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Univalence of New General Integral Operator Defined by the Ruscheweyh Type q-Difference Operator

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Abstract. In this study, by employing the Ruscheweyh type q-analogue operator we consider a new family of integral operators on the space of analytic functions. For this family, we demonstrate some sufficient conditions of univalence criteria on the class of analytical functions.

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Key Words and Phrases: q- analogue of Ruscheweyh operator, integral operators, univalence criteria.

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1. Introduction

Univalence criteria for certain class of analytic functions has attracted many and some of their work can be seen widely in the literature. For example, Pascu [21], [22] studied on the univalence criterion for certain class of functions and improvement of Becker's univalence criteria in 1985 and 1987 respectively. Then, Pescar [23] led on the generalised univalence criteria of Ahlfor's and Becker's. Later, Faisal and Darus [13–15] and Al-Refai and Darus [1] continued to study the same for different operators and classes. Here we are studying similar criteria for a class generated by a q-analogue of Ruscheweyh.

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the following normalized condition:

$$f(0) = f'(0) - 1 = 0.$$

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Additionally, let $\mathcal{S} \subset \mathcal{A}$ be the family of univalent functions in U. The Hadamard product for two analytic functions $f \in \mathcal{A}$ defined in (1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

is given by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Firstly, we will present the concepts and definitions for q-calculus which will later be applied (see [5] and [12]). Let $n \in \mathbb{N}$, $0<q<1$, the q-integer and q-factorial are defined by

$$[n]_q! = \begin{cases} [n]_q[n-1]_q......[1]_q, & n = 1, 2, ..., \\ 1, & n = 0, \end{cases} \tag{2}$$
$$[n]_q = \frac{1-q^n}{1-q}.$$

As $q \to 1$, $[n]_q \to n$.

In 2014, Aldweby and Darus [2] defined the Ruscheweyh type q-operator $\mathcal{R}_q^\upsilon$ as following:

Definition 1. *The q-analogue of Ruscheweyh operator of $f \in \mathcal{A}$ is denoted by $\mathcal{R}_q^\upsilon f(z)$ and defined by*

$$\mathcal{R}_q^\upsilon f(z) = z + \sum_{n=2}^{\infty} \frac{[n+\upsilon-1]_q!}{[\upsilon]_q![n-1]_q!} a_n z^n, \tag{3}$$

where $\upsilon > -1$ and $[n]_q!$ defined by (2).

From the Definition 1, we note that, if $q \to 1$,we have

$$\begin{aligned} \lim_{q \longrightarrow 1} \mathcal{R}_q^\upsilon f(z) &= z + \lim_{q \longrightarrow 1} \left[\sum_{n=2}^{\infty} \frac{[n+\upsilon-1]_q!}{[\upsilon]_q![n-1]_q!} a_n z^n \right] \\ &= z + \sum_{n=2}^{\infty} \frac{(n+\upsilon-1)!}{(\upsilon)!(n-1)!} a_n z^n \\ &= \mathcal{R}^\upsilon f(z), \end{aligned}$$

where $\mathcal{R}^\upsilon f(z)$ is Ruscheweyh operator that was presented in [24] and has been examined by many authors, for instance [19] and [26]. In fact, the q-derivative type of Ruscheweyh operator has been studied recently by Hussain et.al [17], Aldweby and Darus [3] for different properties. Other type of q-derivative can be seen in [16].

Definition 2. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}^{\upsilon}(q,\vartheta)$ if it is satisfying the condition*

$$\left|\frac{z^2\left(\mathcal{R}_q^{\upsilon} f(z)\right)'}{\left[\mathcal{R}_q^{\upsilon} f(z)\right]^2}-1\right|<\vartheta, \qquad (z\in U, 0<\vartheta\leq 1), \tag{4}$$

where $\mathcal{R}_q^{\upsilon} f(z)$ is the operator defined by (3).

Note that, $\mathcal{B}^0(q\to 1,\vartheta)=\mathcal{B}(\vartheta)$, where the analytic and univalent functions class $\mathcal{B}(\vartheta)$ was presented and studied in [11].

