right)} = \frac{|\sin(\lambda\ln(\rho+1).\sin t)|}{\sin\left(\frac{t}{2}\right)} + O(1)$$

as $\rho\to\infty$ *uniformly in* t*".*

Lemma 2. *"([12]). For* $0 < \sigma < \frac{1}{\nu+1}$*,*

$$K_{\nu}^{\lambda}(\sigma) = O\left[\lambda\ln(\nu+1)\right] + O(1)$$

and for $0 < \tau < \frac{1}{\rho+1}$*,*

$$K_{\rho}^{\mu}(\tau) = O\left[\mu\ln(\rho+1)\right] + O(1)$$

".

Lemma 3. *For* $\frac{1}{\nu+1} \leq \sigma \leq \pi$

$$K_{\nu}^{\lambda}(\sigma) = O\left[\frac{1}{\sigma\ \Gamma\lambda}\right].$$

Proof. Using $\sin\frac{\sigma}{2} \geq \frac{\sigma}{\pi}$ and $|\sin(\rho\sigma)| \leq 1$

$$\begin{aligned}|K_{\nu}^{\lambda}(\sigma)| &= \left|\frac{\sum_{p=0}^{\nu}\begin{bmatrix}\nu\\p\end{bmatrix}\lambda^p\sin\left(p+\frac{1}{2}\right)\sigma}{\Gamma(\lambda+\nu)\sin\left(\frac{\sigma}{2}\right)}\right| \\ &\leq \frac{1}{\Gamma(\lambda+\nu)}\sum_{p=0}^{\nu}\begin{bmatrix}\nu\\p\end{bmatrix}\lambda^p\frac{1}{\sin\left(\frac{\sigma}{2}\right)} \\ &= O\left[\frac{1}{\sigma\ \Gamma\lambda}\right].\end{aligned}$$

Lemma 4. *For* $\frac{1}{\rho+1} \leq \tau \leq \pi$

$$K_{\rho}^{\mu}(\tau) = O\left[\frac{1}{\tau\ \Gamma\mu}\right].$$

Proof. This can be proved along the same lines of Lemma 4.3.

5. Proof of the Main Theorem

Let $s_{\nu,\rho}(\zeta,\Theta)$ denote the $(\nu,\rho)^{th}$ partial sum of the series (2), we have

$$s_{\nu,\rho}(\zeta,\Theta)-h(\zeta,\Theta)=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi\Phi(\sigma,\tau)\frac{\sin\left(\nu+\frac{1}{2}\right)\sigma}{\sin\left(\frac{\sigma}{2}\right)}.\frac{\sin\left(\rho+\frac{1}{2}\right)\tau}{\sin\left(\frac{\tau}{2}\right)}\,d\sigma\,d\tau.$$

Denoting $K^{\lambda,\mu}$ means of $\{s_{\nu,\rho}(\zeta,\Theta)\}$ by $s^{\lambda,\mu}_{\nu,\rho}(\zeta,\Theta)$, we get

$$\begin{aligned}
&s^{\lambda,\mu}_{\nu,\rho}(\zeta,\Theta)-h(\zeta,\Theta)\\
&=\frac{\Gamma\lambda}{\Gamma(\lambda+\nu)}.\frac{\Gamma\mu}{\Gamma(\mu+\rho)}\sum_{p=0}^{\nu}\sum_{q=0}^{\rho}\begin{bmatrix}\nu\\p\end{bmatrix}\begin{bmatrix}\rho\\q\end{bmatrix}\lambda^p\mu^q\left(s_{p,q}(\zeta,\Theta)-h(\zeta,\Theta)\right)\\
&=\frac{\Gamma\lambda\Gamma\mu}{\pi^2}\int_0^\pi\int_0^\pi\Phi(\sigma,\tau)\sum_{p=0}^{\nu}\begin{bmatrix}\nu\\p\end{bmatrix}\frac{\lambda^p\sin\left(p+\frac{1}{2}\right)\sigma}{\Gamma(\lambda+\nu)\sin\left(\frac{\sigma}{2}\right)}\sum_{q=0}^{\rho}\begin{bmatrix}\rho\\q\end{bmatrix}\frac{\mu^q\sin\left(q+\frac{1}{2}\right)\tau}{\Gamma(\mu+\rho)\sin\left(\frac{\tau}{2}\right)}\,d\sigma\,d\tau\\
&=\frac{\Gamma\lambda\Gamma\mu}{\pi^2}\int_0^\pi\int_0^\pi\Phi(\sigma,\tau)K^\lambda_\nu(\sigma)K^\mu_\rho(\tau)\,d\sigma\,d\tau\\
&=I_{\nu,\rho}(\zeta,\Theta)\quad\text{(say)}.
\end{aligned}\tag{10}$$

