EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 3, 2020, 697-700
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Quasi-normality of Mrówka spaces

Ibtesam Alshammari[1,*], Lutfi Kalantan[2]

[1] *Department of Mathematics, Faculty of Science, University of Hafr Al Batin, P.O.Box 1803, Hafr Al Batin 31991, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, King Abdulaziz University*

Abstract. A topological space X is called *quasi-normal* if X is regular and any two disjoint π-closed subsets A and B of X are separated. We give a Mrówka space which is not quasi-normal and use the continuum hypothesis (CH) and truly cardinality $\mathfrak{c}$ to present Mrówka spaces which are quasi-normal.

2020 Mathematics Subject Classifications: 54D15, 54D10
Key Words and Phrases: Normal, π-normal, mildly normal, quasi-normal, closed domain, π-closed, Mrówka space, continuum hypothesis (CH), truly cardinality $\mathfrak{c}$

1. Introduction

In this paper, we give a Mrówka space which is not quasi-normal and use the continuum hypothesis (CH) and truly cardinality $\mathfrak{c}$ to present Mrówka spaces which are quasi-normal. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$ and the set of positive integers by $\mathbb{N}$. A T_4 space is a T_1 normal space, a Tychonoff ($T_{3\frac{1}{2}}$) space is a T_1 completely regular space, and a T_3 space is a T_1 regular space. For a subset A of a space X, $\text{int}A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω and the first uncountable ordinal is ω_1.

Definition 1. Two disjoint subsets E and F of a space X are called *separated* if there exist two disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$. A subset A of a space X is called *closed domain* [1], called also *regularly closed, κ-closed,* if $A = \overline{\text{int}A}$. A space X is called *mildly normal* [6], called also *κ-normal* [5], if any two disjoint closed domains A and B of X are separated. In [5], Stchepin required regularity in his definition of κ-normality. A subset A of a space X is called *π-closed* [8] if A is a finite intersection of closed domains. A space X is called *π-normal* [3] if any two disjoint closed subsets A and B of X one of which is π-closed are separated. A space X is called *quasi-normal* [8] if X is regular and any two disjoint π-closed subsets A and B of X are separated, see also [3].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i3.3770

Email addresses: lkalantan@hotmail.com (L. Kalantan), iealshamri@hotmail.com and iealshamri@uhb.edu.sa (I. Alshammari)

Since any closed domain is π-closed and any π-closed is closed, then it is clear from the definitions that

$$\text{normal} \Longrightarrow \pi\text{-normal} \Longrightarrow \text{quasi-normal} \Longrightarrow \text{mildly normal}.$$

Recall that two countably infinite sets are said to be *almost disjoint* [7] if their intersection is finite. Call a subfamily of $[\omega]^\omega = \{ A \subset \omega : A \text{ is infinite} \}$ a *mad family* [7] on ω if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily. Let $\mathcal{A}$ be a pairwise almost disjoint subfamily of $[\omega]^\omega$. The *Mrówka space* $\Psi(\mathcal{A})$ is defined as follows: The underlying set is $\omega \cup \mathcal{A}$, each point of ω is isolated, and a basic open neighborhood of $W \in \mathcal{A}$ has the form $\{W\} \cup (W \setminus F)$, with $F \in [\omega]^{<\omega} = \{B \subseteq \omega \colon B \text{ is finite}\}$.

2. Main Results

It is well known that there exists an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ such that $|\mathcal{A}| > \omega$ and the Mrówka space $\Psi(\mathcal{A})$ is a Tychonoff, separable, first countable, and locally compact space which is neither countably compact nor normal. And $\mathcal{A}$ is a mad family if and only if $\Psi(\mathcal{A})$ is pseudo compact [4].