Using the operator $\mathcal{R}_q^{\upsilon} f(z)$, we now introduce the general integral operator as following:

Definition 3. *Let $m \in \mathbb{N}\cup\{0\}$, let $\gamma_1,\gamma_2,...,\gamma_n, |q|<1$ and $\varrho\in\mathbb{C}\setminus\{0,-1,...\}$, then the integral operator $I_{\gamma_n,\varrho}(\upsilon,q,z):\mathcal{A}\to\mathcal{A}$ is defined by*

$$I_{\gamma_n,\varrho}(\upsilon,q,z)=\left(\varrho\int_0^z t^{\varrho-1}\prod_{n=1}^{m}\left(\frac{\mathcal{R}_q^{\upsilon} f_n(t)}{t}\right)^{\frac{1}{\gamma_n}}dt\right)^{\frac{1}{\varrho}}, \tag{5}$$

where $f_n\in\mathcal{A}$.

Remark 1. *Interestingly, the integral operator $I_{\gamma_n,\varrho}(\upsilon,q,z)$ generalizes a number of operators that have been implemented and studied by several authors, for instance*

- *For $\upsilon=0$ and $\gamma_1,...,\gamma_m=\sigma$, we get the following operator*

$$I_{\sigma,\varrho}(z)=\left(\varrho\int_0^z t^{\varrho-1}\prod_{n=1}^{m}\left(\frac{f_n(t)}{t}\right)^{\frac{1}{\sigma}}dt\right)^{\frac{1}{\varrho}}, \tag{6}$$

that considered by Breaz and Breaz [7].

- *For $\upsilon=0, m=1, \gamma_n=\dfrac{1}{\sigma_n}, \varrho=1, \sigma_1=1, \sigma_2=...=\sigma_m=0$ and $f_1=f_2=...=f_m=f\in\mathcal{S}$, we have the following integral operator developed and studied by Alexander [4],*

$$I(z)=\int_0^z\frac{f(t)}{t}dt. \tag{7}$$

- *For $\upsilon=0, \varrho=1$ and $\gamma_n=\dfrac{1}{\sigma_n}$, we obtain the following integral operator introduced by Breaz and Breaz [6],*

$$f(z)=\int_0^z\left[\frac{f_1(t)}{t}\right]^{\sigma_1}...\left[\frac{f_m(t)}{t}\right]^{\sigma_m}dt. \tag{8}$$

- *For $\upsilon=0, \gamma_n=\dfrac{1}{\sigma-1}$ and $\varrho=m(\sigma-1)+1$, we have the integral operator:*

$$G_{m,\sigma}(z)=\left([m(\sigma-1)+1]\int_o^z (f_1(t))^{\sigma-1}...(f_m(t))^{\sigma-1}dt\right)^{\frac{1}{m(\sigma-1)+1}}, \tag{9}$$

studied by Breaz et al. [9].

- *For* $\upsilon = 0, m = 1, \gamma_n = \dfrac{1}{a_n}, \varrho = 1, \sigma_1 = \sigma, \sigma_2 = ... = \sigma_m = 0$ *and* $f_1 = f_2 = ... = f_m = f \in \mathcal{S}$, *we obtain the integral operator:*

$$I_\sigma(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\sigma dt, \tag{10}$$

introduced by Miller and Mocanu [18].

- *For* $\upsilon = 0, \gamma_n = \dfrac{1}{\sigma - 1}, \varrho = \sigma$ *and* $f_1 = f_2 = ... = f_m = f \in \mathcal{A}$ *where* $\sigma \in \mathbb{C}$ *and* $\Re(\sigma)>0$,*we obtain the following operator:*

$$G_\sigma(z) = \left(\sigma \int_0^z (f(t))^{\sigma-1} dt\right)^{\frac{1}{\sigma}}, \tag{11}$$

studied and introduced by Pescar [23].

- *For* $\upsilon = 1, q \to 1, \gamma_n = \dfrac{1}{\sigma - 1}$ *and* $\varrho = 1 + m(\sigma - 1)$,*we get the integral operator that Selvaraj and Karthikeyan [25] introduced*

$$G_\sigma(z) = \left([m(\sigma-1)+1] \int_o^z t^{m(\sigma-1)} \left(f_1'(t)\right)^{\sigma-1} ... \left(f_m'(t)\right)^{\sigma-1} dt\right)^{\frac{1}{1+m(\sigma-1)}}. \tag{12}$$

- *For* $\upsilon = 1, q \to 1, \gamma_n = \dfrac{1}{\sigma}$ *and* $\varrho = 1$, *we obtain the following integral operator:*

$$G_\sigma(z) = \int_o^z \left(f_1'(t)\right)^\sigma ... \left(f_m'(t)\right)^\sigma dt, \tag{13}$$

studied and introduced by Breaz and Güney [10].