Let us estimate

$$\sup_{\substack{\zeta\neq z,\\ \Theta\neq w}}\frac{|I_{\nu,\rho}(\zeta,\Theta)-I_{\nu,\rho}(z,w)|}{|\zeta-z|^\alpha+|\Theta-w|^\beta}=O(1).\tag{11}$$

Now,

$$\begin{aligned}
&|I_{\nu,\rho}(\zeta,\Theta)-I_{\nu,\rho}(z,w)|\\
&=\frac{\Gamma\lambda\Gamma\mu}{\pi^2}\left|\int_0^\pi\int_0^\pi F(\sigma,\tau)K^\lambda_\nu(\sigma)K^\mu_\rho(\tau)\,d\sigma\,d\tau\right|\\
&\leq\frac{\Gamma\lambda\Gamma\mu}{\pi^2}\left(\int_0^{\frac{1}{\nu+1}}\int_0^{\frac{1}{\rho+1}}+\int_0^{\frac{1}{\nu+1}}\int_{\frac{1}{\rho+1}}^{\pi}+\int_{\frac{1}{\nu+1}}^{\pi}\int_0^{\frac{1}{\rho+1}}+\int_{\frac{1}{\nu+1}}^{\pi}\int_{\frac{1}{\rho+1}}^{\pi}\right)|F(\sigma,\tau)K^\lambda_\nu(\sigma)K^\mu_\rho(\tau)|\,d\sigma\,d\tau\\
&=O\left[\frac{\Gamma\lambda\Gamma\mu}{\pi^2}(J_1+J_2+J_3+J_4)\right].
\end{aligned}\tag{12}$$

Using the fact that $|F(\sigma,\tau)|=O(|\zeta-z|^\alpha+|\Theta-w|^\beta)$ and Lemma 4.2 for $0<\sigma<\frac{1}{\nu+1}$, we obtain

$$\begin{aligned}
J_1&=\int_0^{\frac{1}{\nu+1}}\int_0^{\frac{1}{\rho+1}}|F(\sigma,\tau)K^\lambda_\nu(\sigma)K^\mu_\rho(\tau)|\,d\sigma\,d\tau\\
&=[O\{\lambda\ln(\nu+1)\}+O(1)]\,[O\{\mu\ln(\rho+1)\}+O(1)]\int_0^{\frac{1}{\nu+1}}\int_0^{\frac{1}{\rho+1}}|F(\sigma,\tau)|\,d\sigma\,d\tau\\
&=O(1)[\{\lambda\ln(\nu+1)\}+1][\{\mu\ln(\rho+1)\}+1][(|\zeta-z|^\alpha+|\Theta-w|^\beta)]\int_0^{\frac{1}{\nu+1}}\int_0^{\frac{1}{\rho+1}}\,d\sigma\,d\tau
\end{aligned}$$

$$= O(1)\left\{\frac{[\{\lambda\ln(\nu+1)\}+1]}{\nu+1}\times\frac{[\{\mu\ln(\rho+1)\}+1]}{\rho+1}\right\}(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta}). \tag{13}$$

For $0<\alpha,\beta\leq 1$, by using Lemmas 4.2 for $0<\sigma<\frac{1}{\nu+1}$, 4.4 and the fact that $|F(\sigma,\tau)| = O(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta})$, we get

$$\begin{aligned}
J_2 &= \int_0^{\frac{1}{\nu+1}}\int_{\frac{1}{\rho+1}}^{\pi}|F(\sigma,\tau)K_\nu^\lambda(\sigma)K_\rho^\mu(\tau)|\,d\sigma\,d\tau\\
&= O\{\lambda\log(\nu+1)+1\}\int_0^{\frac{1}{\nu+1}}\int_{\frac{1}{\rho+1}}^{\pi}|F(\sigma,\tau)||K_\rho^\mu(\tau)|\,d\sigma\,d\tau\\
&= O(1)\frac{\{\lambda\log(\nu+1)+1\}}{\nu+1}\int_{\frac{1}{\rho+1}}^{\pi}|F(\sigma,\tau)||K_\rho^\mu(\tau)|\,d\tau\\
&= O\left\{\frac{\{\lambda\ln(\nu+1)\}+1}{\nu+1}\right\}(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta})\int_{\frac{1}{\rho+1}}^{\pi}\frac{1}{\tau\Gamma\mu}\,d\tau
\end{aligned}$$

$$= O\left\{\frac{\{\lambda\ln(\nu+1)\}+1}{(\nu+1)\Gamma\mu}\right\}(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta})\ln\pi(\rho+1). \tag{14}$$