The interesting thing about Mrówka spaces is that some Mrówka spaces are quasi-normal and some are not. In [2, 1.3], a mad family $\mathcal{R} \subset [\omega]^\omega$ was constructed such that the Mrówka space $\Psi(\mathcal{R})$ is not mildly normal. So, such a Mrówka space cannot be quasi-normal. Now, we use the continuum hypothesis (CH) to produce a mad family $\mathcal{A} \subset [\omega]^\omega$ such that its Mrówka space $\Psi(\mathcal{A})$ is quasi-normal. The existence of such a mad family in ZFC is still unsettled.

Proposition 1. *Under* **CH**, *there exists a mad family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is quasi-normal.*

Proof. Let $\mathcal{P} = \{P_i : i < \omega\}$ be a partition of ω such that for each $i < \omega$, P_i is infinite. We will use $\mathcal{P}$ to build our mad family. Let $\mathcal{E} = [[\omega]^\omega]^{<\omega}$. That is, the family of all finite subsets of $[\omega]^\omega$. Consider the family

$$\mathcal{B} = \{\langle C, D\rangle : C, D \in \mathcal{E}, (\cap C) \cap (\cap D) = \emptyset\}.$$

Using **CH**, we can write $\mathcal{B} = \{\langle C_\alpha, D_\alpha\rangle : \alpha < \omega_1\}$. We will build our mad family recursively on $\alpha < \omega_1$. For $\alpha = 0$, $C_0 = \{A_{0,1}, ..., A_{0,n}\}$ and $D_0 = \{B_{0,1}, ..., B_{0,m}\}$ for some $n, m \in \mathbb{N}$. If for each $i \leq n$ and each $j \leq m$ there exist $G_{0,i} \in [A_{0,i}]^\omega$ and $H_{0,j} \in [B_{0,j}]^\omega$ such that $\mathcal{P} \cup \{G_{0,i}\}$ and $\mathcal{P} \cup \{H_{0,j}\}$ are almost disjoint, let $E_0 = (\bigcup_{i=1}^n G_{0,i}) \cup (\bigcup_{j=1}^m H_{0,j})$ and put $\mathcal{A}_0 = \mathcal{P} \cup \{E_0\}$, which is almost disjoint. Otherwise let $\mathcal{A}_0 = \mathcal{P}$.

Now, for each $0 < \alpha < \omega_1$, assume we have built $\mathcal{A}_\beta$ for each $\beta < \alpha$. If α is a limit ordinal, let $\mathcal{A}'_\alpha = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. It is clear that $\mathcal{A}'_\alpha$ is an almost disjoint family. Now consider $\langle C_\alpha, D_\alpha\rangle$, we write $C_\alpha = \{A_{\alpha,1}, ..., A_{\alpha,n}\}$ and $D_\alpha = \{B_{\alpha,1}, ..., B_{\alpha,m}\}$ for some $n, m \in \mathbb{N}$. We proceed as before, if for each $i \leq n$ and each $j \leq m$ there exist $G_{\alpha,i} \in [A_{\alpha,i}]^\omega$ and $H_{\alpha,j} \in [B_{\alpha,j}]^\omega$ such that $\mathcal{A}'_\alpha \cup \{G_{\alpha,i}\}$ and $\mathcal{A}'_\alpha \cup \{H_{\alpha,j}\}$ are almost disjoint, let $E_\alpha =$

$(\bigcup_{i=1}^{n} G_{\alpha,i}) \cup (\bigcup_{j=1}^{m} H_{\alpha,j})$ and put $\mathcal{A}_\alpha = \mathcal{A}'_\alpha \cup \{E_\alpha\}$. Otherwise let $\mathcal{A}_\alpha = \mathcal{A}'_\alpha$. If $\alpha = \beta + 1$, let $\mathcal{A}'_\alpha = \mathcal{A}_\beta$ and consider $\langle C_\alpha, D_\alpha \rangle$. Construct $\mathcal{A}_\alpha$ by doing the process as before.