2. Preliminaries

In order to prove our main results, we need to recall the following.

Lemma 1. (see [21] and [22]) *Let* $\varrho \in \mathbb{C}$ *with* $\Re(\varrho)>0$. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1 - |z|^{2\Re(\varrho)}}{\Re(\varrho)} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad z \in U,$$

then the operator

$$f_\varrho(z) = \left\{\varrho \int_0^z t^{\varrho-1} f'(t) dt\right\}^{\frac{1}{\varrho}},$$

is belonging to $\mathcal{S}$.

Lemma 2. (see [23]) *Let $c \in \mathbb{C}$ with $|c| \leq 1, c \neq -1$, $\varrho \in \mathbb{C}$ with $\Re(\varrho)>0$. If $f \in \mathcal{A}$ satisfies*

$$\left| c|z|^{2\varrho} + (1-|z|^{2\varrho})\frac{zf''(z)}{\varrho f'(z)} \right| \leq 1, \quad z \in U,$$

then the operator

$$f_\varrho(z) = \left\{ \varrho \int_0^z t^{\varrho-1} f'(t)dt \right\}^{\frac{1}{\varrho}},$$

is belonging to $\mathcal{S}$.

Lemma 3. (see [20]) (Generalized Schwarz Lemma) *Let $f \in \mathcal{A}$ within $U_R = \{z : |z|<R\}$, with $|f(z)|<N$ for fixed N. If $f(z)$ has one zero with multiplicity order $> m$ for $z = 0$, thus*

$$|f(z)| \leq \frac{N}{R^m}|z|^m, \qquad (z \in U_R).$$

Equality can only be achieved if

$$f(z) = e^{i\theta}\left(\frac{N}{R^m}\right)z^m,$$

where θ is constant.

3. Main Results

In this part, by utilizing the above lemmas, we find the univalence of this integral operator defined by Ruscheweyh type q-analogue.

Theorem 1. *Let $f_1, ..., f_m \in \mathcal{A}$ and ϱ, $\gamma_1, ..., \gamma_m \in \mathbb{C}$. Let $N \geq 1$ with*

$$\frac{1}{\Re(\varrho)} \sum_{n=1}^{m} \frac{[(1+\vartheta_n)N+1]}{|\gamma_n|} \leq 1. \tag{14}$$

If $f_1, ..., f_m \in \mathcal{B}^\upsilon(q, \vartheta_n)$, $0 < \vartheta_n \leq 1$, $n = 1, ..., m$ and

$$|\mathcal{R}_q^\upsilon f_n(z)| \leq N, \qquad (z \in U),$$

then the function $I_{\gamma_n,\varrho}(\upsilon, q, z)$ given by (5) is univalent.

Proof. From the definition of the operator $\mathcal{R}_q^\upsilon f(z)$ we have

$$\begin{aligned} \frac{\mathcal{R}_q^\upsilon f(z)}{z} &= \frac{z + \sum_{n=2}^{\infty} \frac{[n+\upsilon-1]_q!}{[\upsilon]_q![n-1]_q!} a_n z^n}{z} \\ &= 1 + \sum_{n=2}^{\infty} \frac{[n+\upsilon-1]_q!}{[\upsilon]_q![n-1]_q!} a_n z^{n-1}, \end{aligned}$$

then

$$\frac{\mathcal{R}_q^{\upsilon} f(z)}{z} \neq 0, \qquad (z \in U),$$

and for $z = 0$ and $n = 1, ..., m$, we have

$$\left(\frac{\mathcal{R}_q^{\upsilon} f_1(z)}{z}\right)^{\frac{1}{\gamma_1}} ... \left(\frac{\mathcal{R}_q^{\upsilon} f_m(z)}{z}\right)^{\frac{1}{\gamma_m}} = 1.$$

Define the function

$$f(z) = \int_0^z \prod_{n=1}^{m} \left(\frac{\mathcal{R}_q^{\upsilon} f_n(t)}{t}\right)^{\frac{1}{\gamma_n}} dt, \tag{15}$$

then we have $f(0) = 0$ and $f'(0) = 1$. Therefore

$$f^{'}(z) = \prod_{n=1}^{m} \left(\frac{\mathcal{R}_q^{\upsilon} f_n(z)}{z}\right)^{\frac{1}{\gamma_n}}. \tag{16}$$