Similarly by changing the order of integration in J_3 and using Lemmas 4.2 for $0<\tau<\frac{1}{\rho+1}$ and 4.3, we obtain

$$J_3 = O\left\{\frac{\{\mu\ln(\rho+1)\}+1}{(\rho+1)\Gamma\lambda}\right\}(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta})\ln((\nu+1)\pi). \tag{15}$$

Now, using Lemmas 4.3 and 4.4, we get

$$\begin{aligned}
J_4 &= \int_{\frac{1}{\nu+1}}^{\pi}\int_{\frac{1}{\rho+1}}^{\pi}|F(\sigma,\tau)K_\nu^\lambda(\sigma)K_\rho^\mu(\tau)|\,d\sigma\,d\tau\\
&= \int_{\frac{1}{\nu+1}}^{\pi}K_\nu^\lambda(\sigma)\left(\int_{\frac{1}{\rho+1}}^{\pi}\frac{1}{\tau\Gamma\mu}\,d\tau\right)|F(\sigma,\tau)|\,d\sigma\\
&= O\left\{\frac{1}{\Gamma\mu}\right\}\int_{\frac{1}{\nu+1}}^{\pi}K_\nu^\lambda(\sigma)\ln((\rho+1)\pi)|F(\sigma,\tau)|\,d\sigma\\
&= O\left\{\frac{\ln((\rho+1)\pi).\ln((\nu+1)\pi)}{\Gamma\lambda\Gamma\mu}(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta})\right\}.
\end{aligned} \tag{16}$$

Combining (12) to (16), we obtain

$$\begin{aligned}
&\frac{|I_{\nu,\rho}(\zeta,\Theta)-I_{\nu,\rho}(z,w)|}{(|\zeta-z|^{\alpha}+|\Theta-w|^{\beta})}\\
&= O\left(\frac{[\{\lambda\ln(\nu+1)\}+1][\{\mu\ \ln(\rho+1)\}+1]\Gamma\lambda\Gamma\mu}{(\nu+1)(\rho+1)}+\frac{[\{\lambda\ln(\nu+1)\}+1]\Gamma\lambda\ \ln\pi(\rho+1)}{\nu+1}\right)
\end{aligned}$$

$$+O\left(\frac{[\{\mu\ln(\rho+1)\}+1]\Gamma\mu\ln\pi(\nu+1)}{\rho+1}+\frac{\ln(\rho+1)\pi)\ln((\nu+1)\pi)}{\Gamma\lambda\Gamma\mu}\right). \tag{17}$$

Now, from (10), we have

$$\begin{aligned}
&|I_{\nu,\rho}(\zeta,\Theta)| \\
&= \frac{\Gamma\lambda\Gamma\mu}{\pi^2}\left|\int_0^\pi\int_0^\pi \Phi(\sigma,\tau)K_\nu^\lambda(\sigma)K_\rho^\mu(\tau)\,d\sigma\,d\tau\right| \\
&\leq \frac{\Gamma\lambda\Gamma\mu}{\pi^2}\left[\int_0^{\frac{1}{\nu+1}}\int_0^{\frac{1}{\rho+1}}+\int_0^{\nu+1}\int_{\frac{1}{\rho+1}}^\pi+\int_{\frac{1}{\nu+1}}^\pi\int_0^{\frac{1}{\rho+1}}+\int_{\frac{1}{\nu+1}}^\pi\int_{\frac{1}{\rho+1}}^\pi |\Phi(\sigma,\tau)||K_\nu^\lambda(\sigma)||K_\rho^\mu(\tau)|\,d\sigma\,d\tau\right] \\
&= \frac{\Gamma\lambda\Gamma\mu}{4\pi^2}(I_1+I_2+I_3+I_4).
\end{aligned} \tag{18}$$