Finally, let $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$. Clearly, $\mathcal{A}$ is almost disjoint. In order to show that $\mathcal{A}$ is maximal, let M be any infinite subset of ω. We need to show that there exists $E \in \mathcal{A}$ such that $E \cap M$ is infinite. Suppose that for each $E \in \mathcal{A}$, $|E \cap M| < \omega$. Partition M into two infinite subsets M_1 and M_2. Pick the least $\alpha < \omega_1$ such that

$$\langle C_\alpha, D_\alpha \rangle = \langle \{M_1\}, \{M_2\} \rangle = \langle \{A_{\alpha,1}\}, \{B_{\alpha,1}\} \rangle.$$

Since for each $E \in \mathcal{A}$, $E \cap M$ is finite, we have that for each $E \in \mathcal{A}'_\alpha$, $|E \cap M| < \omega$. Thus, for each $E \in \mathcal{A}'_\alpha$ we have $|E \cap M_1| < \omega$ and $|E \cap M_2| < \omega$. That is, $M_1 \in [A_{\alpha,1}]^\omega$ and $M_2 \in [B_{\alpha,1}]^\omega$ satisfy that $\mathcal{A}'_\alpha \cup \{M_1\}$ and $\mathcal{A}'_\alpha \cup \{M_2\}$ are almost disjoint, hence there exists $E_\alpha \in \mathcal{A}_\alpha \subset \mathcal{A}$ such that $E_\alpha \cap M$ is infinite which is a contradiction. So, $\mathcal{A}$ is mad.

Claim: $\Psi(\mathcal{A})$ is quasi-normal.

Proof of Claim: Let A and B be non-empty disjoint π-closed subsets of $\Psi(\mathcal{A})$. Write $A = \bigcap_{i=1}^{n} A_i$ and $B = \bigcap_{j=1}^{m} B_j$ where each A_i and B_j are closed domains for each $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Observe that if there exists $i \in \{1, ..., n\}$ such that $|A_i \cap \omega| < \omega$, then $A_i \cap \mathcal{A} = \emptyset$ because for each $a \in \mathcal{A}$ we have $\{a\} \cup (a \setminus A_i)$ is an open neighborhood of a disjoint from A_i. Hence, A is a finite closed-and-open subset of $\Psi(\mathcal{A})$ which can be separated from B. Similarly, if there exists $j \in \{1, ..., m\}$ such that $|B_j \cap \omega| < \omega$, then B can be separated from A. So, assume that for each $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$, $|A_i \cap \omega| = \omega = |B_j \cap \omega|$. Take the least $\alpha < \omega_1$ such that for each $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ we have $A_{\alpha,i} = A_i \cap \omega$ and $B_{\alpha,j} = B_j \cap \omega$. Recalling our construction, at stage α, either $\mathcal{A}_\alpha = \mathcal{A}'_\alpha$ or $\mathcal{A}_\alpha = \mathcal{A}'_\alpha \cup \{E_\alpha\}$. But, $\mathcal{A}_\alpha = \mathcal{A}'_\alpha \cup \{E_\alpha\}$ is not possible since $E_\alpha = (\bigcup_{i=1}^{n} G_{\alpha,i}) \cup (\bigcup_{j=1}^{m} H_{\alpha,j})$ for some $G_{\alpha,i} \in [A_{\alpha,i}]^\omega$ and $H_{\alpha,j} \in [B_{\alpha,j}]^\omega$ and that implies E_α is in the closure of each $A_{\alpha,i}$ and each $B_{\alpha,j}$, hence $E_\alpha \in A \cap B$ and this is a contradiction as $A \cap B = \emptyset$. Thus, $\mathcal{A}_\alpha = \mathcal{A}'_\alpha$ and this means that for some $i \in \{1, ..., n\}$, it is the case that for each $G_{\alpha,i} \in [A_{\alpha,i}]^\omega$, $\mathcal{A}'_\alpha \cup \{G_{\alpha,i}\}$ is not almost disjoint or, for some $j \in \{1, ..., m\}$, every infinite $H_{\alpha,j} \subseteq B_{\alpha,j}$, is so that $\mathcal{A}'_\alpha \cup \{H_{\alpha,j}\}$ is not almost disjoint. Without loss of generality, assume that there exists such $i \in \{1, ..., n\}$. Observe that $\mathcal{A}'_\alpha$ is countable, hence $\mathcal{A}'_\alpha \restriction_{A_{\alpha,i}} = \{a \in \mathcal{A}'_\alpha : |a \cap A_{\alpha,i}| = \omega\}$ is either finite or countably infinite. But, it cannot be countably infinite because $\{a \cap A_{\alpha,i} : a \in \mathcal{A}'_\alpha \restriction_{A_{\alpha,i}}\}$ would be a countably infinite almost disjoint family on the set $A_{\alpha,i}$. Hence, it is not maximal and there is $G_{\alpha,i} \in [A_{\alpha,i}]^\omega$ such that $\{a \cap A_{\alpha,i} : a \in \mathcal{A}'_\alpha\} \cup \{G_{\alpha,i}\}$ is almost disjoint, contradicts that for each $G_{\alpha,i} \in [A_{\alpha,i}]^\omega$, $\mathcal{A}'_\alpha \cup \{G_{\alpha,i}\}$ is not almost disjoint. Therefore, $F = \{a \in \mathcal{A}'_\alpha : |a \cap A_{\alpha,i}| = \omega\}$ is finite.