The equality (16) implies

$$\ln f^{'}(z) = \sum_{n=1}^{m} \frac{1}{\gamma_n} \left(ln \frac{\mathcal{R}_q^{\upsilon} f_n(z)}{z}\right).$$

Or equivalently

$$\ln f^{'}(z) = \sum_{n=1}^{m} \frac{1}{\gamma_n} \left(ln \mathcal{R}_q^{\upsilon} f_n(z) - ln z\right).$$

By differentiating the above equality, we have

$$\frac{z f^{''}(z)}{f'(z)} = \sum_{n=1}^{m} \frac{1}{\gamma_n} \left(\frac{z \left(\mathcal{R}_q^{\upsilon} f_n(z)\right)^{'}}{\mathcal{R}_q^{\upsilon} f_n(z)} - 1\right). \tag{17}$$

From (17), we have

$$\left|\frac{z f^{''}(z)}{f^{'}(z)}\right| \leq \sum_{n=1}^{m} \frac{1}{|\gamma_n|} \left(\left|\frac{z \left(\mathcal{R}_q^{\upsilon} f_n(z)\right)^{'}}{\mathcal{R}_q^{\upsilon} f_n(z)}\right| + 1\right) = \sum_{n=1}^{m} \frac{1}{|\gamma_n|} \left(\left|\frac{z^2 \left(\mathcal{R}_q^{\upsilon} f_n(z)\right)^{'}}{[\mathcal{R}_q^{\upsilon} f_n(z)]^2}\right| \left|\frac{\mathcal{R}_q^{\upsilon} f_n(z)}{z}\right| + 1\right). \tag{18}$$

From the hypothesis, we have $|\mathcal{R}_q^{\upsilon} f_n(z)| \leq N,\ f_n \in \mathcal{B}^{\upsilon}(q, \vartheta_n), (n = 1, ..., m, z \in U)$, then by using lemma 3, we get that

$$|\mathcal{R}_q^{\upsilon} f_n(z)| \leq N\,|z|\,,\ (n = 1, ..., m, z \in U).$$

From (18), we get

$$\left|\frac{z f^{''}(z)}{f^{'}(z)}\right| \leq \sum_{n=1}^{m} \frac{1}{|\gamma_n|} \left(\left|\frac{z^2 \left(\mathcal{R}_q^{\upsilon} f_n(z)\right)^{'}}{[\mathcal{R}_q^{\upsilon} f_n(z)]^2}\right| N + 1\right)$$

$$\leq \sum_{n=1}^{m} \frac{1}{|\gamma_n|} \left(\left| \frac{z^2 \left(\mathcal{R}_q^{\upsilon} f_n(z)\right)'}{[\mathcal{R}_q^{\upsilon} f_n(z)]^2} - 1 \right| N + N + 1 \right)$$
$$\leq \sum_{n=1}^{m} \frac{1}{|\gamma_n|} \left(\vartheta_n N + N + 1\right)$$
$$= \sum_{n=1}^{m} \frac{(1+\vartheta_n)N + 1}{|\gamma_n|},$$

which easily shows that

$$\begin{aligned} \frac{1-|z|^{2\Re(\varrho)}}{\Re(\varrho)} \left| \frac{zf''(z)}{f'(z)} \right| &= \frac{1-|z|^{2\Re(\varrho)}}{\Re(\varrho)} \left| \sum_{n=1}^{m} \frac{1}{\gamma_n} \left(\frac{z\left(\mathcal{R}_q^{\upsilon} f_n(z)\right)'}{\mathcal{R}_q^{\upsilon} f_n(z)} - 1 \right) \right| \\ &\leq \frac{1}{\Re(\varrho)} \sum_{n=1}^{m} \frac{(1+\vartheta_n)N + 1}{|\gamma_n|}, \end{aligned}$$

since $\frac{1}{\Re(\varrho)} \sum_{n=1}^{m} \frac{[(1+\vartheta_n)N+1]}{|\gamma_n|} \leq 1$. Using Lemma 1 , we obtain that the integral $I_{\gamma_n,\varrho}(\upsilon, q, z)$ given by (5) is univalent.