Now,

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{\nu+1}}\int_0^{\frac{1}{\rho+1}}|\Phi(\sigma,\tau)||K_\nu^\lambda(\sigma)||K_\rho^\mu(\tau)|\,d\sigma\,d\tau \\
I_1 &= O\left[(\lambda\log(\nu+1)+1)(\mu\log(\rho+1)+1)\int_0^{\frac{1}{\nu+1}}\int_0^{\frac{1}{\rho+1}}(\sigma^\alpha+\tau^\beta)\,d\sigma\,d\tau\right] \\
&= O\left[MN\int_0^{\frac{1}{\nu+1}}\sigma^\alpha\left\{\int_0^{\frac{1}{\rho+1}}d\tau\right\}d\sigma\right]+O\left[MN\int_0^{\frac{1}{\rho+1}}\tau^\beta\left\{\int_0^{\frac{1}{m+1}}d\sigma\right\}d\tau\right] \\
&\text{where } M=\{\lambda\ln(\nu+1)\}+1 \text{ and } N=\{\mu\ln(\rho+1)\}+1 \\
&= \left[\frac{MN}{\rho+1}\int_0^{\frac{1}{\nu+1}}\sigma^\alpha\,d\sigma\right]+\left[\frac{MN}{\nu+1}\int_0^{\frac{1}{\rho+1}}\tau^\beta\,d\tau\right] \\
&= O(1)\frac{MN}{(\nu+1)(\rho+1)}\left[\frac{1}{(\nu+1)^\alpha}+\frac{1}{(\rho+1)^\beta}\right].
\end{aligned} \tag{19}$$

$$\begin{aligned}
I_2 &= \int_0^{\frac{1}{\nu+1}}\int_{\frac{1}{\rho+1}}^\pi|\Phi(\sigma,\tau)||K_\nu^\lambda(\sigma)||K_\rho^\mu(\tau)|\,d\sigma\,d\tau \\
&= O\left[\{\lambda\log(\nu+1)+1\}\int_0^{\frac{1}{\nu+1}}\int_{\frac{1}{\rho+1}}^\pi\frac{1}{\tau\Gamma\mu}(\sigma^\alpha+\tau^\beta)\,d\sigma\,d\tau\right] \\
&= O\left[\frac{}{\Gamma\mu}\int_0^{\frac{1}{\nu+1}}\int_{\frac{1}{\rho+1}}^\pi\frac{1}{\tau}\sigma^\alpha\,d\sigma\,d\tau\right]+O\left[\frac{M}{\Gamma\mu}\int_0^{\frac{1}{\nu+1}}\int_{\frac{1}{\rho+1}}^\pi\frac{1}{\tau}\tau^\beta\,d\sigma\,d\tau\right] \\
&= O(1)\frac{M}{(\nu+1)\Gamma\mu}\left(\frac{\ln(\rho+1)\pi}{(\nu+1)^\alpha}+1\right).
\end{aligned} \tag{20}$$

Similarly,

$$I_3 = O(1)\frac{N}{(\rho+1)\Gamma\lambda}\left(\frac{\ln(\nu+1)\pi}{(\rho+1)^\beta}+1\right) \tag{21}$$

and

$$\begin{aligned}
I_4 &= \int_{\frac{1}{\nu+1}}^{\pi}\int_{\frac{1}{\rho+1}}^{\pi} |\Phi(\sigma,\tau)||K_\nu^\lambda(\sigma)||K_\rho^\mu(\tau)|\,d\sigma\,d\tau \\
&= O\left[\int_{\frac{1}{\nu+1}}^{\pi}\int_{\frac{1}{\rho+1}}^{\pi} \frac{1}{\sigma\tau\Gamma\lambda\Gamma\mu}[\sigma^\alpha+\tau^\beta]\,d\sigma\,d\tau\right] \\
&= O\left[\frac{1}{\Gamma\lambda\Gamma\mu}\int_{\frac{1}{\nu+1}}^{\pi}\frac{\sigma^\alpha}{\sigma}\left\{\int_{\frac{1}{\rho+1}}^{\pi}\frac{d\tau}{\tau}\right\}d\sigma\right] + O\left[\frac{1}{\Gamma\lambda\Gamma\mu}\int_{\frac{1}{\nu+1}}^{\pi}\frac{1}{\sigma}\left\{\int_{\frac{1}{\rho+1}}^{\pi}\frac{\tau^\beta}{\tau}\,d\tau\right\}d\sigma\right].
\end{aligned}$$

$$\begin{aligned}
&= O\left[\frac{1}{\Gamma\lambda\Gamma\mu}\ln(\rho+1)\pi\right] + O\left[\frac{1}{\Gamma\lambda\Gamma\mu}\ln(\nu+1)\pi\right] \\
&= O\left[\frac{1}{\Gamma\lambda\Gamma\mu}\ln(\nu+1)(\rho+1)\pi^2\right] \qquad (22)
\end{aligned}$$