Claim: $|A_{\alpha,i} \setminus \bigcup F| < \omega$. Assume $|A_{\alpha,i} \setminus \bigcup F| = \omega$, then $A_{\alpha,i} \setminus \bigcup F \in [A_{\alpha,i}]^\omega$, and by our hypothesis, $\mathcal{A}'_\alpha \cup \{A_{\alpha,i} \setminus \bigcup F\}$ is not almost disjoint. Thus, there exists $a \in \mathcal{A}'_\alpha \setminus F$ such that $|a \cap (A_{\alpha,i} \setminus \bigcup F)| = \omega$, but that implies $a \in F$, which is a contradiction.

Claim: $\{a \in \mathcal{A} : |a \cap A_{\alpha,i}| = \omega\} = F$. If there exists $a \in \mathcal{A} \setminus F$ such that $|a \cap A_{\alpha,i}| = \omega$, since $|A_{\alpha,i} \setminus \bigcup F| < \omega$, then $|a \cap \bigcup F| = \omega$. Since F is finite, there exists $b \in F \subset \mathcal{A}$ such that $|a \cap b| = \omega$, which is a contradiction.

Hence A_i is compact in the Tychonoff space $\Psi(\mathcal{A})$. Since A is a closed subset of A_i, then A is compact, thus A can be separated from B, see [1, 3.1.6]. Therefore, $\Psi(\mathcal{A})$ is quasi normal.

References

[1] R. Engelking. *General Topology.* PWN, Warszawa, 1977.

[2] L. Kalantan. Results about κ-normality. *Topology and its Applications*, 125:47–62, 2002.

[3] L. Kalantan. π-normal topological spaces. *Filomat*, 22(1):173–181, 2008.

[4] S. MrówkaS. Mrówka. On completely regular spaces. *Fundamenta Mathematicae*, 41:105–106, 1954.

[5] E. V. Shchepin. Real valued functions and spaces close to normal. *Sib. J. Math.*, 13:1182–1196., 1972.

[6] M.K. Singal and A.R. Singal. Mildly normal spaces. *Kyungpook Math J.*, 13:27–31, 1973.

[7] E. K. van Douwen. *The integers and topology.* Handbook of Set-Theoretic Topology, North-Holland, 1984.

[8] V. Zaitsev. On certain classes of topological spaces and their bicompactifications. *Dokl. Akad. Nauk SSSR*, 178:778–779, 1968.