Setting $N = 1, \upsilon = 0, \gamma_n = \frac{1}{\sigma - 1}$, and $\varrho = m(\sigma-1)+1$ in Theorem 1, we get

Corollary 1. *[8] Let $f_1, ..., f_m \in \mathcal{A}$ and $\sigma \in \mathbb{C}$ with*

$$|\sigma - 1| \leq \frac{\Re(\sigma)}{3m},$$

if

$$\left| \frac{z^2 f_k'(z)}{(f_n(z))^2} - 1 \right| < 1, \qquad (z \in U),$$

then the function $G_{m,\sigma}(z)$ defined by (9) is univalent.

Setting $N = 1, \upsilon = 0, \gamma_n = \frac{1}{\sigma - 1}$, $f_1 = ... = f_m = f \in \mathcal{A}$ and $\varrho = \sigma$ where $\sigma \in \mathbb{C}$ in Theorem 1, we get

Corollary 2. *Let $f \in \mathcal{A}$ and $\sigma \in \mathbb{C}$ with*

$$|\sigma - 1| \leq \frac{\Re(\sigma)}{3},$$

if

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1, \qquad (z \in U),$$

then the function $G_{\sigma}(z)$ defined by (11) is univalent.

Next, we prove

Theorem 2. *Let* $f_1, ..., f_m \in \mathcal{A}$, $\gamma_1, ..., \gamma_m \in \mathbb{C}$ *and* $\varrho \in \mathbb{C}$ *with* $\Re(\varrho) > \sum_{n=1}^{m} \frac{[(1+\vartheta_n)N+1]}{|\gamma_n|}$. *Let* $c \in \mathbb{C}$ *and* $N \geq 1$ *with*

$$|c| \leq 1 - \frac{1}{\Re(\varrho)} \sum_{n=1}^{m} \frac{[(1+\vartheta_n)N+1]}{|\gamma_n|}.$$

If $f_1, ..., f_m \in \mathcal{B}^{\upsilon}(q, \vartheta_n)$, $0 < \vartheta_n \leq 1$, $n = 1, ..., m$ *and*

$$|\mathcal{R}_q^{\upsilon} f_n(z)| \leq N, \qquad (z \in U),$$

then the function $I_{\gamma_n,\varrho}(\upsilon, q, z)$ *given by (5) is univalent.*

Proof. Following the proof of Theorem 1, we get

$$\frac{zf''(z)}{f'(z)} = \sum_{n=1}^{m} \frac{1}{\gamma_n} \left(\frac{z \left(\mathcal{R}_q^{\upsilon} f_n(z) \right)'}{\mathcal{R}_q^{\upsilon} f_n(z)} - 1 \right).$$

Then we have

$$\begin{aligned} \left| c|z|^{2\varrho} + (1 - |z|^{2\varrho}) \frac{zf''(z)}{\varrho f'(z)} \right| &= \left| c|z|^{2\varrho} + (1 - |z|^{2\varrho}) \frac{1}{\varrho} \sum_{n=1}^{m} \frac{1}{\gamma_n} \left(\frac{z \left(\mathcal{R}_q^{\upsilon} f_n(z) \right)'}{\mathcal{R}_q^{\upsilon} f_n(z)} - 1 \right) \right| \\ &\leq |c| + \frac{1}{|\varrho|} \sum_{n=1}^{m} \frac{1}{|\gamma_n|} \left(\left| \frac{z^2 \left(\mathcal{R}_q^{\upsilon} f_n(z) \right)'}{[\mathcal{R}_q^{\upsilon} f_n(z)]^2} \right| \frac{\left|\mathcal{R}_q^{\upsilon} f_n(z)\right|}{|z|} + 1 \right). \end{aligned}$$

Now directly from the proof of Theorem 1, we have

$$\begin{aligned} \left| c|z|^{2\varrho} + (1 - |z|^{2\varrho}) \frac{zf''(z)}{\varrho f'(z)} \right| &\leq |c| + \frac{1}{|\varrho|} \sum_{n=1}^{m} \frac{[(1+\vartheta_n)N+1]}{|\gamma_n|} \\ &\leq |c| + \frac{1}{\Re(\varrho)} \sum_{n=1}^{m} \frac{[(1+\vartheta_n)N+1]}{|\gamma_n|}, \end{aligned}$$

since $|c| \leq 1 - \dfrac{1}{\varrho} \sum_{n=1}^{m} \dfrac{[(1+\vartheta_n)N+1]}{|\gamma_n|}$, thus we have

$$\left| c|z|^{2\varrho} + (1 - |z|^{2\varrho}) \frac{zf''(z)}{\varrho f'(z)} \right| \leq 1, \qquad (z \in U).$$

Using Lemma 2 for the function $f(z)$ we obtain that the integral operator $I_{\gamma_n,\varrho}(\upsilon, q, z)$ given by (5) is univalent.