Combining (18) to (22), we get

$$\begin{aligned}
I_{\nu,\rho} &= O\left[\frac{\Gamma\lambda\Gamma\mu}{\pi^2}\left\{\frac{MN}{(\nu+1)(\rho+1)}\left(\frac{1}{(\nu+1)^\alpha}+\frac{1}{(\rho+1)^\beta}\right)+\frac{M}{(\nu+1)\Gamma\mu}\left(1+\frac{\ln(\rho+1)\pi}{(\nu+1)^\alpha}\right)\right\}\right] \\
&+ O\left[\frac{\Gamma\lambda\Gamma\mu}{\pi^2}\left\{\frac{N}{(\nu+1)\Gamma\lambda}\left(1+\frac{\ln(\nu+1)\pi}{(\rho+1)^\beta}\right)+\frac{1}{\Gamma\lambda\Gamma\mu}\log(\nu+1)(\rho+1)\pi^2\right\}\right]. \qquad (23)
\end{aligned}$$

By (17) and (23), we obtain

$$\begin{aligned}
\|I_{\nu,\rho}\|_{\alpha,\beta} &= \|s_{\nu,\rho}^{\lambda;\mu}(\zeta,\Theta)-h(\zeta,\Theta)\|_{\alpha,\beta} = O\left[\frac{MN\Gamma\lambda\Gamma\mu}{(\nu+1)(\rho+1)}\left(\frac{1}{(\nu+1)^\alpha}+\frac{1}{(\rho+1)^\beta}+1\right)\right] \\
&+ O\left[\frac{M\Gamma\lambda\Gamma\mu}{(\nu+1)\Gamma(\mu+\rho)}\left(1+\frac{\log((\rho+1)\pi)}{(\nu+1)^\alpha}+\log((\rho+1)\pi)\right)\right] \\
&+ \left[\frac{N\Gamma\lambda\Gamma\mu}{(\nu+1)\Gamma(\lambda+\nu)}\left(1+\frac{\log((\rho+1)\pi)}{(\rho+1)^\beta}+\log((\nu+1)\pi)\right)\right] \\
&+ O\left[\frac{\Gamma\lambda\Gamma\mu}{\Gamma(\lambda+\nu)\Gamma(\mu+\rho)}\left(\log\big((\nu+1)(\rho+1)\pi^2\big)+\{\log((\nu+1)\pi)\}\{\log((\rho+1)\pi)\big)\right\}\right]
\end{aligned}$$

where $M=\{\lambda\ln(\nu+1)\}+1$ and $N=\{\mu\ln(\rho+1)\}+1$.

6. Verification

We calculate error by putting some values of $\nu,\rho,\alpha,\beta,\lambda,\mu$.

6.1. $\nu=\rho=10, \alpha=\beta=0.5, \lambda=2, \mu=3$

error $E_1 \sim 1.25827$

6.2. $\nu = \rho = 20, \alpha = \beta = 0.5, \lambda = 2, \mu = 3$

error $E_2 \sim 0.46798$

6.3. $\nu = \rho = 50, \alpha = \beta = 0.5, \lambda = 2, \mu = 3$

error $E_3 \sim 0.111632$

6.4. $\nu = \rho = 500, \alpha = \beta = 0.5, \lambda = 2, \mu = 3$

error $E_4 \sim 0.0011456$

6.5. $\nu = \rho = 1000, \alpha = \beta = 0.5, \lambda = 2, \mu = 3$

error $E_5 \sim 6.83193 \times 10^{-4}$.

6.6. $\nu = \rho = 10000, \alpha = \beta = 0.5, \lambda = 2, \mu = 3$

error $E_6 \sim 1.13411 \times 10^{-5}$.

6.7. $\nu = \rho = 100000, \alpha = \beta = 0.5, \lambda = 2, \mu = 3$

error $E_7 \sim 1.1191 \times 10^{-7}$.

6.8. $\nu = \rho = 10^{10}, \alpha = \beta = 0.5, \lambda = 2, \mu = 3$

error $E_8 \sim 5.36301 \times 10^{-15}$.

7. Conclusion

From above verification, we observed that error estimation approaches to zero rapidly as ν, ρ increase infinitely. Thus, we arrive at the best approximation of the function.

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