Corollary 3. *Let* $f_1, ..., f_m \in \mathcal{A}$*,* $\gamma \in \mathbb{C}$ *and* $\varrho \in \mathbb{C}$ *with* $\Re(\varrho) > \frac{m[(1+\vartheta_n)N+1]}{|\gamma|}$*. Let* $N \geq 1$ *with*

$$|c| \leq 1 - \frac{1}{\Re(\varrho)} \frac{m[(1+\vartheta_n)N+1]}{|\gamma|}, \qquad (c \in \mathbb{C}).$$

If for all $n = 1, .., m$*,* $f_n \in \mathcal{B}^{\upsilon}(q, \vartheta_n)$*,* $0 < \vartheta_n \leq 1$*, and*

$$|\mathcal{R}_q^{\upsilon} f_n(z)| \leq N, \qquad (z \in U).$$

Then the integral operator

$$I_{\gamma_n,\varrho}(\upsilon, q, z) = \left(\varrho \int_0^z t^{\varrho - 1} \prod_{n=1}^{m} \left(\frac{\mathcal{R}_q^{\upsilon} f_n(t)}{t} \right)^{\frac{1}{\gamma}} dt \right)^{\frac{1}{\varrho}},$$

is univalent.

Proof. In Theorem 2, we consider $\gamma_1 = \gamma_2 = ... = \gamma_m = \gamma$.

Corollary 4. *Let* $f_1, ..., f_m \in \mathcal{A}$*,* $\gamma_n \in \mathbb{C}$ *and* $\varrho \in \mathbb{C}$ *with* $\Re(\varrho) > \sum_{n=1}^{m} \frac{[\vartheta_n+2]}{|\gamma_n|}$*. Let* $c \in \mathbb{C}$ *with*

$$|c| \leq 1 - \frac{1}{\Re(\varrho)} \sum_{n=1}^{m} \frac{[\vartheta_n + 2]}{|\gamma_n|}.$$

If for all $n = 1, .., m$*,* $f_n \in \mathcal{B}^{\upsilon}(q, \vartheta_n)$*,* $0 < \vartheta_n \leq 1$*, and*

$$|\mathcal{R}_q^{\upsilon} f_n(z)| \leq 1, \qquad (z \in U),$$

then the function $I_{\gamma_n,\varrho}(\upsilon, q, z)$ *given by (5) is univalent.*

Proof. In Theorem 2, we consider $N = 1$.

Setting $\upsilon = 0, \gamma_n = \dfrac{1}{\sigma - 1}$, and $\varrho = m(\sigma - 1) + 1$ where $\sigma \in \mathbb{R}$ in Theorem 2, we have

Corollary 5. *Let* $f_1, ..., f_m \in \mathcal{A}$ *,* $\sigma \in \mathbb{R}, c \in \mathbb{C}$ *and* $N \geq 1$ *with*

$$|c| \leq 1 + \left(\frac{1 - \sigma}{(\sigma - 1)m + 1} \right) (2N + 1)m,$$

and

$$\sigma \in \left[1, \frac{2mN + 1}{2mN} \right],$$

if for all $n = 1, ..., m$

$$\left| \frac{z^2 f_k'(z)}{(f_n(z))^2} - 1 \right| < 1, \qquad (z \in U),$$

and

$$|f_n(z)| \leq N, \qquad (z \in U),$$

then the function $G_{m,\sigma}(z)$ *defined by (9) is univalent.*

4. Conclusion

In our present investigation, we have considered a new integral operator $I_{\gamma_n,\varrho}(\upsilon, q, z)$ by using the Ruscheweyh type q-analogue operator. Additionally, some sufficient conditions of univalence for this operator are determined.

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Conflict of interest

We declare that there is no conflict of interest.

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